EXACT RUIN PROBABILITIES

FOR A CLASS Of LÉVY PROCESSES

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Exact Ruin Probabilities

1. Jump-diffusion (I)

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2. Rational transform positive jumps (II)

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   - Proof: Baxter-Donsker (complex analysis)

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The problem: Ruin and Maxima

Mathematical model:

$X_t\colon \Omega \to \mathbb{R}$ $(t \geq 0)$, is a *stochastic process* defined $(\Omega, \mathcal{F}, \mathbf{P}_x)$.

Notation: $X_0 = x$, $\mathbf{P}_0 = \mathbf{P}$.

Problem:

Compute the *Ruin Probability*

$$R(x) = \mathbf{P}_x(\exists t \geq 0\colon X_t \leq 0) = \mathbf{P}_x(\inf X_t \leq 0)$$

Equivalent Problem:

Define the *maximum*

$$M := \sup\{Y_t \colon t \geq 0\}$$

of the symmetric process Y, and find

$$1 - F_M(x) = \mathbf{P}(M > x) = R(x).$$

Ruin for X = maximum for Y

Generalized Problem:

Take

$$\begin{cases} \tau(q) \sim \exp(q) \text{ indep. of } X \\ \tau(0) = +\infty \end{cases}$$

and consider now

$$M_q := \sup\{Y_t \colon 0 \leq t < \tau(q)\}$$

We want to find

$$F_{M_q}(x) = \mathbf{P}(M_q \leq x).$$

Lévy Processes: Definition

- It starts at $X_0 = 0$,

- *Independent Increments*:

  If $0 \leq t_1 \leq \cdots \leq t_n$, then

  $$X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$$

  are independent random variables

- Stationary Increments:

  $$X_{t+h} - X_t \text{ same distribution as } X_h$$

Also called PIIS: Processes with independent and stationary increments

Lévy Processes: Characterization

Lévy-Kinchine Formula:

$$\mathbf{E}\, e^{zX_t} = e^{t\psi(z)},$$

where

$$\psi(z) = az + \frac{1}{2}\sigma^2 z^2 + \int_{\mathbb{R}} (e^{zy} - 1 - zy\mathbf{1}_{\{|y|<1\}})\Pi(dy)$$

- $a \in \mathbb{R}$ is the *drift*
- $\sigma \geq 0$ is the *variance of the Gaussian part*
- Π is a positive measure on $\mathbb{R} \setminus \{0\}$, with $\int (1 \wedge y^2)\Pi(dy) < +\infty$, governing the *jumps* of X: is the *jump measure*

Example: Brownian motion with drift

Let $B = (B_t)$ be a Brownian motion. Define

$$X_t = at + \sigma B_t$$

We have

$$\mathbf{E}\, e^{zX_t} = e^{t\psi(z)},$$

where

$$\psi(z) = az + \frac{1}{2}\sigma^2 z^2$$

Conclusion: In Levy-Khinchine formula:

- $\Pi = 0$ indicates *absence of jumps*

Example: Compound Poisson Process

Taking $T = (T_k)$ i.i.d.r.v. with distribution exp(λ), define

$$N_t = \inf\{k\colon T_1 + T_2 + \dots T_k \le t\}.$$

$N = (N_t)$ is a *Poisson process*. Consider

$$X_t = \sum_{k=1}^{N_t} Y_k$$

where $Y = (Y_k)$ are i.i.d.r.v. with distribution $F(dy)$ (independent of T). Simple computations give:

$$\psi(z) = \lambda \int_{\mathbb{R}} (e^{zy} - 1) F(dy),$$

In this case

- $a = \sigma = 0$

- $\Pi(dy) = \lambda F(dy)$.

Example: Jump-diffusion processes

$$X_t = at + \sigma B_t + \sum_{k=1}^{N_t} Y_k$$

with B, N, Y independent, gives

$$\psi(z) = az + \frac{1}{2}\sigma^2 z^2 + \lambda \int_{\mathbb{R}} (e^{zy} - 1) F(dy),$$

If Y_1 has density

$$f(x) = \begin{cases} p\alpha e^{-\alpha y} & \text{when } y > 0 \\ (1-p)\beta e^{\beta y} & \text{when } y < 0 \end{cases}$$

for some positive α, β and $p \in (0,1)$, then

$$\begin{aligned} \psi(z) = & az + \frac{1}{2}\sigma^2 z^2 \\ & + \lambda p \frac{z}{\alpha - z} - \lambda(1-p)\frac{z}{\beta + z} \end{aligned}$$

Denote

$$M_q = \sup\{t \geq 0\} \qquad I_q = \inf\{t \geq 0\}$$

Theorem 1. *If X is a jump-diffusion process, the densities of M_q and I_q are:*

$$f_{M_q}(x) = A_1 \exp(-p_1 x) + A_2 \exp(-p_2 x), \quad x > 0$$

$$f_{I_q}(x) = B_1 \exp(-r_1 x) + B_2 \exp(-r_2 x), \quad x < 0$$

where equation $q - \psi(z) = 0$ has

- p_1 *and* p_2 *positive roots*
- r_1 *and* r_2 *negative roots*

and coefficients are:

$$A_1 = \frac{1 - p_1/\alpha}{1 - p_1/p_2} \qquad A_2 = \frac{1 - p_2/\alpha}{1 - p_2/p_1}$$

$$B_1 = \frac{1 + r_1/\beta}{1 - r_1/r_2} \qquad B_2 = \frac{1 + r_2/\beta}{1 - r_2/r_1}$$

Proof: Consider the characteristic function of M_q and I_q:

$$\phi_q^+(z) = \mathbf{E}\,\mathsf{exp}(zM_q)$$

$$\phi_q^-(z) = \mathbf{E}\,\mathsf{exp}(zI_q)$$

and use Rogozin's (1966) WH factorization:

$$\frac{q}{q-\psi(z)} = \phi_q^+(z)\phi_q^-(z)$$

Uniqueness in WH and complex analysis arguments give the result (roots of $q - \psi(z)$ are poles, use residue's calculus)

LP with rational transform negative jumps

Let X be a LP with jump measure

$$\Pi(dx) = \begin{cases} \pi^+(dx) & \text{if } \ x > 0, \\ \\ \pi^-(dx) = \lambda p(x) dx & \text{if } \ x < 0. \end{cases} \tag{1}$$

where π^+ arbitrary on $(0, +\infty)$ (arbitrary positive jumps), and

$$p(x) = \sum_{k=1}^{n} \sum_{j=1}^{m_k} c_{kj} \Big(\alpha_k\Big)^j \frac{(-x)^{j-1}}{(j-1)!} e^{\alpha_k x}, \quad x < 0,$$

is the more general form of a r.v. with rational Laplace transform.

$P = m_1 + \cdots + m_n$ is the Pole count.

Characteristic exponent ψ

$$\widehat{p}(z) = \int_{-\infty}^{0} e^{zy} p(y)dy = \sum_{k=1}^{n} \sum_{j=1}^{m_k} c_{kj} \Big(\frac{\alpha_k}{\alpha_k - z}\Big)^j.$$

So

$$\psi(z) = az + \frac{1}{2}\sigma^2 z^2$$

$$+ \int_0^{\infty} (e^{zy} - 1 - zh(y))\pi^+(dy)$$

$$+ \lambda(\widehat{p}(z) - 1)$$

Theorem 2 (A. Lewis - EM).
(a) *On the half space* $\Re z > 0$ *the equation* $q - \psi(z) = 0$ *has roots*

$$\beta_1, \dots, \beta_N$$

with multiplicities

$$n_1, \dots, n_N$$

and such that $n_1 + \cdots + n_N = P + 1$ *(*$\sigma > 0$*)*

(b) *The characteristic function* ϕ_q^- *of the infimum* I_q *is*

$$\phi_q^-(z) = \prod_{k=1}^{n} \left(\frac{\alpha_k - z}{\alpha_k} \right)^{m_k} \prod_{j=1}^{N} \left(\frac{\beta_j}{\beta_j - z} \right)^{n_j},$$

Proof: (1) Study the roots of $q-\psi(z)=0$ on $\Re z>0$ with Rouche's Theorem (we know the poles).

(2) Consider Baxter-Donsker (1957) Formula:

$$\begin{aligned}\phi_q^-(iu) &= \mathbf{E}\, e^{-uI_q} \\ &= \exp\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{u}{\xi(\xi-iu)}\log\left(\frac{q}{q-\psi(\xi)}\right)d\xi\right\}\end{aligned}$$

(3) Compute the integral over a convenient contour in $\Re z>0$, computing the residues at the poles, and take limit.

Open Questions:

- Solve a two barrier problem for this class of Processes: Probability of hitting level $a > 0$ before than hitting level $b < 0$. (now π^+ should also be of rational type)

- Compute the Green Kernel for this class of Processes

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