EXACT RUIN PROBABILITIES

FOR LÉVY PROCESSES

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Consider a Lévy process X with exponent

$$\psi(z) = az + \frac{1}{2}\sigma^2 z^2 + \int_{\mathbf{R}} (e^{zx} - 1 - zx\mathbf{1}_{|x|\leq 1})\Pi(dx)$$

We are interested in the distribution of the random variable

$$I = \inf_{0\leq t<\tau(r)} X_t$$

where $\tau(r)$ is an exponential random variable with parameter $r \geq 0$, independent of X, taking $\tau(0) = \infty$. Ruin probabilities are given by

$$R(x) = P(I \leq -x), \qquad x \geq 0$$

The motivation of finding exact formulae for ruin probabilities for Lévy processes, despite the interest by themselfs, is to compute the exact solution for pricing a perpetual put option in a market with a stock driven by a Lévy process X.

More precisely, if we have interest rate $r > 0$ and the stock is $S_t = S_0 e^{X_t}$, in [1] the following result was obtained:

Theorem 1 (Pricing a Perpetual Put).

$$\sup_{\tau \geq 0} E\Big(e^{-r\tau}(K - S_\tau)^+\Big) = \frac{E[KE(e^I) - S_0 e^I]^+}{E(e^I)},$$

with optimal stopping time

$$\tau_p^* = \inf\{t \geq 0 \colon S_t \leq KE(e^I)\}$$

Here I is the infimum of the process, the knowledge of its distribution is equivalent to the knowledge of the ruin probabilities for a Lévy process.

In consequence we present three approaches to the problem, that give closed formulae for the distribution of I with some "exponential" specification on the structure of the negative jumps, modelled by the jump measure Π:

- *Martingale approach*: main ingredient is Itô formula, with an argument of the ruin of the gambler problem type.

- *Random walk approach*: We imbedd the Lévy process in an associated random walk, in such a way that the infimum is preserved, and use results of infima of random walks.

- *Analitic approach*: We apply classical results of Baxter and Donsker [2], and invert the Fourier Transform of I.

Martingale approach [3]

Theorem 2. *Assume*

$$\Pi(dy) = \begin{cases} \pi^+(dy) & \text{if } y > 0, \\ \lambda \sum_{k=1}^n a_k \alpha_k e^{\alpha_k y} dy & \text{if } y < 0, \end{cases}$$

where π^+ *is an arbitrary Lévy measure concentrated on* $(0, \infty)$*. Then*

$$R(x) = \sum_{j=1}^{n+1} A_j e^{-p_j x}, \quad x \geq 0,$$

where $-p_1, \ldots, -p_{n+1}$ *are the negative roots of the equation* $\psi(z) = r$*, and the constants* $A_1, \ldots, A_{n+1}$ *are given by*

$$A_j = \frac{\prod_{k=1}^n \left(1 - \frac{p_j}{\alpha_k}\right)}{\prod_{k=1, k\neq j}^{n+1} \left(1 - \frac{p_j}{p_k}\right)}, \quad j = 1, \ldots, n+1.$$

Idea of the proof

We verify that

$$\mathcal{L}R(x) = rR(x), \quad x > 0$$

with $\mathcal{L}$ the infinitesimal generator of X, and apply Itô-Meyer formula to the process $Y_t = x + X_t$, to obtain:

$$\begin{aligned} e^{-r(t\wedge\tau_0)} & R(Y_{t\wedge\tau_0}) - R(x) \\ &= \int_0^{\tau_0\wedge t} e^{-rs}[\mathcal{L}R(Y_{s-}) - rR(Y_{s-})]ds \\ &\qquad + \int_0^{\tau_0\wedge t} e^{-rs} dM_s \end{aligned}$$

with M a martingale. As $Y_{s-} = x + X_{s-} > 0$ on $[0, \tau_0]$ and the function R is bounded, taking expectations and limits as $t \to \infty$

$$\begin{aligned} R(x) &= E(e^{-r\tau_0} R(x + X_{\tau_0})) \\ &= E(e^{-r\tau_0}) = P(\tau_0 < \tau(r)) \\ &= P(\exists t \in [0, \tau(r)) \colon x + X_t \le 0) = P(I \le -x) \end{aligned}$$

concluding the proof.

Random walk approach

The Lévy measure is now

$$\Pi(dx) = \begin{cases} \pi^{+}(dx) & \text{if } x > 0 \\ \lambda b(x), & \text{if } x < 0, \end{cases}$$

where $b(x) = \pi \exp(Tx)t$ is the density of a **phase type** distribution with initial distribution π, d phases, and intensity matrix T, the constant $\lambda > 0$ is the intensity of negative jumps, and $\pi^{+}(dy)$ is, as before, an arbitrary Lévy measure supported on the set $(0, \infty)$. The result ([4]), with $\sigma > 0$ is the following

Theorem 3. *The random variable I is of phase type, with $d+1$ phases and initial probabilities and intensity matrix that are computed in terms of π and T.*

Idea of the proof

The proof divides in two steps:

- We obtain that the infimum coincides with the infimum of an associated random walk, resulting of observing the Lévy process at certain specified random times. Independence of increments of the random walk is obtained based on fluctuation theory.

- We identify the distributions of the increments of the random walk using properties of phase type distribution, and apply Asmussen results [5] to identify the distribution of the associated random walk.

Analytic approach

In this case we assume that the Lévy measure is given by

$$\Pi(dx) = \begin{cases} \pi^+(dx) \\ \lambda \sum_{k=1}^{n} \sum_{j=1}^{m_k} c_{kj} \Big(\alpha_k\Big)^j \frac{(-x)^{j-1}}{(j-1)!} e^{\alpha_k x} dx \end{cases}$$

The result [6] is the following.

Theorem 4. *Assume $\sigma > 0$, and $r > 0$. Then the equation $\psi(z) = r$ has, in the half-plane $\Re(z) > 0$, N distinct roots $\beta_1, \ldots, \beta_N$, with multiplicities $n_1, \ldots, n_N$. and ordered such that $0 < \Re(\beta_1) \leq \Re(\beta_2) \leq \cdots \leq \Re(\beta_N)$. The root β_1 is purely real.*

Furthermore, the negative Wiener-Hopf factor is given by

$$E(e^{zI}) = \prod_{k=1}^{n} \left(\frac{z - \alpha_k}{-\alpha_k} \right)^{m_k} \prod_{j=1}^{N} \left(\frac{-\beta_j}{z - \beta_j} \right)^{n_j},$$

Idea of the proof

The first task is to identify and locate the roots of the equation $\psi(z) = r$, and this is done through some contour integrations, applying Rouche's Theorem.

After this, we apply Baxter-Donsker result to identify the Fourier transform of I.

It should be noticed that in order to obtain the result in full generality for π^+ an approximation argument (cutting short and long jumps of X) should be applied.

References

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