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The Volatility Smile and Its Implied Tree

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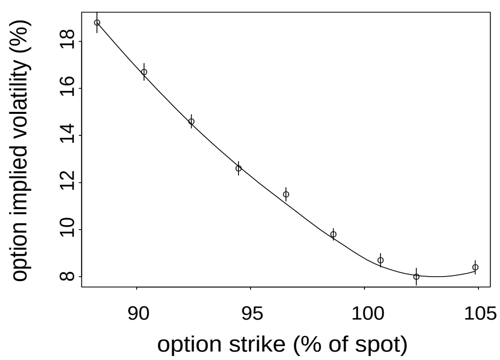
SUMMARY

The market implied volatilities of stock index options often have a skewed structure, commonly called “the volatility smile.” One of the long-standing problems in options pricing has been how to reconcile this structure with the Black-Scholes model usually used by options traders. In this paper we show how to extend the Black-Scholes model so as to make it consistent with the smile.

The Smile: Implied Volatilities of S&P 500 Options on Jan 31, 1994.

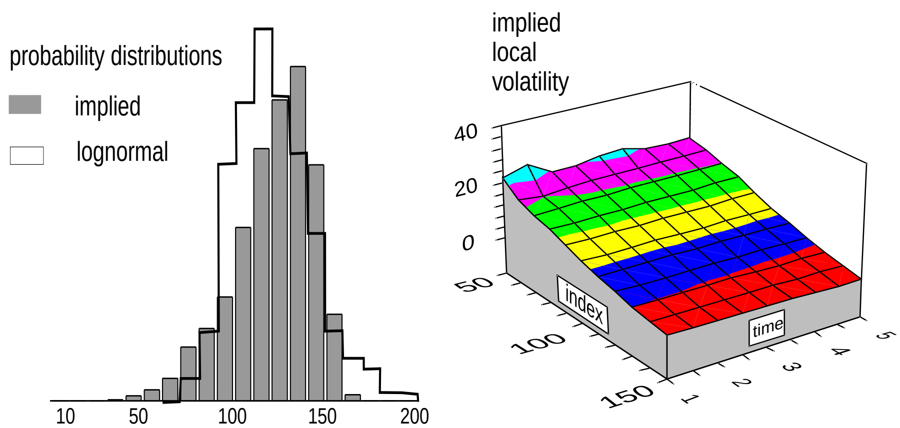


The Black-Scholes model assumes that the index level executes a random walk with a *constant* volatility. If the Black-Scholes model is correct, then the index distribution at any options expiration is lognormal, and all options on the index must have the same implied volatility. But, ever since the ‘87 crash, the market’s implied Black-Scholes volatilities for index options have shown a negative relationship between implied volatilities and strike prices – out-of-the-money puts trade at higher implied volatilities than out-of-the-money calls. The graph above illustrates this behavior for 47-day European-style March options on the S&P 500, as of January 31, 1994. The data for strikes above (below) spot comes from call (put) prices.

By empirically varying the Black-Scholes volatility with strike level, traders are implicitly attributing a unique *non-lognormal* distribution to the index. You can think of this non-lognormal distribution as a consequence of the index level executing a *modified* random walk – modified in the sense that the index has a *variable* volatility that depends on both stock price and time. To value European-style options consistently by calculating the expected values of their payoffs, you then need to know the exact form of the non-lognormal distribution. To value American-style or more exotic options, you must know the exact nature of the modified random walk – that is, how the volatility varies with stock price and time.

In this paper we show how you can use the smile – the prices of known, liquid, European-style index options of all available strikes and expirations – as inputs to deduce the form of the index's random walk. More specifically, we show how you can systematically extract, from the smile, a *unique binomial tree* for the index corresponding to the modified random walk mentioned above. We call this *the implied tree*. When you use this tree to value any of the options on which it is based, it produces values that match the observed market prices.

From this tree you can calculate both the distribution and the volatility of the index at future times and market levels, as implied by options prices. The chart below illustrates some of the information that follows from the implied tree.



You can use this implied tree to value other derivatives whose prices are not readily available from the market – standard but illiquid European-style options, American-style options and exotic options that depend on the details of the index distribution – secure in the knowledge that the model is valuing all your hedging instruments consistently with the market.

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THE BLACK-SCHOLES THEORY AND ITS DISCONTENTS

In this paper we propose an extension of the Black-Scholes (BS) options model in order to accommodate the structure of market implied volatilities for index options.

There are two important but independent features of the Black-Scholes theory. The primary feature of the theory is that it is *preference-free* — the values of contingent claims do not depend upon investors' risk preferences. Therefore, you can value an option *as though* the underlying stock's expected return is riskless. This *risk-neutral valuation* is allowed because you can hedge an option with stock to create an instantaneously riskless portfolio.

A secondary feature of the BS theory is its assumption that stock prices evolve lognormally with a constant *local volatility* σ at any time and market level. This stock price evolution over an infinitesimal time dt is described by the stochastic differential equation

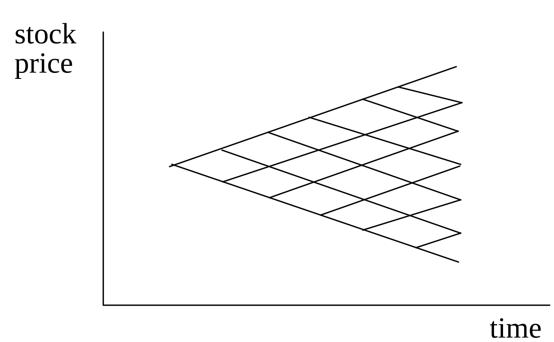
$$\frac{dS}{S} = \mu dt + \sigma dZ \qquad \text{(EQ 1)}$$

where S is the stock price, μ is its expected return and dZ is a Wiener process with a mean of zero and a variance equal to dt.

The Black-Scholes formula $C_{BS}(S, \sigma, r, t, K)$ for a call with strike K and time to expiration t, when the riskless rate is r, follows from applying the general method of risk-neutral valuation to a stock whose evolution is specifically assumed to follow Equation 1.

In the Cox-Ross-Rubinstein (CRR) binomial implementation of the process in Equation 1, the stock evolves along a risk-neutral binomial tree with constant logarithmic stock price spacing, corresponding to constant volatility, as illustrated schematically in Figure 1.

FIGURE 1. Schematic Risk-Neutral Stock Tree with Constant Volatility



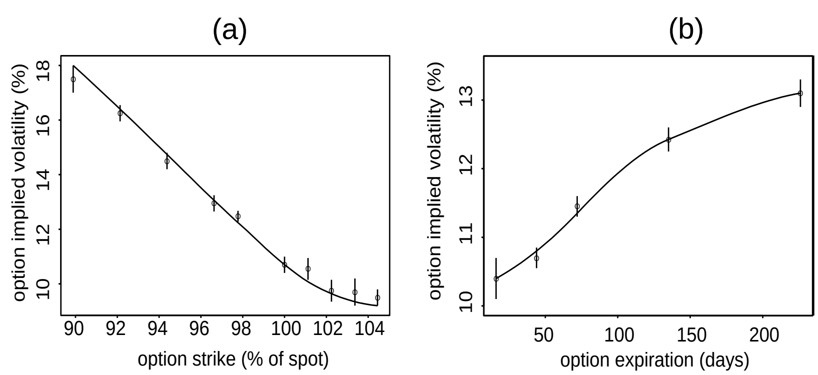
The binomial tree corresponding to the risk-neutral stock evolution is the same for all options on that stock, irrespective of their strike level or time to expiration. The stock tree cannot “know” about which option we are valuing on it.

Market options prices are not exactly consistent with theoretical prices derived from the BS formula. Nevertheless, the success of the BS framework has led traders to quote a call option’s market price in terms of whatever constant local volatility σ_{imp} makes the BS formula value equal to the market price. We call σ_{imp} the *Black-Scholes-equivalent* or *implied volatility*, to distinguish it from the theoretically constant local volatility σ assumed by the BS theory. In essence, σ_{imp} is a means of quoting prices.

The Smile

How consistent are market option prices with the BS formula? Figure 2(a) shows the decrease of σ_{imp} with the strike level of options on the S&P 500 index with a fixed expiration of 44 days, as observed on May 5, 1993. This asymmetry is commonly called the volatility “skew.” Figure 2(b) shows the increase of σ_{imp} with the time to expiration of at-the-money options. This variation is generally called the volatility “term structure.” In this paper we will refer to them collectively as the volatility “smile.”

FIGURE 2. **Implied Volatilities of S&P 500 Options on May 5, 1993**



In Figure 2(a) the data for strikes above (below) spot comes from call (put) prices. In Figure 2(b) the average of at-the-money call and put implied volatilities are used. You can see that σ_{imp} falls as the strike level increases. Out-of-the-money puts trade at higher implied volatilities than out-of-the-money calls. Though the exact shape and magnitude vary from day-to-day, the asymmetry persists and belies the BS theory, which assumes constant local (and therefore, constant implied) volatility for all options. Its persistence suggests a discrepancy between theory and the market. It may be convenient to keep *quoting* options prices in terms of Black-Scholes-equivalent volatilities, but it is probably incorrect to *calculate* options prices using the BS formula.

There have been various attempts to extend the BS theory to account for the volatility smile. One approach incorporates a stochastic volatility factor[1]; another allows for discontinuous jumps in the stock price[2]. These extensions cause several practical difficulties. First, since there are no securities with which to directly hedge the volatility or the jump risk, options valuation is in general no longer preference-free. Second, in these multifactor models, options values depend upon several additional parameters whose values must be estimated. This often makes confident option pricing difficult.

1. See, for instance, J. Hull and A. White, *Journal of Finance 42, 281-300, 1987.*
2. See R. Merton, *Journal of Financial Economics 3, 125-144, 1976.*

THE IMPLIED TREE

We want to develop an arbitrage-free model that fits the smile, is preference-free, avoids additional factors and can be used to value options from easily observable data.

The most natural and minimal way to extend the BS model is to replace Equation 1 above by

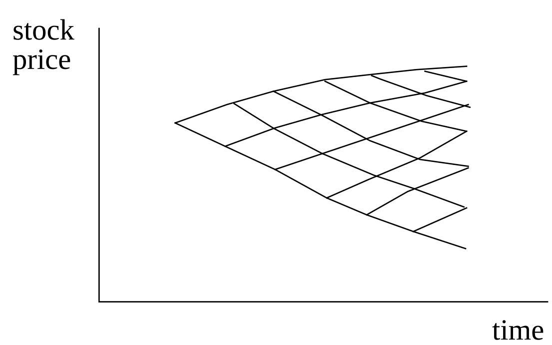
$$\frac{dS}{S} = \mu dt + \sigma(S, t) dZ \qquad \text{(EQ 2)}$$

where $\sigma(S,t)$ is the *local volatility function* that is dependent on both stock price and time.

Other models of this type often involve a special parametric form for $\sigma(S,t)$. In contrast, our approach is to deduce $\sigma(S,t)$ numerically from the smile[3]. We can completely determine the unknown function $\sigma(S,t)$ by requiring that options prices calculated from this model fit the smile.

In the binomial framework in which we work, the regular binomial tree of Figure 1 will be replaced by a distorted or *implied tree*, drawn schematically in Figure 3. Options prices for all strikes and expirations, obtained by interpolation from known options prices, will determine the position and the probability of reaching each node in the implied tree.

FIGURE 3. The Implied Tree



3. We have become aware of two recent papers with similar aims. See Mark Rubinstein, *Implied Binomial Trees*, talk presented to the American Finance Association, January 1993, and Bruno Dupire, *Pricing With A Smile*, RISK, January 1994, pages 18-20.

CONSTRUCTING THE TREE

We use induction to build an implied tree with uniformly spaced levels, Δt apart. Assume you have already constructed the first n levels that match the implied volatilities of all options with all strikes out to that time period. Figure 4 shows the n^{th} level of the tree at time t_n, with n implied tree nodes and their already known stock prices s_i.

Notation

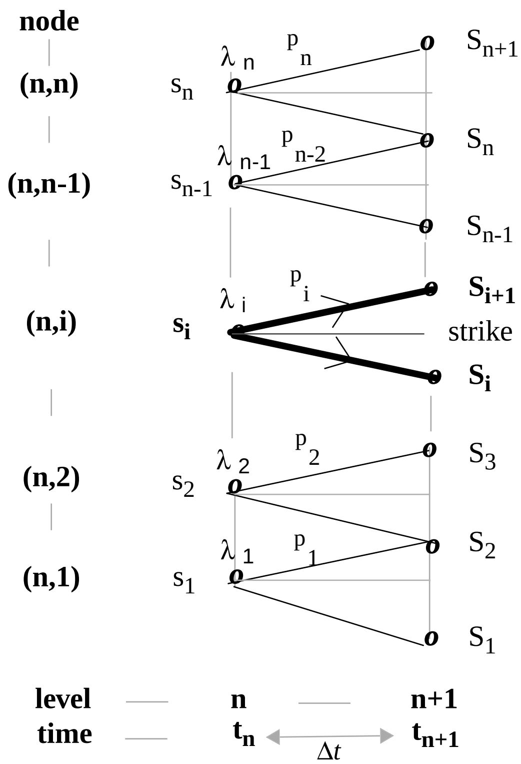
We call the continuously compounded forward riskless interest rate at the n^{th} level r. In general this rate is time-dependent and can vary from level to level; for notational simplicity we avoid attaching an explicit level index to this and other variables used here. We want to determine the nodes of the $(n+1)^{th}$ level at time t_{n+1}. There are $n+1$ nodes to fix, with $n+1$ corresponding unknown stock prices S_i. Figure 4 shows the i^{th} node at level n, denoted *by* **(n,i)** in boldface. It has a known stock price s_i and evolves into an "up" node with price S_{i+1} and a "down" node with price S_i at level $n+1$, where the forward price corresponding to s_i is $F_i = e^{r\Delta t}s_i$. We call p_i the probability of making a transition into the up node. We call λ_i the Arrow-Debreu price at node (n,i); it is computed by forward induction as the sum over all paths, from the root of the tree to node (n,i), of the product of the risklessly-discounted transition probabilities at each node in each path leading to node (n,i). All λ_i at level n are known because earlier tree nodes and their transition probabilities have already been implied out to level n.

There are $2n+1$ parameters that define the transition from the n^{th} to the $(n+1)^{th}$ level of the tree, namely the $n+1$ stock prices S_i and the n transition probabilities p_i. We show how to determine them using the smile.

Implying the Nodes

We imply the nodes at the $(n+1)^{th}$ level by using the tree to calculate the theoretical values of $2n$ known quantities – the values of n forwards and n options, all expiring at time t_{n+1} – and requiring that these theoretical values match the interpolated market values. This provides $2n$ equations for these $2n+1$ parameters. We use the one remaining degree of freedom to make the center of our tree coincide with the center of the standard CRR tree that has constant local volatility. If the number of nodes at a given level is odd, choose the central node's stock price to be equal to spot today; if the number is even, make the average of the natural logarithms of the two central nodes' stock prices equal to the logarithm of today's spot price. We now derive the $2n$ equations for the theoretical values of the forwards and the options.

FIGURE 4. Constructing the (n+1)th Level of the Implied Tree



NOTATION

r: known forward riskless interest rate at this level

s_i: known stock price at node *(n,i)* at level *n;* node *i;* also the strike for options expiring at level *n+1*

F_i: known forward price at level *n+1* of the known price s_i at level *n*

S_i: unknown stock price at node *(n+1,i)*

λ_i: known Arrow-Debreu price at node *(n,i)*

p_i: unknown risk-neutral transition probability from node *(n,i)* to node *(n+1,i+1)*

The implied tree is risk-neutral. Consequently, the expected value, one period later, of the stock at any node *(n,i)* must be its known forward price. This leads to the equation

$$F_i = p_i S_{i+1} + (1 - p_i) S_i \qquad \textbf{(EQ 3)}$$

where F_i is known. There are n of these forward equations, one for each i.

The second set of equations expresses the values of the n independent options[4], one for each strike s_i equal to the known stock prices at the n^{th} level, that expire at the $(n+1)^{th}$ level. The strike level s_i splits

4. There are only n independent options because puts and calls with the same strike are related through put-call parity, which holds in our model because the implied tree is constrained to value all forwards correctly.

the up and down nodes, S_{i+1} and S_i, at the next level, as shown in Figure 4. This ensures that only the up (down) node and all nodes above (below) it contribute to a call (put) struck at s_i. These n equations for options, derived below, together with Equation 3 and our choice in centering the tree, will determine both the transition probabilities p_i that lead to the $(n+1)^{th}$ level and the stock prices S_i at the nodes at that level.

Let $C(s_i,t_{n+1})$ and $P(s_i,t_{n+1})$, respectively, be the known market values for a call and put struck today at s_i and expiring at t_{n+1}. We know the values of each of these calls and puts from interpolating the smile curve at time t_{n+1}. The theoretical binomial value of a call struck at K and expiring at t_{n+1} is given by the sum over all nodes j at the $(n+1)^{th}$ level of the discounted probability of reaching each node $(n+1, j)$ multiplied by the call payoff there, or

$$C(K, t_{n+1}) = e^{-r\Delta t}\sum_{j=1}^{n}\{\lambda_j p_j + \lambda_{j+1}(1-p_{j+1})\}\max(S_{j+1}-K, 0) \quad \textbf{(EQ 4)}$$

When the strike K equals s_i, the contribution from the transition to the first in-the-money up node can be separated from the other contributions, which, using Equation 3, can be rewritten in terms of the known Arrow-Debreu prices, the known stock prices s_i and the known forwards $F_i = e^{r\Delta t}s_i$, so that

$$e^{r\Delta t}C(s_i, t_{n+1}) = \lambda_i p_i(S_{i+1}-s_i) + \sum_{j=i+1}^{n}\lambda_j(F_j - s_i) \quad \textbf{(EQ 5)}$$

The first term depends upon the unknown p_i and the up node with unknown price S_{i+1}. The second term is a sum of already known quantities.

Since we know both F_i and $C(s_i,t_{n+1})$ from the smile, we can simultaneously solve Equation 3 and Equation 5 for S_{i+1} and the transition probability p_i in terms of S_i:

$$S_{i+1} = \frac{S_i[e^{r\Delta t}C(s_i, t_{n+1}) - \Sigma] - \lambda_i s_i(F_i - S_i)}{[e^{r\Delta t}C(s_i, t_{n+1}) - \Sigma] - \lambda_i(F_i - S_i)} \quad \textbf{(EQ 6)}$$

$$p_i = \frac{F_i - S_i}{S_{i+1} - S_i} \quad \text{(EQ 7)}$$

where Σ denotes the summation term in Equation 5.

We can use these equations to find iteratively the S_{i+1} and p_i for all nodes above the center of the tree if we know S_i at one initial node. If the number of nodes at the $(n+1)^{th}$ level is odd (that is, n is even), we can identify the initial S_i, for $i = n/2 + 1$, with the central node whose stock price we choose to be today's spot value, as in the CRR tree. Then we can calculate the stock price S_{i+1} at the node above from Equation 6, and then use Equation 7 to find the p_i. We can now repeat this process moving up one node at a time until we reach the highest node at this level. In this way we imply the upper half of each level.

If the number of nodes at the $(n+1)^{th}$ level is even (that is, n is odd), we start instead by identifying the initial S_i and S_{i+1}, for $i = (n+1)/2$, with the nodes just below and above the center of the level. The logarithmic CRR centering condition we chose is equivalent to choosing these two central stock prices to satisfy $S_i = S^2/S_{i+1}$, where $S = s_i$ is today's spot price corresponding to the CRR-style central node at the previous level. Substituting this relation into Equation 6 gives the formula for the upper of the two central nodes for even levels:

$$S_{i+1} = \frac{S[e^{r\Delta t}C(S, t_{n+1}) + \lambda_i S - \Sigma]}{\lambda_i F_i - e^{r\Delta t}C(S, t_{n+1}) + \Sigma} \quad \text{for } i = n/2 \quad \text{(EQ 8)}$$

Once we have this initial node's stock price, we can continue to fix higher nodes as shown above.

In a similar way we can fix all the nodes below the central node at this level by using known put prices. The analogous formula that determines a lower node's stock price from a known upper one is

$$S_i = \frac{S_{i+1}[e^{r\Delta t}P(s_i, t_{n+1}) - \Sigma] + \lambda_i s_i(F_i - S_{i+1})}{[e^{r\Delta t}P(s_i, t_{n+1}) - \Sigma] + \lambda_i(F_i - S_{i+1})} \quad \text{(EQ 9)}$$

where here Σ denotes the sum $\sum_{j=1}^{i-1} \lambda_j (s_i - F_j)$ over all nodes below the one with price s_i at which the put is struck. If you know the value of the stock price at the central node, you can use Equation 9 and Equation 7 to find, node by node, the values of the stock prices and transition probabilities at all the lower nodes.

By repeating this process at each level, we can use the smile to find the transition probabilities and node values for the entire tree. If we do this for small enough time steps between successive levels of the tree, using interpolated call and put values from the smile curve, we obtain a good discrete approximation to the implied risk-neutral stock evolution process.

Avoiding Arbitrages

The transition probabilities p_i at any node in the implied tree must lie between 0 and 1. If $p_i > 1$, the stock price S_{i+1} at the up-node at the next level will fall below the forward price F_i in Figure 4. Similarly, if` $p_i < 0$, the stock price S_i at the down-node at the next level will fall above the forward price F_i. Either of these conditions allows riskless arbitrage. Therefore, as we move through the tree node by node, we demand that each newly determined node's stock price must lie between the neighboring forwards from the previous level, that is $F_i < S_{i+1} < F_{i+1}$.

If the stock price at a node violates the above inequality, we override the option price that produced it. Instead we choose a stock price that keeps the logarithmic spacing between this node and its adjacent node the same as that between corresponding nodes at the previous level. This procedure removes arbitrage violations (in this one-factor model) from input option prices, while keeping the implied local volatility function smooth.

HOW IT WORKS: A DETAILED EXAMPLE

We now illustrate the construction of a complete tree from the smile. To keep life simple, we build the tree for levels spaced one year apart. You can do it for more closely spaced levels on a computer.

We assume that the current value of the index is 100, its dividend yield is zero, and that the annually compounded riskless interest rate is 3% per year for all maturities. We assume that the annual implied volatility of an at-the-money European call is 10% for all expirations, and that implied volatility increases (decreases) linearly by 0.5 percentage points with every 10 point drop (rise) in the strike. This defines the smile.

Figure 5 shows the standard (not implied) CRR binomial stock tree for a local volatility of 10% everywhere. This tree produces no smile and is the discrete binomial analog of the continuous-time BS equation. We use it to convert implied volatilities into quoted options prices. Its up and down moves are generated by factors $exp(\pm\sigma/100)$. The transition probability at every node is 0.625.

FIGURE 5. Binomial Stock Tree with Constant 10% Stock Volatility

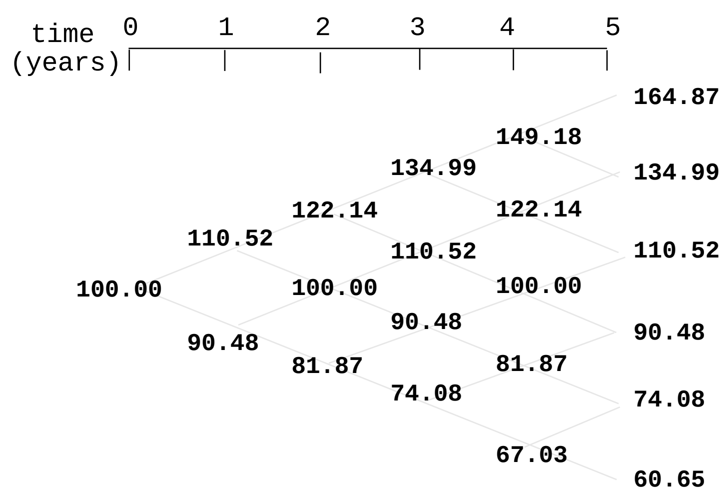
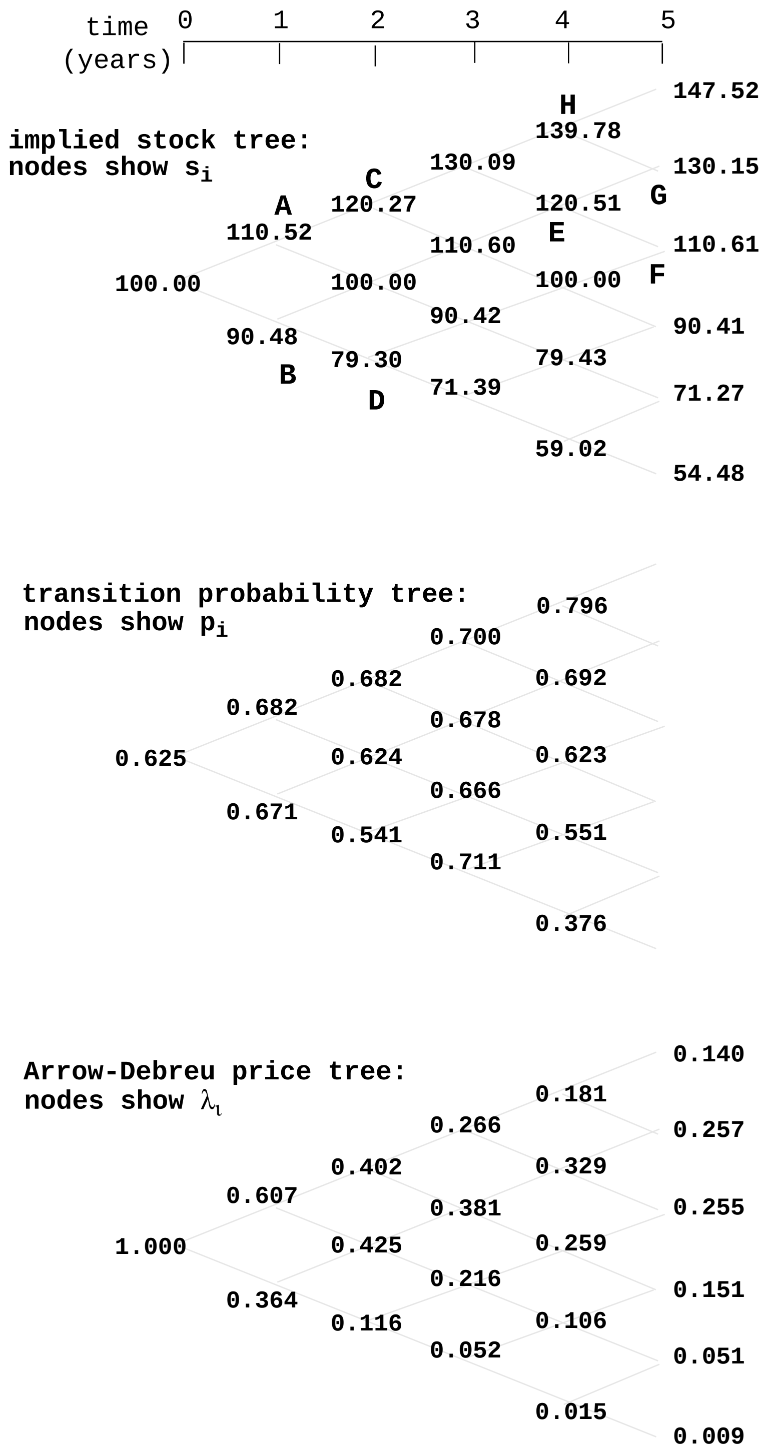


Figure 6 displays the implied stock tree, the tree of transition probabilities and the tree of Arrow-Debreu prices that fits the smile. We illustrate how a few representative node parameters are fixed in our model.

FIGURE 6. **The Implied Tree, Probability Tree and Arrow-Debreu Tree**



First, the assumed 3% interest rate means that the forward price one year later for any node is 1.03 times that node's stock price.

Today's stock price at the first node on the implied tree is 100, and the corresponding initial Arrow-Debreu price $\lambda_0 = 1.000$. Now let's find the node A stock price in level 2 of Figure 6. Using Equation 8 for even levels, we set $S_{i+1} = S_A$, $S = 100$, $e^{r\Delta t} = 1.03$ and $\lambda_1 = 1.000$. Then

$$S_A = \frac{100[1.03 \times C(100, 1) + 1.000 \times 100 - \Sigma]}{1.000 \times 103 - 1.03 \times C(100, 1) + \Sigma}$$

where $C(100,1)$ is the value today of a one-year call with strike 100. Σ must be set to zero because there are no higher nodes than the one with strike above 100 at level 0. According to the smile, we must value the call $C(100,1)$ at an implied volatility of 10%. In the simplified binomial world we use here, $C(100, 1) = 6.38$ when valued on the tree of Figure 5. Inserting these values into the above equation yields $S_A = 110.52$. The price corresponding to the lower node B in Figure 6 is given by our chosen centering condition $S_B = S^2 / S_A = 90.48$. From Equation 7, the transition probability at the node in year 0 is

$$p = \frac{(103 - 90.48)}{(110.52 - 90.48)} = 0.625$$

Using forward induction, the Arrow-Debreu price at node A is given by $\lambda_A = (\lambda_0 p)/1.03 = (1.00 \times 0.625)/1.03 = 0.607$, as shown on the bottom tree in Figure 6. In this way the smile has implied the second level of the tree.

Now let's look at the nodes in year 2. We *choose* the central node to lie at 100. The next highest node C is determined by the one-year forward value $F_A = 113.84$ of the stock price $S_A = 110.52$ at node A, and by the two-year call $C(S_A,2)$ struck at S_A. Because there are no nodes with higher stock values than that of node A in year 1, the Σ term is again zero and Equation 8 gives

$$S_C = \frac{100[1.03 \times C(S_A, 2)] - 0.607 \times S_A \times (F_A - 100)}{1.03 \times C(S_A, 2) - 0.607 \times (F_A - 100)}$$

The value of $C(S_A,2)$ at the implied volatility of 9.47% corresponding to a strike of 110.52 is 3.92 in our binomial world. Substituting these values into the above equation yields S_C = 120.27. Equation 7 for the transition probability gives

$$p_A = \frac{(113.84 - 100)}{(120.27 - 100)} = 0.682$$

We can similarly find the new Arrow-Debreu price λ_C. We can also show that the stock price at node D must be 79.30 to make the put price $P(S_B,2)$ have an implied volatility of 10.47% consistent with the smile.

The implied local one-year volatility at node A in the tree is

$$\sigma_A = \sqrt{p_A(1 - p_A)}\log(120.27 / 100) = 8.60\%$$

Similarly, σ_B = 10.90%. You can see that fitting the smile causes local volatility one year out to be greater at lower stock prices.

To leave nothing in doubt, we show how to find the value of one more stock price, that at node G in year 5 of Figure 6. Let's suppose we have already implied the tree out to year 4, and also found the value of S_F at node F to be 110.61, as shown in Figure 6. The stock price S_G at node G is given by Equation 8 as

$$S_G = \frac{S_F[1.03 \times C(S_E, 5) - \Sigma] - \lambda_E \times S_E \times (F_E - 110.61)}{[1.03 \times C(S_E, 5) - \Sigma] - \lambda_E \times (F_E - 110.61)}$$

where S_E = 120.51 and $F_E = 120.51 \times 1.03 = 124.13$ and λ_E = 0.329. The smile's interpolated implied volatility at a strike of 120.51 is 8.86%, corresponding to a call value $C(120.51,5)$ = 6.24. The value of the Σ term in the above equation is given by the contribution to this call from the node H above node E in year 4. From Equation 5 and Figure 6 it is

$$\begin{aligned}\Sigma &= \lambda_H(F_H - S_E)\\ &= 0.181 \times (1.03 \times 139.78 - 120.51)\\ &= 4.247\end{aligned}$$

Substituting these values gives S_G = 130.15.

SOME DISTRIBUTIONS

Once you have an implied tree that fits the smile, you can look at distributions of future stock prices in the risk-neutral world. If you take the model seriously, these are the distributions the market is attributing to the stock through its quoted options prices.

The implied distributions in Figure 7 result from fitting an implied five-year tree with 500 levels to the following smile: for all expirations, at-the-money (strike=100) implied volatility is 10%, and increases (decreases) by one percentage point for every 10% drop (rise) in the strike. We assume a continuously compounded interest rate of 3% per year, and no stock dividends.

Figure 7(a) shows the implied risk-neutral stock price distribution at five years, as computed from the implied tree. The mean stock price is 116.18; the standard deviation is 21.80%.

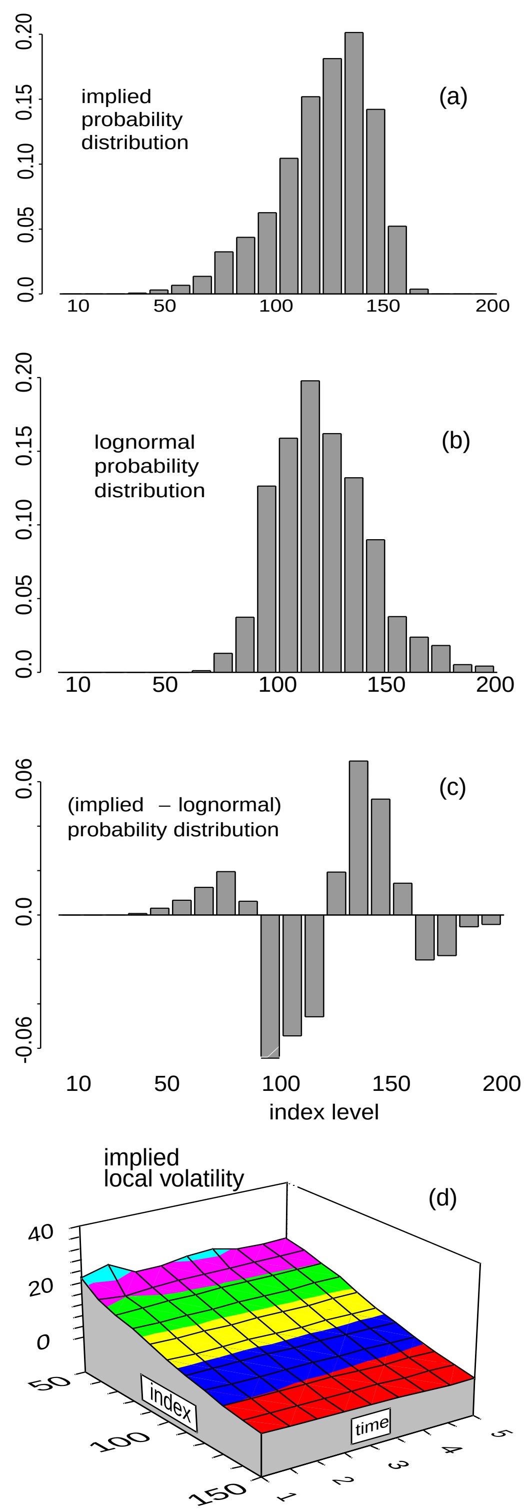
Figure 7(b) shows the lognormal distribution with the same mean and standard deviation. You can see that the implied tree has a distribution that is shifted towards low stock prices.

Figure 7(c) shows the difference between the two distributions.

Figure 7(d) shows the local volatility $\sigma(S,t)$ in the implied tree at all times and stock price levels. To explain this smile the local volatility must decrease sharply with increasing stock price and vary slightly with time.

In this example we have found the implied tree and its distributions resulting from a smile whose shape is independent of expiration time. We can do the same for more complex smiles, where volatility changes with time to expiration.

FIGURE 7. The Implied Distributions



CONCLUSION

We have shown that you can use the volatility smile of liquid index options, as observed at any instant in the market, to construct an entire implied tree. This tree will correctly value all standard calls and puts that define the smile. In the continuous time limit, the risk-neutral stochastic evolution of the stock price in our model has been completely determined by market prices for European-style standard options.

You can use this tree to value other derivatives whose prices are not readily available from the market – standard but illiquid European-style options, American-style options and exotic options – secure in the knowledge that the model is valuing all your hedging instruments consistently with the market. We believe the model may be especially useful for valuing barrier options, where the probability of striking the barrier is sensitive to the shape of the smile. You can also use the implied tree to create static hedge portfolios for exotic options[5], and to generate Monte Carlo distributions for valuing path-dependent options.

Finally, it would be interesting to see to what extent the implied tree's local volatility function $\sigma(S,t)$ forecasts index volatility at future times and market levels.

5. E. Derman, *RISK Exotic Options Conference*, London, December 1993.

APPENDIX: The Continuous-Time Theory

In this section we will investigate the continuous time theory associated with the stock price diffusion process

$$\frac{dS}{S} = r(t)dt + \sigma(S, t)dZ \tag{A 1}$$

where $r(t)$ is the expected instantaneous stock price return, which is assumed to be a deterministic function of time, and $\sigma(S, t)$ is local volatility function which is assumed to be a (path-independent) function of stock price and time. Here $Z(t)$ denotes the standard Brownian motion. Let $\Phi(S', S, t)$ denote the transition probability function associated with the diffusion Equation A1. It is defined as the probability that the stock price reaches the value S' at time t given its starting value S at time 0. It is well known that this function satisfies both the backward and forward Kolmogorov equations together with the boundary condition $\Phi(S, S', 0) = \delta(S' - S)$, where $\delta(x)$ is the Dirac delta function. The backward equation reads

$$\frac{1}{2}\sigma^2(S, t)S^2\frac{\partial^2\Phi}{\partial S^2} + r(t)S\frac{\partial\Phi}{\partial S} - \frac{\partial\Phi}{\partial t} = 0 \tag{A 2}$$

while the forward equation is its formal adjoint

$$\frac{1}{2}\frac{\partial^2}{\partial S'^2}(\sigma^2(S', t)S'^2\Phi) - r(t)\frac{\partial}{\partial S'}(S'\Phi) - \frac{\partial\Phi}{\partial t} = 0 \tag{A 3}$$

Let $D(t)$ denote the discount function

$$D(t) = exp\left(-\int_0^t r(t')dt'\right) \tag{A 4}$$

Then the value of a standard European call option with spot price S, strike price K and time to expiration t is given by

$$C(S, K, t) = D(t)\int_K^\infty \Phi(S, S', t)(S' - K)dS' \tag{A 5}$$

Differentiating Equation A5 once with respect to strike price K leads to the following relationship between a *strike spread* and the integrated distribution function:

$$D(t)\int_K^{\infty}\Phi(S, S', t)\,dS' = \frac{\partial}{\partial K}C(S, K, t) \quad \textbf{(A 6)}$$

Differentiating Equation A5 twice with respect to strike price K leads to the following relationship between a *butterfly spread* and the distribution function:

$$D(t)\Phi(S, K, t) = \frac{\partial^2}{\partial K^2}C(S, K, t) \quad \textbf{(A 7)}$$

The left side of this equation is the familiar Arrow-Debreu price in this theory. It is the price of a security whose payoff function is given by $\delta(S' - K)$. If, for a given stock level, the prices (and therefore, all partial derivatives with respect to the strike) of call options of all strikes and all maturities were to be available, Equation A7 would entirely specify the distribution functions of this theory. However, the stock distribution function is not necessarily sufficient to determine the diffusion process completely. Different diffusion processes can have the same distribution functions. Remarkably, though, all the parameters of the diffusion process in Equation A1 are uniquely specified by the stock price distribution.

To show this, we will establish that the standard European call option prices $C(S, K, t)$ in this theory satisfy the following “forward” equation:

$$\frac{1}{2}\sigma^2(K, t)K^2\frac{\partial^2 C}{\partial K^2} - r(t)K\frac{\partial C}{\partial K} - \frac{\partial C}{\partial t} = 0 \quad \textbf{(A 8)}$$

Our proof here is a variation of the original proof by Dupire[3]. Multiplying both sides of Equation A3 by $(S' - K)$ and integrating with respect to S' leads to:

$$\begin{aligned}&\frac{1}{2}D(t)\int_K^{\infty}\frac{\partial^2}{\partial S'^2}[\sigma^2(S', t)S'^2\Phi(S, S', t)](S' - K)\,dS' \\ &-r(t)D(t)\int_K^{\infty}\frac{\partial}{\partial S'}[S'\Phi(S, S', t)](S' - K)\,dS' \\ &-D(t)\int_K^{\infty}\frac{\partial}{\partial t}\Phi(S, S', t)(S' - K)\,dS' = 0\end{aligned} \quad \textbf{(A 9)}$$

Integrating the first term on the left side of Equation A9 by parts, and then substituting from Equation A6 leads to

$$\frac{1}{2}D(t)\int_K^{\infty}\frac{\partial^2}{\partial S'^2}[\sigma^2(S',t)S'^2\Phi(S,S',t)](S'-K)dS' =$$
$$\frac{1}{2}\sigma^2(K,t)K^2\frac{\partial^2}{\partial K^2}C(S,K,t) + \text{boundary terms at infinity} \tag{A 10}$$

Integrating the second term on the left side of Equation A9 by parts, and then substituting from Equation A7 leads to

$$r(t)D(t)\int_K^{\infty}\frac{\partial}{\partial S'}[S'\Phi(S,S',t)](S'-K)dS' =$$
$$-r(t)D(t)\int_K^{\infty}S'\Phi(S,S',t)dS' + \text{boundary terms at infinity} = \tag{A 11}$$
$$-r(t)\left[C(S,K,t)-K\frac{\partial}{\partial K}C(S,K,t)\right] + \text{boundary term at infinity}$$

Finally, using Equation A5, the last term on the left side of Equation A9 can be written in the form

$$D(t)\int_K^{\infty}\frac{\partial}{\partial t}\Phi(S,S',t)(S'-K)dS' =$$
$$r(t)C(S,K,t) + \frac{\partial}{\partial t}C(S,K,t) \tag{A 12}$$

Let us *assume* that $\Phi(S, S', t)$ approaches zero sufficiently fast for large values of S' so that all the boundary terms above vanish. Then Equations A10 through A12 can be combined to yield Equation A8.

Equation A5 shows that, in the theory defined by the diffusion of Equation A1, the distribution function $\Phi(S, K, t)$ completely determines call option prices $C(S, K, t)$ for all values of strike K and time t. Conversely, from Equation A7, call prices determine the distribution. Furthermore Equation A9 can be used in this theory to derive local volatility function $\sigma(S, t)$ from the known call option prices (and their known derivatives). Combining these facts we can see that the stock price diffusion process of Equation A1 is entirely determined from the knowledge of the stock price distribution function, as we asserted earlier.

In a more general theory, knowledge of the stock price distributions do not necessarily allow the unique deduction of the diffusion process. This is the case, for example, where the drift in the diffusion process depends on the path the stock price takes as well as on time, and therefore call option prices cannot be described in terms of a distribution function alone. If the drift function is an *a priori* known (path-independent) function of spot price and time, we can show that the knowledge of call option prices is in fact sufficient to derive the underlying diffusion, as we will now demonstrate.

Consider a diffusion process whose drift is any known function $r(S, t)$ of spot price and time, satisfying the following diffusion equation:

$$\frac{dS}{S} = r(S, t)dt + \sigma(S, t)dZ \quad \text{(A 13)}$$

The Arrow-Debreu price $\Lambda(S, S', t)$ is the price of a security which pays one dollar if the stock price $S(t)$ at time t attains value S', and zero otherwise. $\Lambda(...)$ can be computed as the expected discounted value of its payoff as follows:

$$\Lambda(S, S', t) = E_{(S, 0)}\left[exp\left(-\int_0^t r(S(t'), t')dt'\right)\delta(S(t) - S')\right] \quad \text{(A 14)}$$

where $E_{(S,0)}[...]$ is the expectation conditional on the initial stock price being S at $t = 0$. The price of a standard European call option with spot price S, strike price K and time to expiration t is by definition given by:

$$C(S, K, t) = E_{(S, 0)}\left[exp\left(-\int_0^t r(S(t'), t')dt'\right)(S(t) - K)^+\right] \quad \text{(A 15)}$$

Using Equation A14 we can rewrite this in terms of Arrow-Debreu prices as

$$C(S, K, t) = \int_K^\infty \Lambda(S, S', t)(S' - K)dS' \quad \text{(A 16)}$$

Differentiating this equation once with respect to K leads to the following more general form of Equation A6:

$$\int_K^{\infty} \Lambda(S, S', t)\,dS' = -\frac{\partial}{\partial K}C(S, K, t) \qquad \text{(A 17)}$$

and differentiating twice leads to the more general form of Equation A7,

$$\Lambda(S, K, t) = \frac{\partial^2}{\partial K^2}C(S, K, t) \qquad \text{(A 18)}$$

It is known that $\Lambda(S, S', t)$ satisfies the following forward Kolmogorov differential equation:

$$\frac{1}{2}\frac{\partial^2}{\partial S'^2}(\sigma^2(S', t)S'^2\Lambda) - r(t)\frac{\partial}{\partial S'}(S'\Lambda) - \frac{\partial \Lambda}{\partial t} = r(S', t)\Lambda \qquad \text{(A 19)}$$

This equation is analogous to Equation A3 satisfied by the transition probability function, and can be used in the same manner to derive a forward equation for European call option prices similar to Equation A8. So, multiplying both sides of Equation A19 by $(S' - K)$ and integrating with respect to S', and then assuming similar boundary conditions at infinity, leads to the following equation:

$$\frac{1}{2}\sigma^2(K, t)K^2\frac{\partial^2 C}{\partial K^2} - r(K, t)K\frac{\partial C}{\partial K} + \int_K^{\infty} \frac{\partial}{\partial S'}C(S, S', t)\frac{\partial}{\partial S'}r(S', t) - \frac{\partial C}{\partial t} = 0 \qquad \text{(A 20)}$$

For a given spot price S, if the local drift function $r(S, t)$ and European call option prices corresponding to all strikes and expirations are known, then we can use Equation A20 to find the local volatility $\sigma(S, t)$ for all values of S and t. This completes the specification of the diffusion process associated with Equation A13.

Selected Quantitative Strategies Publications

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