Statistical Methods and Calibration in Finance and Insurance

MA6622 Assignment of Lectures 12-13

1. Search on the web under the keywords "ARCH" and "Hang Seng Index". How many results do you find?

2. Estimate from one year of daily returns the skewness and kurtosis of the HSI index.

3. Show that a log-return of the form

$$X(t) = \mu + \sigma\varepsilon(t), \quad \text{where } \{\varepsilon(t)\} \text{ gaussian WN with unit variance.}$$

gives a price process of the form

$$S(t) = S(0) \exp\left[n\mu + \sigma\{\varepsilon(1) + \cdots + \varepsilon(t)\}\right].$$

4. The conditional variance given the information $\mathcal{F}(t-1) = (X(t-1), X(t-2), \dots)$, is defined as

$$\mathbf{var}(X(t) \mid \mathcal{F}(t-1)) = \mathbf{E}\left[[X(t) - \mathbf{E}(X(t) \mid \mathcal{F}(t))]^2 \mid \mathcal{F}(t-1)\right].$$

Prove

$$\mathbf{var}(X(t) \mid \mathcal{F}(t-1)) = \mathbf{E}\left[X(t)^2 \mid \mathcal{F}(t-1)\right] - \left[\mathbf{E}[X(t) \mid \mathcal{F}(t)]\right]^2$$

5. Verify that the density $f(x)$ of a random variable with mean μ and standard deviation σ can be written as

$$f(x) = \frac{1}{\sigma}\varphi\Big(\frac{x-\mu}{\sigma}\Big)$$

where $\varphi(x)$ is the density of a standard normal distribution.

6. Suppose given two random vectors $\mathbf{X}, \mathbf{Y}$, with joint density $f(x, y)$. The conditional density of $\mathbf{X}$ given $\mathbf{Y}$, denoted by $f(x \mid y)$ is defined by the equation

$$f(x, y) = f(x \mid y) f(y).$$

(Assuming that $f(y)$, the density of $\mathbf{Y}$ does not vanish.)

Prove for an ARCH process (where $X(t)$ depends only on $X(t-1)$) that the conditional density of the vector $(X(t), \dots, X(1))$ given the initial value $X(0) = x_0$ satisfies:

$$f(x_n, x_{n-1}, \dots, x_1 \mid x_0) = f(x_n \mid x_{n-1}) f(x_{n-1} \mid x_{n-1}) \dots f(x_1 \mid x_0).$$

7. Assume that your portfolio daily log-returns satisfy an ARCH model with parameters $\omega = 0.5$ and $\alpha = 0.1$, and with standard normal errors.

Compute the value at risk of your portfolio over one day, through the following procedure:

1. Simulate 100 steps for the returns, according to the equations

$$x(k) = \sqrt{\omega + \alpha x(k-1)^2}\varepsilon(t),$$

   simulating a standard normal variable $\varepsilon(t)$, departing from $x(0) = 0$.

2. Repeat the previous procedure 1000 times, and compute the 0.05 VaR.

8. Implement the ARCH model to the HSI index daily data. Use the last 3 years of daily returns.

9. Read the article (from the course website):

- R. F. Engle. *GARCH 101: The Use of ARCH/GARCH Models in Applied Econometrics*, Journal of Economic Perspectives 15(4):157-168, 2001.