

Optimal stopping, ruin probabilities and prophet inequalities for Lévy processes *

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Abstract

Solution to the optimal stopping problem $V(x) = \sup_{\tau} E(x + X_{\tau})^+$ is given, where $X = \{X_t\}_{t \geq 0}$ is a Lévy process, and the supremum is taken over the class of stopping times. Results are expressed in terms of the distribution of the random variable $M = \sup_t X_t$, under the hypothesis $E(M) < +\infty$, and simple conditions for this hypothesis to hold are given. Based on this, the prophet inequality

$$V(x) \leq E(x + M) \leq eV(x)$$

is obtained. Closed form solutions of the distribution of M are given for a wide class of Lévy processes.

1 Introduction and main results

Let $X = \{X_t\}_{t \geq 0}$ be a real valued stochastic process defined on a stochastic basis $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t \geq 0}, P)$ that satisfy the usual conditions. Assume that X is càdlàg, adapted, $X_0 = 0$, and for $0 \leq s \leq t$ the random variable $X_t - X_s$ is independent of the σ -field \mathcal{F}_s with a distribution that only depends on the difference $t - s$. X is a process with stationary independent increments (PIIS), or a Lévy process. For general reference on the subject see Jacod and Shiryaev (1987), Skorokhod (1991), or Bertoin (1996).

The characteristic function of X_t is given by the Lévy-Khintchine formula, that states

$$E(e^{izX_t}) = e^{t\psi(z)},$$

where the function $\psi: \mathbf{R} \rightarrow \mathbf{C}$ is given by

$$(1) \quad \psi(z) = ibz - \frac{1}{2}\sigma^2 z^2 + \int_{\mathbf{R}} (e^{izy} - 1 - izy\mathbf{1}_{\{|y|<1\}}) \Pi(dy)$$

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where b and $\sigma \geq 0$ are real constants, and Π is a positive measure on $\mathbf{R} - \{0\}$ such that $\int (1 \wedge y^2) \Pi(dy) < +\infty$, called the Lévy measure. The function ψ is the characteristic exponent and completely determines the law of X . τ is a stopping time relative to \mathbf{F} if

$$\tau: \Omega \rightarrow [0, +\infty] \text{ and } \{\tau \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0.$$

We say that X drifts to $-\infty$ if $P(\lim_{t \rightarrow +\infty} X_t = -\infty) = 1$. Denote by \mathcal{M} the class of all stopping times relative to $(\mathcal{F}_t)_{t \geq 0}$ and by $x^+ = \max(x, 0)$.

1.1. Consider now the optimal stopping problem: given the process X and the reward function x^+ find a stopping time τ^* in \mathcal{M} and a cost function $V: \mathbf{R} \rightarrow \mathbf{R}$ such that

$$(2) \quad V(x) = \sup_{\tau \in \mathcal{M}} E(x + X_\tau)^+ = E(x + X_{\tau^*})^+.$$

The solution to this problem can be expressed in terms of the distribution of the overall supremum of X , that is the random variable $M = \sup_{t \geq 0} X_t$ whenever this variable is proper and has finite expected value. Denote by

$$B(x) = P(M \geq x)$$

the probability of crossing the barrier at level x , and $x^* = E(M)$.

Theorem 1 *Let X be a Lévy process such that*

$$E(M) = \int_0^{+\infty} B(y) dy < +\infty.$$

Then, the solution to the optimal stopping problem (2) is

$$(3) \quad V(x) = E(x + M - E(M))^+ = \int_{E(M)-x}^{+\infty} B(y) dy,$$

$$\tau^* = \inf\{t \geq 0: x + X_t \geq E(M)\}.$$

Theorem 1 essentially says, that although in general, the optimal stopping problem for a Markov process leads to a free boundary problem (Theorem III§8.15 in Shiryaev (1978)), in the case considered, this problem reduces to the barrier problem (i.e. finding $B(x)$) for X , that has the advantage of having fixed boundary conditions. The conclusion is, that for a Lévy process, (as happened for random walks in Darling, Liggett and Taylor (1972)) the optimal stopping problem with reward x^+ can be explicitly solved once the distribution of M is known. Observe that the barrier-crossing probability for the process X is equivalent to the computation of the ruin probability for the process $\{x - X_t\}_{t \geq 0}$. In section §2 we state conditions on the characteristic exponent ψ of the process in order to have $E(M) < +\infty$.

1.2. The second result states a prophet inequality for the class of Lévy processes, similar to one the obtained for random walks by Darling, Ligget and Taylor (1972).

Theorem 2 *Let X be an arbitrary Lévy process, and $x \geq 0$.*

(i) *If $E(M) < +\infty$, the following inequality holds:*

$$V(x) \leq E(x + M) \leq eV(x).$$

(ii) *If $E(M) = +\infty$, then $V(x) = +\infty$.*

This result is called a prophet inequality in the sense that it compares the expected reward of a gambler that has complete foresight, or a “prophet” that can choose the moment at which the maximum is attained, obtaining $x + M$, with the expectation of the best possible strategy in the class \mathcal{M} . For references on analogous results for sequences of random variables see Hill and Kertz (1990).

1.3. Finally we address to the question of the distribution of M . Next result presents a wide class of processes that admits simple closed solutions. Let X be a Lévy process with characteristic exponent

$$(4) \quad \psi(z) =iaz - \frac{1}{2}\sigma^2 z^2 + \lambda \frac{iz}{\alpha - iz} + \int_{-\infty}^0 (e^{izy} - 1 - izy\mathbf{1}_{\{-1 < y < 0\}})\pi(dy)$$

where π is an arbitrary Lévy measure concentrated on $(-\infty, 0)$, $\lambda > 0$ and $\alpha > 0$. As

$$\int_0^{+\infty} (e^{izy} - 1)\alpha e^{-\alpha y} dy = \frac{iz}{\alpha - iz}$$

we conclude that the process considered has positive exponential jumps with parameter α at poissonian moments with rate λ , and arbitrary negative jumps. Taking

$$(5) \quad \Pi(dy) = \begin{cases} \lambda \alpha e^{-\alpha y} dy & \text{if } y > 0, \\ \pi(dy) & \text{if } y < 0. \end{cases}$$

and

$$(6) \quad b = a + \frac{\lambda}{\alpha} [1 - (1 + \alpha)e^{-\alpha}]$$

in (1) we obtain (4).

Consider now, for $p \geq 0$, $p \neq \alpha$ the function

$$(7) \quad \kappa(p) = ap + \frac{1}{2}\sigma^2 p^2 + \lambda \frac{p}{\alpha - p} + \int_{-\infty}^0 (e^{py} - 1 - py\mathbf{1}_{\{-1 < y < 0\}})\pi(dy).$$

The function $z \mapsto \psi(z)$ in (4) can be extended analytically in such a way that, for $0 \leq p < \alpha$, $\kappa(p) = \psi(-ip)$, and the Laplace transform of X_t is given by

$$E(e^{pX_t}) = e^{t\kappa(p)}.$$

Following Bertoin (1996) we call $\kappa = \kappa(p)$ the Laplace exponent of X .

As the integrand in (7) is convex in p for fixed y , we obtain that $\kappa(p)$ is convex on $[0, \alpha)$. From this follows, that when

$$(8) \quad \kappa'(0+) = \lim_{p \rightarrow 0+} \frac{1}{p} \kappa(p) = a + \frac{\lambda}{\alpha} + \int_{-\infty}^{-1} y \pi(dy) < 0,$$

(the integral can take the value $-\infty$) equation $\kappa(p) = 0$ has two positive roots p_1, p_2 that satisfy $0 < p_1 < \alpha < p_2$.

Theorem 3 *Let X be a Lévy process with characteristic exponent given by (4). Denote by $M = \sup_{t \geq 0} X_t$ and by $B(x) = P(M \geq x)$. Assume that condition (8) holds. Then*

(i) *The process drifts to $-\infty$, and the barrier problem for X has solution*

$$(9) \quad B(x) = \begin{cases} A_1 e^{-p_1 x} + A_2 e^{-p_2 x}, & x > 0, \\ 1, & x \leq 0, \end{cases}$$

where p_1 and p_2 are the two positive solutions of the equation $\kappa(p) = 0$, and the constants A_1 and A_2 are given by

$$(10) \quad A_1 = \frac{p_2(\alpha - p_1)}{\alpha(p_2 - p_1)}, \quad A_2 = \frac{p_1(p_2 - \alpha)}{\alpha(p_2 - p_1)}.$$

(ii) *The random variable M has finite expectation,*

$$x^* = E(M) = A_1/p_1 + A_2/p_2,$$

and the optimal stopping problem (2) for the process X with reward x^+ has solution

$$V(x) = \begin{cases} \frac{A_1}{p_1} e^{p_1(x-x^*)} + \frac{A_2}{p_2} e^{p_2(x-x^*)}, & x < x^*, \\ x, & x \geq x^*. \end{cases}$$

2 Conditions for $E(M) < +\infty$

In this section we give conditions in order to have $E(M) < +\infty$ in terms of the characteristic exponent ψ of a Lévy process X . More precisely, we state conditions in terms of the triplet (b, σ^2, Π) similar to the ones for random walks, i.e. (i) negative drift, and (ii) finite second moment of the positive part (Theorem 5 in Kiefer and Wolfowitz (1956)). The analysis is based on the decomposition of the process X as a semimartingale, (see Protter (1992) or Jacod and Shiryaev (1987)). The first question addressed is the case of bounded positive jumps.

Lemma 1 *Let $X = \{X_t\}_{t \geq 0}$ be a Lévy process. Denote $M = \sup_t X_t$. If the positive jumps of X are bounded, and drifts to $-\infty$, then $E(M) < +\infty$.*

Proof. Without loss of generality we can assume that the positive jumps are bounded by 1. Denote $B(x) = P(M \geq x)$. We claim that

$$(11) \quad B(x+y+1) \leq B(x)B(y) \quad \text{for all positive } x, y.$$

Denote $\tau_x = \inf\{t \geq 0: X_t \geq x\}$. We have

$$\begin{aligned} B(x+y+1) &= P(\sup_t X_t \geq x+y+1) = P(\sup_t X_t \geq x+y+1, \tau_x < +\infty) \\ &= P(\sup_t X_t - X_{\tau_x} \geq x+y+1 - X_{\tau_x}, \tau_x < +\infty) \\ &= P(\sup_t (X_{\tau_x+t} - X_{\tau_x}) \geq y+x+1 - X_{\tau_x}, \tau_x < +\infty) \\ &\leq P(\sup_t (X_{\tau_x+t} - X_{\tau_x}) \geq y, \tau_x < +\infty) = B(x)B(y) \end{aligned}$$

based on $x+1 - X_{\tau_x} \geq 0$ on the set $\{\tau_x < +\infty\}$, and the strong Markov property (I.6 in Bertoin (1996)). In this way (11) is proved.

As the process drifts to $-\infty$ there exists x_0 such that $B(x_0) < 1$. As $E(M) = \int_0^{+\infty} B(x)dx$, to conclude the proof is enough to verify

$$\int_{2x_0+1}^{+\infty} B(x)dx < +\infty.$$

As $B(x)$ is non-increasing, taking $x = y = x_0$ in (11), by induction, we obtain that if $x \in [nx_0 + n - 1, (n+1)x_0 + n]$, $n \in \mathbf{N}$

$$B(x) \leq B(nx_0 + n - 1) \leq B(x_0)^n.$$

So

$$\begin{aligned} \int_{2x_0+1}^{+\infty} B(x)dx &= \sum_{n=2}^{+\infty} \int_{nx_0+n-1}^{(n+1)x_0+n} B(x)dx \leq \sum_{n=2}^{+\infty} (x_0+1)B(x_0)^n \\ &= (x_0+1) \frac{B(x_0)^2}{1-B(x_0)} < +\infty. \end{aligned}$$

and the proof is concluded. \square

In order to analyze the general case consider the condition

$$(12) \quad b + \int_{\{|y| \geq 1\}} y \Pi(dy) < 0.$$

This means $\int_{\{y \geq 1\}} y \Pi(dy) < +\infty$ but the integral can take the value $-\infty$.

Lemma 2 *Let X be a Lévy process with characteristic exponent given by (1). Then*

(i) *If condition (12) holds the process drifts to $-\infty$, that is*

$$P(\lim_{t \rightarrow +\infty} X_t = -\infty) = 1.$$

(ii) *If (12) holds, and furthermore*

$$(13) \quad \int_1^{+\infty} y^2 \Pi(dy) < +\infty,$$

we have

$$E(M) < +\infty.$$

Proof. Consider the decomposition of X as

$$(14) \quad X_t = \sigma B_t + \int_0^t \int_{\{|y| < 1\}} y * (\mu - \nu) + bt + \sum_{0 < s \leq t} \Delta X_s \mathbf{1}_{\{|\Delta X_s| \geq 1\}}.$$

(Theorem I.42 in Protter (1992)) where $B = \{B_t\}_{t \geq 0}$ is a Brownian motion, $\mu = \mu(\omega, dy, dt)$ is the random measure of the process X , $\nu = \nu(dy, dt) = \Pi(dy)dt$ its compensator, and $b = E(X_1 - \int_0^1 \int_{\{|y| \geq 1\}} y * \mu)$. By condition (12) we can take $\varepsilon > 0$ such that

$$\int_{\{|y| \geq 1\}} y \Pi(dy) + b + \varepsilon < 0.$$

Let $X_t = X'_t + X''_t$ with

$$X'_t = \sigma B_t + \int_0^t \int_{\{|y| < 1\}} y * (\mu - \nu) - \varepsilon t,$$

$$X''_t = (b + \varepsilon)t + \sum_{0 < s \leq t} \Delta X_s \mathbf{1}_{\{|\Delta X_s| \geq 1\}}.$$

Denote now

$$M' = \sup_{t \geq 0} X'_t, \quad M'' = \sup_{t \geq 0} X''_t.$$

We have $M \leq M' + M''$. The process X' has bounded jumps. From Theorem I.34 in Protter (1992) follows that X' has finite moments of any order, giving $E(X'_t) = -\varepsilon t < 0$. This fact allows to conclude that X' drifts to $-\infty$ (16.3.3 in Koroliuk et al.(1988)). Then Lemma 1 applies, giving $E(M') < +\infty$.

Let us analyze X'' . Define the sequence of stopping times

$$T_0 = 0, \quad T_{k+1} = \inf\{t > T_k : |\Delta X_t| \geq 1\}.$$

The process $N = \{N_t\}_{t \geq 0}$ defined by $N_t = \sup\{k: T_k \leq t\}$ is a Poisson process with parameter $\lambda = \int_{\{|y| \geq 1\}} \Pi(dy)$. Denote $\Delta X_{T_0} = T_{-1} = X_{T_0-} = 0$. X_t'' can be represented as

$$X_t'' = \sum_{k=0}^{N_t} \Delta X_{T_k} + (b + \varepsilon)t.$$

In consequence if $b + \varepsilon < 0$

$$\sup_t X_t'' = \sup_n X_{T_n}'' = \sup_{n \geq 0} \left\{ \sum_{k=0}^n (\Delta X_{T_k} + (b + \varepsilon)(T_k - T_{k-1})) \right\},$$

and if $b + \varepsilon \leq 0$

$$\sup_t X_t'' = \sup_n X_{T_n-}'' = \sup_n \left\{ \sum_{k=0}^n (\Delta X_{T_{k-1}} + (b + \varepsilon)(T_k - T_{k-1})) \right\},$$

As both cases are similar we analyze the first. Denoting

$$Z_k = \Delta X_{T_k} + (b + \varepsilon)(T_k - T_{k-1}),$$

from Theorem 18§4.3 in Skorokhod (1991) we deduce

$$E(Z_k) = \frac{1}{\lambda} \left(\int_{\{|y| \geq 1\}} y \Pi(dy) + b + \varepsilon \right) < 0.$$

concluding (i) based on the fact that the random walk $\sum_{k=1}^n Z_k$ drifts to $-\infty$.

If (13) holds, we obtain

$$E((Z_k^+)^2) \leq \frac{2}{\lambda} \int_{\{y \geq 1\}} y^2 \Pi(dy) + \frac{4(b + \varepsilon)^2}{\lambda^2} < +\infty.$$

So Theorem 5 in Kiefer and Wolfowitz (1956) applies giving $E(M'') < \infty$, concluding the proof of (ii). \square

3 Examples

Theorem 1 gives the solution to the optimal stopping problem for a Lévy process X through the function $B(x) = P(M \geq x)$ and the value $x^* = E(M)$. We present some examples where explicit formulae for this quantities can be found, and discuss whether the bound in Theorem 2 is attained.

3.1 Lévy Processes with no positive jumps

Let $X = \{X_t\}_{t \geq 0}$ be a Lévy process with no positive jumps, that is, a process with Lévy measure π with support on the set $(-\infty, 0)$. It is known, that for this class of processes the distribution of the overall supremum can be computed explicitly (Prabhu (1980), Bertoin (1996)). Let us remark in advance, that in the case considered, the bound e in Theorem 2 is attained. The characteristic exponent of the process is

$$(15) \quad \psi(z) = iaz - \frac{1}{2}\sigma^2 z^2 + \int_{-\infty}^0 (e^{izy} - 1 - izy\mathbf{1}_{\{-1 < y < 0\}})\pi(dy).$$

The Laplace exponent is given by

$$(16) \quad \kappa(p) = ap + \frac{1}{2}\sigma^2 p^2 + \int_{-\infty}^0 (e^{py} - 1 - py\mathbf{1}_{\{-1 < y < 0\}})\pi(dy).$$

Corollary VII.2(ii) in Bertoin (1996) states that, under the condition

$$(17) \quad \kappa'(0+) = a + \int_{-\infty}^{-1} y\pi(dy) < 0,$$

there exists an unique positive root p_0 of the equation $\kappa(p) = 0$, and that $M = \sup_t X_t$ has exponential distribution with parameter p_0 . In other words, for $x \geq 0$

$$B(x) = P(M \geq x) = e^{-p_0 x} \quad \text{and} \quad E(M) = \frac{1}{p_0}.$$

Observe that the bound in the prophet inequality (Theorem 2) is attained at $x = 0$, that is

$$E(M) = eV(0).$$

Consider some particular cases.

3.1.1 Wiener process with trend

Let $X = \{X_t\}_{t \geq 0}$ be given by

$$X_t = \sigma B_t + at,$$

with $B = \{B_t\}_{t \geq 0}$ a Wiener process. In this case $\pi \equiv 0$, and (17) reads $a < 0$. The root is $p_0 = -\frac{2a}{\sigma^2}$ (This is example in III§3 in Shiryaev (1978), see also Taylor (1978)).

3.1.2 Risk processes

Consider a Lévy process $X = \{X_t\}_{t \geq 0}$ with

$$X_t = at - \sum_{k=1}^{N_t} Y_k,$$

where $N = \{N_t\}_{t \geq 0}$ is a Poisson process with intensity λ , and $Y = \{Y_k\}_{k \in \mathbf{N}}$ is a sequence of independent positive random variables with distribution $F = F(y)$. X is called a Risk process (Embrechts and Klüpelberg (1993)), and is a Lévy process with characteristic exponent

$$\psi(z) = ia z + \lambda \int_0^{+\infty} (e^{-izy} - 1) dF(y) = ia z + \lambda \int_{-\infty}^0 (e^{izy} - 1) d\pi(y)$$

with $\pi(y) = 1 - F(-y^-)$. Condition (17) is

$$a < \lambda \int_0^{+\infty} y F(dy),$$

where the integral can take the value $+\infty$. p_0 is the Cramer-Lundberg constant, i.e. the positive root of the equation

$$ap + \lambda \int_0^{+\infty} (e^{-py} - 1) F(dy) = 0.$$

We are interested in two particular cases, the first considered in Mordecki (1998)

- **Exponential jumps.**

If $F(x) = 1 - e^{-\beta x}$ we obtain $p_0 = \frac{\lambda}{a} - \beta$.

- **Negative Poisson process.**

Taking $F(dx) = \delta_\rho(dx)$, the probability measure concentrated at $\rho > 0$, our process results

$$X_t = at - N_t.$$

Condition (17) is $a < \rho\lambda$, and p_0 is the solution of the equation

$$ap + \lambda(e^{-\rho p} - 1) = 0.$$

3.2 Processes with positive jumps

3.2.1 Poisson processes

Consider a Lévy process $X = \{X_t\}_{t \geq 0}$ with

$$X_t = N_t - at$$

where $N = \{N_t\}_{t \geq 0}$ is a Poisson process with parameter λ . The characteristic exponent is

$$\psi(z) = -iaz + \lambda(e^{iz} - 1).$$

The crossing barrier probability $B(x)$ was obtained by Pyke (1959), and when $\lambda < a$ is

$$B(x) = 1 - \left(1 - \frac{\lambda}{a}\right) \sum_{k=0}^{[x]} \left(\frac{\lambda}{a}\right)^k \frac{(k-x)^k}{k!} e^{-\lambda(k-x)/a}.$$

Then the problem is solved according to Theorem 1. By Lemma 1, under the condition $\lambda < a$, $x^* = E(M) < +\infty$. Let us compute its value. The function $B(x)$ is the solution to the problem (VI.5 in Feller (1966))

$$(18) \quad B(x) - B(x-1) = \frac{a}{\lambda} B'(x), \quad B(x) = 1 \quad \text{for} \quad -1 \leq x \leq 0.$$

Denote for $x \geq -1$

$$w(x) = \int_0^x B(y) dy$$

We compute $x^* = E(M) = \lim_{x \rightarrow +\infty} w(x)$. Integrating (18) over $[x, +\infty)$,

$$(19) \quad w(x) - w(x-1) = \frac{a}{\lambda} B(x)$$

From the boundary condition in (18) follows

$$(20) \quad w(x) = x \quad \text{if} \quad -1 \leq x \leq 0.$$

Integrating (19) over $[0, x]$, taking into account (20)

$$\int_{x-1}^x w(y) dy = \frac{a}{\lambda} w(x) - \frac{1}{2}.$$

As $w = w(x)$ is increasing, we obtain

$$w(x-1) \leq \frac{a}{\lambda} w(x) - \frac{1}{2} \leq w(x).$$

Taking limit as $x \rightarrow +\infty$, results

$$x^* = \frac{\lambda}{2(a-\lambda)}.$$

3.3 Signed exponential jumps

We consider a particular case of Theorem 3. Take $X = \{X_t\}_{t \geq 0}$ with

$$X_t = at + \sigma B_t + \sum_{k=1}^{N_t^\lambda} Y_k^\alpha - \sum_{k=1}^{N_t^\mu} Y_k^\beta$$

where $B = \{B_t\}_{t \geq 0}$ is a Wiener process, $N^\lambda = \{N_t^\lambda\}_{t \geq 0}$, $N^\mu = \{N_t^\mu\}_{t \geq 0}$ are Poisson processes with parameters λ and μ respectively, and $Y^\alpha = \{Y_k^\alpha\}_{k \in \mathbf{N}}$, $Y^\beta = \{Y_k^\beta\}_{k \in \mathbf{N}}$ two sequences of independent exponentially distributed random variables with parameters α and β respectively. The five processes B , N^λ , N^μ , Y^α , Y^β are independent. The process X has exponentially distributed positive (negative) jumps with rate λ (μ) and parameter α (β). The characteristic exponent of the process is

$$\psi(z) = ia z - \frac{1}{2} \sigma^2 z^2 + \lambda \frac{iz}{\alpha - iz} - \mu \frac{iz}{\beta + iz}$$

with Lévy measure

$$\Pi(dy) = \begin{cases} \lambda \alpha e^{-\alpha y} dy & y > 0, \\ \mu \beta e^{\beta y} dy & y < 0. \end{cases}$$

In consequence, the Laplace exponent is

$$\kappa(p) = ap + \frac{1}{2} \sigma^2 p^2 + \frac{\lambda p}{\alpha - p} - \frac{\mu p}{\beta + p}.$$

The conclusion follows from Theorem 3. This example is motivated by example VI.8 (b) in Feller (1966). The case with only positive jumps was considered in Mordecki (1997).

3.3.1 Compound Poisson process with exponential jumps

In the following example we have $P(M = 0) > 0$. This fact makes the bound in Theorem 2 not to be attained.

Consider $X = \{X_t\}_{t \geq 0}$ with

$$X_t = \sum_{k=1}^{N_t} Y_k - at,$$

where $N = \{N_t\}_{t \geq 0}$ is a Poisson process with intensity λ , and $Y = \{Y_k\}_{k \in \mathbf{N}}$ form a sequence of independent random variables with exponential distribution and parameter α . The characteristic exponent is

$$\psi(z) = -iaz + \lambda \frac{iz}{\alpha - iz}.$$

Condition (8) reads $\lambda < a\alpha$. It is known (see for instance Embrechts and Klüpelberg (1993)) that for $x > 0$

$$B(x) = \frac{\lambda}{a\alpha} e^{-\rho x}$$

with $\rho = \alpha - \frac{\lambda}{a} > 0$. Simple computations show, that

$$x^* = E(M) = \frac{\lambda}{\alpha(a\alpha - \lambda)}$$

In particular, taking $\frac{\lambda}{a\alpha} = \frac{1}{2}$, follows that

$$\sup_{x \geq 0} \frac{E(x + M)}{V(x)} = 2 < e.$$

and the bound in Theorem 2 is not attained.

4 Proof of the Theorems

Proof of Theorem 1. It is enough to verify the following two conditions

- (i) $\sup_{\tau \in \mathcal{M}} E(x + X_\tau)^+ = E(x + M - E(M))^+.$
- (ii) $E(x + X_{\tau^*})^+ = E(x + M - E(M))^+.$

In order to prove (i) we consider our problem as the limit of a sequence of optimal stopping problems for the random walks $X^n = \{X_{k2^{-n}} : k \in \mathbf{N}\}$, $n \in \mathbf{N}$.

Denote $g(x) = x^+$ and $T_t g(x) = Eg(x + X_t)$. For each fixed n , let

$$Q_n g(x) = \max\{g(x), T_{2^{-n}} g(x)\}, \quad M^n = \sup_k X_{k2^{-n}}.$$

The sequence $\{M^n\}$ increases with n , and

$$M^n \rightarrow M = \sup_t X_t \quad P\text{-a.s.}$$

because X is càdlàg. Denote $\mathbf{F}^n = (\mathcal{F}_k^n)$ the filtration generated by the random walk X^n , that is $\mathcal{F}_k^n = \sigma\{X_{i2^{-n}} : 1 \leq i \leq k\}$, and \mathcal{M}^n the class of stopping times relative to \mathbf{F}^n , that take values $k2^{-n}$, $k \in \mathbf{N}$. Observe that if $\tau \in \mathcal{M}^n$, $X_\tau^n = X_\tau$. Q_n^N is the N^{th} iteration of the operator Q_n .

Now

$$\begin{aligned} \sup_{\tau \in \mathcal{M}} Eg(x + X_\tau) &= \lim_n \left(\lim_N Q_n^N g(x) \right) = \lim_n \left(\sup_{\tau \in \mathcal{M}_n} Eg(x + X_\tau) \right) \\ &= \lim_n E(x + M^n - E(M^n))^+ = E(x + M - E(M))^+. \end{aligned}$$

The first equality follows from Lemma III.3 and Theorem II.3, in Shiryaev (1978). The second from Theorem II.3 also in Shiryaev (1978). The third from Darling, Ligget and Taylor (1972) and the fourth by dominated convergence, because

$$-E(M) \leq M^n - E(M^n) \leq M.$$

(ii) is proved as in Darling, Ligget and Taylor (1972). For the sake of completeness we rewrite the proof. Consider τ^* defined by (3), and $A = \{\tau^* < +\infty\}$. We have

$$E(x + M)\mathbf{1}_A = E(M - X_{\tau^*} + x + X_{\tau^*})\mathbf{1}_A = E(M - X_{\tau^*})\mathbf{1}_A + E(x + X_{\tau^*})\mathbf{1}_A.$$

On the set A , applying the strong Markov property

$$E(M - X_{\tau^*})\mathbf{1}_A = E(\sup_{t \geq 0} (X_{\tau^*+t} - X_{\tau^*})\mathbf{1}_A) = E(M)P(A).$$

In conclusion, as the process drifts to $-\infty$

$$\begin{aligned} E(x + X_{\tau^*})^+ &= E(x + X_{\tau^*})\mathbf{1}_A = E(x + M)\mathbf{1}_A - E(M)P(A) = \\ &= E(x + M - E(M))\mathbf{1}_A = E(x + M - E(M))^+ \end{aligned}$$

concluding the proof. \square

Proof of Theorem 2. (i) Let X be an arbitrary Lévy process such that $E(M) < +\infty$. Inequality $V(x) \leq E(x + M)$ is immediate. We have to prove, that for $x \geq 0$

$$E(x + M) \leq eV(x).$$

As $V(x) = E(x + M - E(M))^+$, this amounts to say, that for $x \geq 0$

$$(21) \quad q(x) = \frac{E(x + M)}{E(x + M - E(M))^+} \leq e.$$

If $x \geq E(M)$, $E(x + M - E(M))^+ = x$, and $q(x) \leq 2 < e$. Let now $x \in \mathbf{R}$, and denote $\tau_x = \inf\{t \geq 0: X_t \geq x\}$, and as before, $B(x) = P(M \geq x)$. We now prove, for all x, y positive

$$(22) \quad B(x + y) \geq B(x)B(y).$$

Taking into account that $X_{\tau_x} - x \geq 0$, we obtain

$$\begin{aligned} B(x + y) &= P(M \geq x + y) = P(M \geq x + y, \tau_x < +\infty) \\ &= P(\sup_t (X_t - x) \geq y, \tau_x < +\infty) = P(\sup_t (X_{\tau_x+t} - x) \geq y, \tau_x < +\infty) \\ &\geq P(\sup_t (X_{t+\tau_x} - X_{\tau_x}) \geq y, \tau_x < +\infty) = B(x)B(y) \end{aligned}$$

and (22) is proved. Consider now the auxiliary function $f = f(x)$ given by

$$f(x) = e^{\frac{x}{x^*}} \int_x^{+\infty} B(y)dy, \quad x \geq 0.$$

Differentiation gives

$$f'(x) = e^{\frac{x}{x^*}} \left(\frac{1}{x^*} \int_x^{+\infty} B(x)dx - B(x) \right)$$

In order to conclude that $f'(x) \geq 0$ for all $x \geq 0$, we integrate (22)

$$\int_x^{+\infty} B(y)dy = \int_0^{+\infty} B(x + y)dy \geq B(x) \int_0^{+\infty} B(y)dy = B(x)x^*$$

and deduce that $f(x)$ is an increasing function. As $f(0) = x^*$ we conclude

$$(23) \quad f(x) = e^{\frac{x}{x^*}} \int_x^{+\infty} B(y) dy \geq x^*.$$

Now, applying (23) with $0 \leq x \leq x^*$

$$q(x) = \frac{x + x^*}{\int_{x^*-x}^{+\infty} B(y) dy} \leq \frac{x + x^*}{x^*} \exp\left(\frac{x^* - x}{x^*}\right) = c(x).$$

It is direct to see that the function $c(x)$ is decreasing, and that $c(0) = e$, concluding the proof of (i).

(ii) Assume $E(M) = +\infty$. If $P(\limsup_{t \rightarrow +\infty} X_t = +\infty) = 1$, denoting $\tau_a = \inf\{t \geq 0: x + X_t \geq a\}$ for any level a , we have $P(\tau_a < +\infty) = 1$ and

$$V(x) \geq E(x + X_{\tau_a}) \geq a.$$

As a is arbitrary $V(x) = +\infty$ follows.

By VI.3.12 in Bertoin (1996) it remains to consider the case $P(\lim_{t \rightarrow +\infty} X_t = -\infty) = 1$. Introduce X_t^a as

$$X_t^a = X_t - \sum_{0 < s \leq t} \Delta X_s \mathbf{1}_{\{\Delta X_s > a\}}.$$

As $P(X_t^a \leq X_t \forall t) = 1$, we deduce $P(\lim_{t \rightarrow +\infty} X_t^a = -\infty) = 1$ and

$$(24) \quad V^a(x) = \sup_{\tau} E(x + X_{\tau}^a)^+ \leq V(x).$$

Furthermore, as the Lévy process $X^a = \{X_t^a\}_{t \geq 0}$ has bounded positive jumps and drifts to $-\infty$, denoting $M^a = \sup_t X_t^a$ by Lemma 1 (i) we have $E(M^a) < +\infty$. Then by part (i)

$$1 \leq \frac{E(x + M^a)}{V^a(x)} \leq e.$$

As

$$M^a \nearrow M \quad P\text{-a.s.} \quad \text{as } a \rightarrow +\infty,$$

and $E(M) = +\infty$, we conclude that $V^a(x) \nearrow +\infty$ as $a \rightarrow +\infty$. By (24) the proof of (ii) is concluded. \square

Proof of Theorem 3. First observe, that (ii) follows from (i) and Theorem 1.

Let us prove (i). In order to see that the process drifts to $-\infty$ we apply Lemma 2. By (5) and (6)

$$b + \int_{\{|y| \geq 1\}} y \Pi(dy) = b + \lambda \int_1^{+\infty} y \alpha e^{-\alpha y} dy + \int_{-\infty}^{-1} y \pi(dy)$$

$$= a + \frac{\lambda}{\alpha} + \int_{-\infty}^{-1} y\pi(dy) < 0$$

so (12) follows from (8), and $\int_1^{+\infty} y^2\Pi(dy) < +\infty$ follows from (5).

Fix now $x > 0$, and consider the auxiliary function

$$(25) \quad A(z) = B(x - z),$$

with B given in (9). In order to prove (i) we apply Meyer-Itô's formula to X with characteristic exponent given by (4) and the function A .

As the process considered has a finite number of positive jumps on $[0, t]$ we consider a slightly modification in the decomposition of X in (14). Denoting

$$M_t = \sigma B_t + \int_0^t \int_{\{-1 < y < 0\}} y * (\mu - \nu), \quad S_t = \sum_{0 < s \leq t} \Delta X_s \mathbf{1}_{\{\Delta X_s \notin (-1, 0)\}},$$

with $M = \{M_t\}_{t \geq 0}$ a local martingale and $S = \{S_t\}_{t \geq 0}$ a process with finite variation, we have

$$X_t = M_t + S_t + at.$$

As the function A is not $\mathbf{C}^2(\mathbf{R})$ we apply Meyer-Itô formula (IV.51 of Protter (1992)) denoting $m(da)$ the signed measure that is the second derivative of A in the sense of distributions (when restricted to compacts).

$$(26) \quad \begin{aligned} A(X_t) - A(X_0) &= \int_0^t A'(X_{s-})dX_s + \frac{1}{2} \int_{\mathbf{R}} l^X(a, t)m(da) \\ &+ \sum_{0 < s \leq t} \left(A(X_s) - A(X_{s-}) - A'(X_{s-})\Delta X_s \right) \end{aligned}$$

where $l^X(a, t)$ is the local time of the process X at level a and time t .

We examine the three terms in (26) one by one.

$$\int_0^t A'(X_{s-})dX_s = \int_0^t A'(X_{s-})dM_s + \int_0^t A'(X_{s-})dS_s + \int_0^t A'(X_{s-})adt.$$

In particular,

$$\int_0^t A'(X_{s-})dS_s = \sum_{0 < s \leq t} A'(X_{s-})\Delta X_s \mathbf{1}_{\{\Delta X_s \notin (-1, 0)\}}.$$

For the second term, observe that

$$m(da) = A''(a)da - A'(x-)\delta_x(da)$$

with $A''(a)$ the second derivative of A if $a \neq x$, $A'(x-)$ the left derivative of A at the point x and $\delta_x(da)$ the point mass at x . Applying Corollary 1 to Theorem VI.51 of Protter (1992),

$$\int_{\mathbf{R}} l^X(a, t) m(da) = \int_0^t A''(X_{s-}) d\langle X, X \rangle_s - A'(x-) l^X(x, t).$$

If

$$\tau_x = \inf\{t \geq 0: X_t \geq x\},$$

we have $l^X(x, \tau_x) = 0$ because $X_t < x$ on the set $[0, \tau_x)$. So,

$$\int_{\mathbf{R}} l^X(a, t \wedge \tau_x) m(da) = \int_0^{t \wedge \tau_x} A''(X_{s-}) d\langle X, X \rangle_s = \int_0^{t \wedge \tau_x} A''(X_{s-}) \frac{1}{2} \sigma^2 ds.$$

The third term is transformed with stochastic integrals with respect to the measures μ and ν . First

$$\begin{aligned} & \sum_{0 < s \leq t} \left(A(X_s) - A(X_{s-}) - A'(X_{s-}) \Delta X_s \right) \\ &= \sum_{0 < s \leq t} \left(A(X_s) - A(X_{s-}) - A'(X_{s-}) \Delta X_s \mathbf{1}_{\{-1 < \Delta X_{s-} < 0\}} \right) - \int_0^t A'(X_{s-}) dS_s. \end{aligned}$$

Now, by Proposition II.1.28 in Jacod and Shiryaev (1978)

$$\begin{aligned} & \sum_{0 < s \leq t} \left(A(X_s) - A(X_{s-}) - A'(X_{s-}) \Delta X_s \mathbf{1}_{\{-1 < \Delta X_{s-} < 0\}} \right) = \\ &= \int_0^t \int_{\mathbf{R}} \left(A(X_{s-} + y) - A(X_{s-}) - A'(X_{s-}) y \mathbf{1}_{\{-1 < y < 0\}} \right) * (\mu - \nu) \\ & \quad + \int_0^t \int_{\mathbf{R}} \left(A(X_{s-} + y) - A(X_{s-}) - A'(X_{s-}) y \mathbf{1}_{\{-1 < y < 0\}} \right) * \nu, \end{aligned}$$

and, as $\nu(dy, ds) = \Pi(dy) ds$

$$\begin{aligned} & \int_0^t \int_{\mathbf{R}} \left(A(X_{s-} + y) - A(X_{s-}) - A'(X_{s-}) y \mathbf{1}_{\{-1 < y < 0\}} \right) * \nu \\ &= \int_0^t \left[\int_{\mathbf{R}} \left(A(X_{s-} + y) - A(X_{s-}) - A'(X_{s-}) y \mathbf{1}_{\{-1 < y < 0\}} \right) \Pi(dy) \right] ds. \end{aligned}$$

In conclusion, evaluating (26) at time $t \wedge \tau_x$ we obtain

$$(27) \quad A(X_{t \wedge \tau_x}) - A(0) = \int_0^{t \wedge \tau_x} (L^X A)(X_{s-}) ds + M(A)_{t \wedge \tau_x}$$

where

$$(L^X A)(x) = \frac{1}{2} \sigma^2 A''(x) + a A'(x) + \int_{\mathbf{R}} \left(A(x+y) - A(x) - y A'(x) \mathbf{1}_{\{-1 < y < 0\}} \right) \Pi(dy)$$

is the infinitesimal generator of X , and the process $M(A) = \{M(A)_t\}_{t \geq 0}$ given by

$$M(A)_t = \int_0^t A'(X_{s-}) dM_s + \int_0^t \int_{\mathbf{R}} \left(A(X_{s-} + y) - A(X_{s-}) - A'(X_{s-})y \mathbf{1}_{\{-1 < y < 0\}} \right) * (\mu - \nu).$$

is a local martingale. Suppose that

$$(28) \quad L^X A(z) = 0 \quad \text{for } z < x.$$

As $X_{s-} < x$ on the set $[0, t \wedge \tau_x]$,

$$\int_0^{t \wedge \tau_x} (L^X A)(X_{s-}) ds = 0.$$

Furthermore, as the function A is bounded, from (27) follows that $M(A)$ is a martingale. Taking expectations, and limit as $t \rightarrow +\infty$ in (27), in view of (25)

$$B(x) = A(0) = EA(X_{\tau_x}) = P(\tau_x < +\infty) = P(M \geq x).$$

because $X_{\tau_x} = -\infty$ on the set $\{\tau_x = +\infty\}$ and $\lim_{x \rightarrow -\infty} A(y) = 0$. To conclude the proof we verify (28).

Denote $A(z) = A_x(z) = B(x - z)$. As the operator L^X is homogeneous in space, taking $t = z - x$, (28) is equivalent to

$$L^X A_0(t) = 0 \quad \text{for } t < 0.$$

As $A_1 + A_2 = 1$, we have $A_0(t) = \sum_{k=1}^2 A_k e^{p_k t^-}$, where $t^- = \min(t, 0)$. By linearity

$$L^X A_0(t) = \sum_{k=1}^2 A_k L^X (e^{p_k t^-}).$$

For $t < 0$,

$$L^X (e^{pt}) = e^{pt} \left(ap + \frac{1}{2} \sigma^2 p^2 + \int_{-\infty}^0 (e^{py} - 1 - py \mathbf{1}_{\{-1 < y < 0\}}) \pi(dy) \right) + \lambda \int_0^{+\infty} (e^{p(t+y)^-} - e^{pt}) \alpha e^{-\alpha y} dy.$$

The second integral equals,

$$\int_0^{-t} e^{p(t+y)} \alpha e^{-\alpha y} dy + \int_{-t}^{+\infty} \alpha e^{-\alpha y} dy - e^{pt} = (e^{\alpha t} - e^{pt}) \left(\frac{p}{p - \alpha} \right).$$

In conclusion, for $t < 0$

$$\begin{aligned}
LA_0(t) &= \sum_{k=1}^2 A_k e^{p_k t^-} \left(ap_k + \frac{1}{2} \sigma^2 p_k^2 + \lambda \frac{p}{\alpha - p} \right. \\
&+ \left. \int_{-\infty}^0 (e^{p_k y} - 1 - p_k y \mathbf{1}_{\{-1 < y < 0\}}) \pi(dy) \right) + \lambda e^{\alpha t} \sum_{k=1}^2 A_k \frac{p_k}{p_k - \alpha} \\
&= \sum_{k=1}^2 A_k e^{p_k t^-} \kappa(p_k) + \lambda e^{\alpha t} \sum_{k=1}^2 A_k \frac{p_k}{p_k - \alpha} = 0
\end{aligned}$$

by (10), concluding that (28) holds, and the proof of the Theorem 3. \square

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