Optimal Stopping and Maximal Inequalities for Poisson Processes

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Abstract

Closed form solutions for some optimal stopping problems for stochastic processes driven by a Poisson processes $N = (N_t)_{t \geq 0}$ are given.

First, cost functions and optimal stopping rules are described for the problems

$$s(x) = \sup_{\tau} E(\max[x, \sup_{0 \leq t \leq \tau} (N_t - at)] - c\tau),$$
$$v(x) = \sup_{\tau} E(\max[x, \sup_{0 \leq t \leq \tau} (bt - N_t)] - c\tau),$$

with $a, b, c$ positive constants and $\tau$ a stopping time.

Based on the obtained results, maximal inequalities in the spirit of [1] are obtained.

To complete the picture, solutions to the problems

$$c(x) = \sup_{\tau} E(x + N_\tau - a\tau)^+, \quad p(x) = \sup_{\tau} E(x + b\tau - N_\tau)^+$$

are given.
1 Introduction and main results

1.1. Let be given a Poisson process $N = (N_t)_{t \geq 0}$ with intensity $\lambda > 0$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$. Denote by $\mathcal{M}$ the class of all stopping times (that can take the value $+\infty$), by $\mathcal{M}$ the class of finite valued stopping times, and by $\mathcal{M}_0$ the class of stopping times with finite expectation. All stopping times are considered with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$.

In this paper we face the problem of giving the cost functions and optimal stopping times in the following optimal stopping problems:

(1) $s(x) = \sup_\tau E(\max[x, \sup_{0 \leq t \leq \tau} (N_t - at)] - c\tau)$,

(2) $v(x) = \sup_\tau E(\max[x, \sup_{0 \leq t \leq \tau} (bt - N_t)] - c\tau)$,

where $a$, $b$ and $c$ are positive constants, ($c$ is the “price” of one unity of observation), and $x \in \mathbb{R}$.

Problems (1) and (2) are related to the pricing of “Russian Options” introduced by L. Shepp and A.N. Shiryaev in [2]. In our case, the price process for the risky asset is driven by a Poisson process instead of a geometric Brownian motion.

The solutions to these problems are then used to obtain maximal inequalities of the form

(3) $\sup_{\tau \in \mathcal{M}_0} E[\sup_{0 \leq t \leq \tau} (N_t - at)] \leq \phi(E\tau)$

(4) $\sup_{\tau \in \mathcal{M}_0} E[\sup_{0 \leq t \leq \tau} (bt - N_t)] \leq \psi(E\tau)$

where $\phi = \phi(x)$, $\psi = \psi(x)$, $x > 0$ are the minimal possible functions, that satisfy (3) and (4).

In relation with this, we refer to the paper of L. Dubbins, L.A. Shepp and A.N. Shiryaev [1] devoted to the study of optimal stopping rules and maximal inequalities for Bessel Processes, the work of L. Dubbins and G. Schwartz [2], and to some recent results for linear diffusions of S.E. Graversen and G. Peskir [4].

Finally, to complete the picture, we give the cost function and optimal stopping time in the following problems

(5) $c(x) = \sup_{\tau \in \mathcal{M}} E(x + N_\tau - a\tau)^+$
where $x \in \mathbb{R}$, $z^+$ denotes $\max(z, 0)$.

1.2. In order to formulate our results, let us introduce the function $u = u(x) = u(x; d), x \geq 0$ with $d$ a positive constant, defined by

\begin{equation}
(7) \quad u(x; d) = \sum_{k=0}^{+\infty} [e^{\frac{x-k}{d}} P_k(\frac{x-k}{d}) - (k+1)] I_{[k,k+1)}(x)
\end{equation}

where $P_k = P_k(x)$ is a polynomial of order $k$:

\begin{equation}
(8) \quad P_k(x) = \sum_{l=0}^{k} d_{k-l} \frac{(-1)^l x^l}{l!},
\end{equation}

and the coefficients $d_k, k \geq 0$ are defined by the following recurrence relation:

\begin{equation}
(9) \quad d_0 = 1, \quad d_{k+1} = 1 + e^{1/d} \sum_{l=0}^{k} \frac{(-1)^l}{d^l l!} = 1 + e^{1/d} P_k(1/d).
\end{equation}

Some properties of the function $u = u(x)$ are revisted in section 2.

Denote now by $\tau^*_s$ the optimal stopping time in the problem (1), that is, the stopping time for which the supremum is realized. The price function $s = s(x)$ and the optimal stopping time $\tau^*_s$ are given in the following

**Theorem 1**

(i) If $a + c \leq \lambda$, then $s(x) = +\infty$, for all $x \geq 0$.

(ii) If $a > \lambda$ and $c = 0$, then

\begin{equation}
(10) \quad s(x) = E(\max[x, \sup_{0 \leq t \leq +\infty} (N_t - at)]) = \frac{\lambda}{2(a - \lambda)} + \frac{a - \lambda}{\lambda} u(x; \frac{a}{\lambda})
\end{equation}

(iii) If $a > 0$ and $c \geq \lambda$, then $s(x) = x$ and $\tau^*_s = 0$.

(iv) If $c < \lambda < a + c$, then

\begin{equation}
(11) \quad s(x) = \begin{cases} x, & \text{if } x \geq x^* \\ x_s^* + \frac{a+c-\lambda}{\lambda}[u(x; \frac{a}{\lambda}) - u(x_s^*; \frac{a}{\lambda})], & \text{if } 0 \leq x < x^* \end{cases}
\end{equation}

and

\begin{equation}
(12) \quad \tau^*_s = \inf \{ t \geq 0 : X_t \geq x_s^* \},
\end{equation}

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where \( X = (X_t)_{t \geq 0} \) is a stochastic process defined by

\[
X_t = \max[x, \sup_{0 \leq r \leq t} (N_r - ar)] - (N_t - at),
\]

and the positive constant \( x^*_s \) is the solution of the equation

\[
\frac{a + c - \lambda}{\lambda} u'(x; \frac{a}{\lambda}) = 1.
\]

We also have, when \( x < x^*_s \)

\[
E(\tau^*_s) = \frac{1}{\lambda}[u(x^*_s, \frac{a}{\lambda}) - u(x; \frac{a}{\lambda})].
\]

Denote now by \( \tau^*_v \) the optimal stopping time for the problem (2). The price function \( v = v(x) \) and the optimal stopping time are given in the following

**Theorem 2**

(i) If \( c + \lambda \leq b \), then \( v(x) = +\infty \), for all \( x \geq 0 \).

(ii) If \( \lambda > b \) and \( c = 0 \), then

\[
v(x) = E(\max[x, \sup_{0 \leq t < +\infty} (bt - N_t)]) = x + \frac{1}{\alpha^*} e^{-\alpha^* x},
\]

where the constant \( \alpha^* \) is the unique positive root of the equation

\[
\frac{b}{\lambda} \alpha + e^{-\alpha} - 1 = 0.
\]

(iii) If \( c \geq b \), then \( v(x) = x \) and \( \tau^*_v = 0 \).

(iv) If \( c < b < c + \lambda \), then

\[
v(x) = \begin{cases} 
  x, & \text{if } x \geq x^*_v, \\
  x + \frac{x^*_v}{\lambda} u(x^*_v - x, \frac{b}{\lambda}), & \text{if } 0 \leq x < x^*_v,
\end{cases}
\]

and

\[
\tau^*_v = \inf\{t \geq 0: Y_t \geq x^*_v\},
\]

where \( Y = (Y_t)_{t \geq 0} \) is a stochastic process defined by

\[
Y_t = \max[x, \sup_{0 \leq r \leq t} (br - N_r)] - (bt - N_t),
\]
and the positive constant $x_v^*$ is the solution of the equation

\begin{equation}
\frac{c}{\lambda} u'(x; \frac{b}{\lambda}) = 1.
\end{equation}

We also have, when $0 \leq x < x_v^*$

\begin{equation}
E(\tau_v^*) = \frac{1}{c + \lambda - b} \left[ x_v^* - x - \frac{c}{\lambda} u(x_v^* - x, \frac{b}{\lambda}) \right].
\end{equation}

1.3. Let $\mathcal{M}_T = \{ \tau \in \mathcal{M}_0: E(\tau) \leq T \}$ denote the set of all the stopping times $\tau$ with expected value less or equal than $T > 0$. Functions $\phi$ and $\psi$ in (3) and (4) can be defined in the following way

\begin{equation}
\phi(T) = \sup_{\tau \in \mathcal{M}_T} E( \sup_{0 \leq t \leq \tau} (N_t - at)),
\end{equation}

\begin{equation}
\psi(T) = \sup_{\tau \in \mathcal{M}_T} E( \sup_{0 \leq t \leq \tau} (bt - N_t)).
\end{equation}

In what follows, the stopping times such that the supremum in (22) and (23) is realized will be called $\mathcal{M}_T$-optimal.

**Theorem 3** Let $0 < T < +\infty$.

(i) Denote by $x_s^*(T)$ the root of the equation

\begin{equation}
u(x; \frac{a}{\lambda}) = \lambda T.
\end{equation}

Then

\begin{equation}
\phi(T) = x_s^*(T) + (\lambda - a)T
\end{equation}

and the stopping time

$\tau_s^*(T) = \inf\{ t \geq 0: X_t \geq x_s^*(T) \}$,

is $\mathcal{M}_T$-optimal for the problem (22), where in the process $X = (X_t)_{t \geq 0}$ defined in (13) we take $x = 0$.

Furthermore, when $\lambda = a$ we have

\begin{equation}
\frac{\sqrt{4T + 1} - 1}{2} \leq \phi(T) \leq \sqrt{T}
\end{equation}
and in consequence

\[
\lim_{T \to +\infty} \frac{\phi(T)}{\sqrt{T}} = 1.
\]

(ii) Denote by \(x_v^*(T)\) the positive root of the equation

\[
\frac{xu'(x; \frac{b}{\lambda}) - u(x; \frac{b}{\lambda})}{1 + (1 - \frac{b}{\lambda})u'(x; \frac{b}{\lambda})} = \lambda T,
\]

Then

\[
\psi(T) = x_v^*(T) + (b - \lambda)T,
\]

and the stopping time

\[
\tau_v^*(T) = \inf\{t \geq 0: Y_t \geq x_v^*(T)\},
\]

is \(M_T\)-optimal for the problem (23), where in the process \(Y = (Y_t)_{t \geq 0}\) defined in (19) we take \(x = 0\).

Furthermore, when \(\lambda = b\) we have

\[
\lim_{T \to +\infty} \frac{\psi(T)}{\sqrt{T}} = 1.
\]

1.4. Denote by \(\tau_c^*\) the optimal stopping time in the problem (5). The price function and the optimal stopping time for this problem are given in the following

**Theorem 4**

(i) If \(a \leq \lambda\), then \(c(x) = +\infty\) for all \(x \in \mathbb{R}\).

(ii) If \(\lambda < a\), then

\[
c(x) = \begin{cases} x, & \text{if } x \geq x_c^*, \\ x + \left(\frac{a}{\lambda} - 1\right)u(x_c^* - x, \frac{a}{\lambda}), & \text{if } x < x_c^*, \end{cases}
\]

with \(x_c^* = \frac{\lambda}{2(a-\lambda)}\). The optimal stopping time is

\[
\tau_c^* = \inf\{t \geq 0: x + N_t - at \geq x_c^*\}.
\]

Denote finally by \(\tau_p^*\) the optimal stopping time in the problem (6). The cost function and the optimal stopping time for this problem are given in the following
Theorem 5
(i) If \( b \geq \lambda \), then \( p(x) = +\infty \) for all \( x \in \mathbb{R} \).
(ii) Assume \( b < \lambda \). Denote by \( \alpha^* \) the positive root of the equation (17). Then, the cost function is

\[
p(x) = \begin{cases} 
  x, & \text{if } x \geq x_p^* \\
  x_p^* e^{\alpha^*(x-x_p^*)} & \text{if } x < x_p^*,
\end{cases}
\]

with \( x_p^* = \frac{1}{\alpha^*} \), and the optimal stopping time is

\[
\tau^*_p = \inf\{t \geq 0: x + bt - N_t \geq x_p^*\}.
\]

2 Some auxiliar results

2.1. In this section we formulate some technical results concerning the function \( u = u(x), x \geq 0 \) defined in (7). Let us introduce the operator \( \mathcal{K} = \mathcal{K}(d) \) defined on the set of continuously differentiable functions by the following relation

\[
\mathcal{K}w(x) = d w'(x) + w(x - x \wedge 1) - w(x), \quad x \geq 0.
\]

As will be shown in section 3, the operator \( \lambda \mathcal{K} \) is the infinitesimal operator associated to the process \( X_t = (X_t)_{t \geq 0} \) defined in (13) with \( d = \frac{a}{\lambda} \).

Lemma 1 The function \( u = u(x) \) defined in (7) is continuosly differentable, and satisfies the following relation

\[
\mathcal{K}u(x) = du'(x) - u(x) + u(x - x \wedge 1) = 1, \quad x \geq 0.
\]

Proof. From the definition of the function \( u = u(x) \) (see (7), (8) and (9)) follows, that this function is smooth on the intervals \((k, k + 1)\) for any non-negative integer \( k \).

If \( x \in [0, 1) \), we have \( u(x) = e^{x/d} - 1 \), giving that \( u(0) = 0 \) and

\[
\mathcal{K}u(x) = du'(x) - u(x) = 1.
\]

If \( k \geq 1 \) and \( x \in (k, k + 1) \), equation (36) is

\[
\begin{align*}
  du'(x) - u(x) &= 1 - u(x - 1).
\end{align*}
\]
In view of (7) for $x \in (k, k+1)$

$$u(x) = e^{\frac{x-k}{d}}P_k\left(\frac{x-k}{d}\right) - (k+1),$$

and from (8) follows that $P'_k(x) = -P_{k-1}(x)$. Then

$$du'(x) - u(x) = e^{\frac{x-k}{d}}P'_k\left(\frac{x-k}{d}\right) + k + 1$$

(37)

$$= 1 - e^{\frac{x-k}{d}}P_{k-1}\left(\frac{x-k}{d}\right) + k = 1 - u(x-1)$$

In this way (36) is proved for $x \in (k, k+1)$. It is clear from (35), that in order to verify (36) when $x = 1, 2, \ldots$ it is enough to see that the functions $u = u(x)$ and $u' = u'(x)$ are continuous in these points.

Let us examine the continuity of $u$. In view of (7)

$$u(k) = P_k(0) - (k+1) = d_k - (k+1)$$

and by (9)

$$\lim_{x \to k} u(x) = e^{1/d}P_{k-1}(1/d) - k = d_k - (k+1),$$

and the continuity of $u$ follows. For the function $u'$ in view of (37)

$$\lim_{x \to k} u'(x) = \frac{1}{d} \lim_{x \to k} [1 + u(x-1) - u(x)]$$

$$= \frac{1}{d} [1 + u(k-1) - u(k)],$$

and

$$\lim_{x \to k} u'(x) = \frac{1}{d} \lim_{x \to k} [1 + u(x-x \wedge 1) - u(x)]$$

$$= \frac{1}{d} [1 + u(k-1) - u(k)],$$

concluding the proof. \hfill \Box

2.2. In the following Lemma we study the behaviour of $u = u(x)$ and its derivative.

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Lemma 2 The functions $u = u(x)$ and $u' = u'(x)$ satisfy

a) $u(0) = 0$, $u'(0) = \frac{1}{d}$.

b) $u$ and $u'$ are strictly increasing.

c) $\lim_{x \to +\infty} u(x) = +\infty$, $\lim_{x \to +\infty} u'(x) = \begin{cases} \frac{1}{d-1}, & d > 1 \\ +\infty, & 0 < d \leq 1 \end{cases}$.

d) If $d = 1$, for all $x \geq 0$ we have

$$x^2 \leq u(x) \leq x^2 + x.$$  

e) If $d = 1$, then

$$\lim_{x \to +\infty} \frac{xu'(x) - u(x)}{x^2} = 1.$$  

Proof.

a) As $u(x) = e^{x/d} - 1$ when $x \in [0, 1)$ we obtain

$$u(0) = 0 \quad \text{and} \quad u'(0) = \frac{1}{d}.$$  

b) In order to prove that $u$ and $u'$ are strictly increasing, as they are both absolutely continuous, we will see that

(38) $u''(x)$ is positive and continuous for all $x \neq 1$, $x \geq 0$.

As $u(x) = e^{x/d} - 1$ if $x \in (0, 1)$, by differentiation $u''(x) = \frac{1}{x^2}e^{x/d}$ is continuous and positive if $x \in [0, 1)$. Take now $x > 1$. From (36), for $x > 1$ we obtain

(39) $du''(x) = \int_{x-1}^{x} u''(y)dy$.

As $u''$ is bounded on compact intervals, the continuity if $x \neq 1$ follows. In view of (39) follows that

$$u''(x) > 0, \quad x > 1.$$  

In fact, let $x_0 = \inf\{x \geq 0: u''(x) = 0\}$. We have $u''(x_0) = 0$ and $u''(x) > 0$ when $x < x_0$ contradicting (39).

So $u = u(x)$ and $u' = u'(x)$ are both strictly increasing functions.

c) Let us now compute the limits at infinite. If $x > 1$, then from (36) follows that

$$du'(x) = u(x) - u(x - 1) + 1 = 1 + \int_{x-1}^{x} u'(y)dy.$$  

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From this, taking into account the monotonicity of the function \( u' = u'(x) \) we obtain
\[
1 + u'(x - 1) < du'(x) < 1 + u'(x)
\]
and taking limits we obtain the second limit in c). Finally, this ensures the convergence of \( u(x) \to +\infty \) when \( x \to +\infty \), proving c).

d) Let \( d = 1 \). Denote \( \Delta(x) = u(x) - x^2 \). We want to establish
\[
\Delta(x) \geq 0 \quad \text{for all} \quad x \geq 0.
\]
For this function
\[
K\Delta(x) = Ku - K(x^2) = 0.
\]
This means
\[
\Delta'(x) = \Delta(x) - \Delta(x - 1) = \int_{x-1}^{x} \Delta'(t) dt.
\]
If \( x \in (0,1) \), we know \( \Delta(x) = e^x - 1 - x^2 \), and in consequence \( \Delta'(x) > 0 \) on this interval. So, with the same argument as in the proof of (38) we obtain that \( \Delta'(x) > 0 \) for all \( x > 0 \). As \( \Delta(0) = 0 \), (40) is proved.

Denote now
\[
\delta(x) = x^2 + x - u(x).
\]
In a similar way, as was done for \( \Delta \), we obtain that
\[
\delta'(x) = 2x + 1 - e^x > 0 \quad \text{for} \quad 0 < x < 1,
\]
and for \( x > 1 \)
\[
\delta'(x) = \int_{x-1}^{x} \delta'(t) dt.
\]
concluding that \( \delta(x) \geq 0 \) for all \( x \geq 0 \) and the proof of d).

e) From d) we obtain
\[
\lim_{x \to +\infty} \frac{u(x)}{x^2} = 1
\]
Now, denoting \( w(x) = xu'(x) - u(x) \), we have
\[
w'(x) = xu''(x).
\]
So, by L'Hôpital rule
\[
\lim_{x \to +\infty} \frac{w(x)}{x^2} = \lim_{x \to +\infty} \frac{xu''(x)}{2x} = \lim_{x \to +\infty} \frac{u''(x)}{2}.
\]
But, in view of (41), and L'Hôpital rule again
\[
\lim_{x \to +\infty} \frac{w''(x)}{2} = \lim_{x \to +\infty} \frac{u'(x)}{2x} = \lim_{x \to +\infty} \frac{u(x)}{x^2} = 1.
\]
proving d). \qed

3. We will also need the following

Lemma 3 Let \( d > 1 \). Then the function \( w = w(x) \) defined by
\[
w(x) = x - (d - 1)u(x), \quad x \geq 0,
\]
is strictly increasing and
\[
\lim_{x \to +\infty} w(x) = \frac{1}{2(d - 1)}.
\]

Proof. By part c) of Lemma 2
\[
w'(x) = 1 - (d - 1)u'(x) > 0 \quad \text{for all} \quad x \geq 0
\]
so the function \( w \) is strictly increasing. In consequence the limit in (42) exists.

Now, by (36)
\[
Kw(x) = dw'(x) - w(x) = 1 - x, \quad x < 1,
\]
and
\[
Kw(x) = dw'(x) - w(x) + w(x - 1) = 0, \quad x \geq 1.
\]
Integrating (43) over \([1, x]\), follows
\[
d[w(x) - w(1)] = \int_1^x [w(y) - w(y - 1)] dy
\]
\[
= \int_{x-1}^x w(y) dy + \int_0^1 w(y) dy
\]
Taking into account (36)
\[
dw(x) = \int_{x-1}^x w(y) dy + \frac{1}{2}.
\]
Now, the monotonicity of \( w(x) \) gives
\[
w(x - 1) + \frac{1}{2} < dw(x) < w(x) + \frac{1}{2},
\]
and taking limits as \( x \to +\infty \) we conclude the proof. \qed
3 Proofs of the Theorems

3.1. The process \( X = (X_t)_{t \geq 0} \), defined in (13), is an homogeneous Markov process with stochastic differential

\[
\mathrm{d}X_t = a\mathrm{d}t - (X_t - 1)\mathrm{d}N_t.
\]

If \( w = w(x), \ x \geq 0, \) is a continuously differentiable function, then, Itô’s formula for pure jump processes gives (see [6] ch. 3 §6)

\[
(44) \quad w(X_t) = w(x) + \lambda \int_0^t K w(X_{r-})\mathrm{d}r + M_t, \quad t \geq 0,
\]

with \( K = K(\frac{d}{X}) \) in (35) and the process \( M = (M_t)_{t \geq 0} \) given by

\[
M_t = \int_0^t [w(X_{r-} - X_{r-} - 1) - w(X_{r-})]\mathrm{d}(N_r - \lambda r)
\]

is a local martingale. From (44) and Lemma 1 follows, that the process \( (u(X_t) - \lambda t)_{t \geq 0} \) is a local martingale, with the function \( u = u(x; \frac{a}{X}) \) defined in (7) and \( d = \frac{a}{X} \).

Proof of Theorem 1.

(i) Assume \( a + c \leq \lambda \). As \( E(N_t - (a + c)t) = (\lambda - a - c)t \geq 0 \) and the process \( \{N_t - (a + c)t\} \) has stationary independent increments, it follows that \( P(\sup_{t \geq 0} (N_t - (a + c)t) = +\infty) = 1 \) (see [7]). This means, that for the stopping time

\[
\tau_H = \inf\{t \geq 0; N_t - (a + c)t \geq H\},
\]

we have \( P(\tau_H < +\infty) = 1 \). Then

\[
s(x) \geq s(0) \geq E(\sup_{0 \leq r \leq \tau_H} (N_r - ar) - c\tau_H)
\]

\[
\geq E((N_{\tau_H} - (a + c)\tau_H) \geq H.
\]

As \( H \) is arbitrary, the proof of (i) is concluded.

(ii) Let \( c = 0 \) and \( a > \lambda \). For \( t > 0 \) denote

\[
s_t(x) = E(\max[x, \sup_{0 \leq r \leq t} (N_r - ar)]).
\]

By monotnonous convergence

\[
s_t(x) \uparrow s(x), \quad t \to +\infty.
\]
Taking into account, that \( E(N_t) = \lambda t, t \geq 0 \), we obtain that

\[
s_t(x) = E(X_t - (a - \lambda)t), \quad X_0 = x,
\]

with the process \( X = (X_t)_{t \geq 0} \) defined in (13).

Denote \( w(x) = x - \frac{a-\lambda}{\lambda} u(x) \). By Lemma 3, we have

\[
\lim_{x \to +\infty} w(x) = \frac{\lambda}{2(a - \lambda)}.
\]

Furthermore, as \( X_t \geq at - N_t \), and \( a > \lambda \), then, \( X_t \to +\infty \) with probability one as \( t \to +\infty \). As the process \( \{\frac{a-\lambda}{\lambda} u(X_t) - (a - \lambda)t\} \) is a martingale, we obtain

\[
\begin{align*}
\lim_{t \to +\infty} E(w(x)) &= \lim_{t \to +\infty} E(X_t - \frac{a-\lambda}{\lambda} u(X_t)) = \frac{\lambda}{2(a - \lambda)}.
\end{align*}
\]

by bounded convergence, concluding the proof of (ii).

(iii) First we observe, that when \( c > 0 \) and \( a + c > \lambda \) the supremum in (1) can be taken over \( M_0 \). To see this, take an arbitrary stopping time \( \tau \in M \) with \( E(\tau) = +\infty \). Consider \( \delta \) such that \( \lambda - a < \delta < c \).

\[
E(\max[x, \sup_{0 \leq t \leq \tau} (N_t - at)] - c\tau) \leq x + E[\sup_{0 \leq t < +\tau} (N_t - at) - c\tau]
\]

\[
\leq x + E[\sup_{0 \leq t < +\tau} (N_t - (a + \delta)t) - (c - \delta)\tau] = -\infty
\]

because, in accordance with (10)

\[
E(\sup_{0 \leq t < +\infty} (N_t - (a + \delta)t)) = \frac{\lambda}{a + \delta - \lambda} < +\infty.
\]

Now, if \( c \geq \lambda \), \( a > 0 \) and \( E(\tau) < +\infty \), then \( E(N_\tau) = \lambda E(\tau) \) and

\[
E(\max[x, \sup_{0 \leq t \leq \tau} (N_t - at)] - c\tau)) \leq x + E(\sup_{0 \leq t \leq \tau} (N_t - at) - c\tau)
\]

\[
\leq x + E(N_\tau - c\tau) \leq x - (c - \lambda)E(\tau) \leq x.
\]
On the other hand, taking \( \tau^*_s = 0 \)
\[
E(\max[x, \sup_{0 \leq t \leq \tau^*_s} (N_t - at)] - ct^*_s) = x,
\]
and follows that \( \tau^*_s = 0 \) is the optimal stopping time, and \( s(x) = x, \ x \geq 0. \)

(iv) Let \( c < \lambda < a + c. \) In view of Lemma 2 (14) has an unique solution \( x^*_s \) such that \( 0 < x^*_s < +\infty. \)

Let us first prove, that for \( \tau^*_s \) defined by (12), identity (15) holds.

As \( u(x) \leq u(x^*_s) \) if \( x \leq x^*_s, \) then
\[
-\lambda t \leq u(X_{\tau^*_s \wedge t}) - \lambda (\tau^*_s \wedge t) \leq u(x^*_s)
\]
and, as a consequence, the local martingale \( \{u(X_{\tau^*_s \wedge t}) - \lambda (\tau^*_s \wedge t)\}_{t \geq 0} \) is in fact a martingale. Therefore

(45) \[
\lambda E(\tau^*_s \wedge t) = E(u(X_{\tau^*_s \wedge t})) - u(x)
\]
From this, we deduce that \( E(\tau^*_s) < +\infty, \) so \( P(\tau^*_s < +\infty) = 1, \) concluding that
\[
u(X_{\tau^*_s \wedge t}) \to u(x^*_s) \quad \text{as} \quad t \to +\infty.
\]
Now, taking limits in (45) as \( t \to +\infty \) we obtain (15).

We know by (iii) that we can take the supremum over \( \mathcal{M}_0. \) If \( \tau \in \mathcal{M}_0, E(\tau) < +\infty, \) then \( E(N_\tau) = \lambda E(\tau). \) In consequence
\[
E(\max[x, \sup_{0 \leq t \leq \tau} (N_t - at)] - ct) = E(X_\tau - (a + c - \lambda)\tau).
\]
Denote by \( \tilde{s} = \tilde{s}(x) \) by
\[
s(x) = \begin{cases} x, & \text{if } x \geq x^*_s \\ x^*_s + \frac{x + c - \lambda}{\lambda}[u(x; \frac{a}{\lambda}) - u(x^*_s; \frac{a}{\lambda})], & \text{if } 0 \leq x < x^*_s \end{cases}
\]
We want to prove \( \tilde{s} = s \) with \( s \) in (1). The following relation takes place
(A) \( s(x) = \max(x, x^*_s) - (a + c - \lambda)E(\tau^*_s) = E(X_{\tau^*_s} - (a + c - \lambda)\tau^*_s), \)
that is direct if \( x \geq x^*_s \) and follows from (15) when \( x < x^*_s. \) In consequence, in order to complete the proof of the Theorem, it remains to see that for any stopping time \( \tau \in \mathcal{M}_0 \)
(B) \( \tilde{s}(x) \geq E(X_\tau - (a + c - \lambda)\tau). \)
In view of Lemma 2 the function \( u = u(x) \) is convex, so \( s(x) \geq x \). To prove (B) it is enough to see that the process \( Z = (Z_t)_{t \geq 0} \) defined by

\[
Z_t = \tilde{s}(X_{\tau \wedge t}) - (a + c - \lambda)(\tau \wedge t)
\]
is a supermartingale. From (44) follows that the process \( Z \) is a local supermartingale, if

\[
\lambda \mathcal{K}\tilde{s}(x) - (a + c - \lambda) \leq 0, \quad x \geq 0,
\]
with \( \mathcal{K} = \mathcal{K}(\frac{x}{\lambda}) \) defined in (35).

For \( x \leq x^*_s \) (46) takes places in view of Lemma 1 (in fact we have an identity). If \( x > x^*_s \), then \( \tilde{s}(x) = x \) and

\[
\lambda \mathcal{K}\tilde{s}(x) - (a + c - \lambda) = a + \lambda \tilde{s}(x - x \wedge 1) - \lambda x - (a + c - \lambda)
\]
\[
\leq a + \lambda \tilde{s}(x^*_s - x^*_s \wedge 1) - \lambda x^*_s - (a + c - \lambda) = \lambda \mathcal{K}\tilde{s}(x^*_s) - (a + c - \lambda) = 0,
\]
where we used the convexity of the function \( \tilde{s} \). In this way the process \( Z = (Z_t)_{t \geq 0} \) is a local supermartingale, and as \( Z_t \geq -(a + c - \lambda)t, \quad t \geq 0 \), \( E(\tau) < +\infty \), we deduce that \( Z \) is a supermartingale and the proof is concluded. \( \square \)

3.2. The process \( Y = (Y_t)_{t \geq 0} \) defined in (19) is an homogeneous Markov process with stochastic differential

\[
dY_t = dN_t - \mathbf{1}_{\{Y_t \geq 0\} \lambda} dW_t.
\]

If \( w = w(x), \ x \geq 0 \), is a continously differentiable function, Itô’s formula gives (see [6])

\[
w(Y_t) = w(x) + \lambda \int_0^t \mathcal{L}w(Y_r-)dr + M_t,
\]
where

\[
\mathcal{L}w(x) = -\frac{b}{\lambda} \mathbf{1}_{\{x \geq 0\} \lambda} w'(x) + w(x + 1) - w(x),
\]
and the process \( M = (M_t)_{t \geq 0}, \) given by

\[
M_t = \int_0^t \left[ w(Y_{r-} + 1) - w(Y_{r-}) \right] d(N_r - \lambda r)
\]
is a local martingale.

Proof of Theorem 2.

(i) Let \( \lambda + c \leq b \). Take \( H > 0 \) and define

\[
\tau_H = \inf\{t \geq 0: (b - c)t - N_t \geq H\}.
\]

As on the proof of (i) in Theorem 1 we conclude that \( P(\tau_H < +\infty) = 1 \). We have

\[
v(x) \geq E(\max[x, \sup_{0 \leq t \leq \tau_H} (bt - N_t)] - c\tau_H) \geq E((b - c)\tau_H - N_{\tau_H}) \geq H.
\]

As \( H \) is arbitrary, the proof of (i) is concluded.

(ii) Let \( c = 0 \) and \( b < \lambda \). For \( t > 0 \) denote

\[
v_t(y) = E(\max[x, \sup_{0 \leq r \leq t} (br - N_r)]).
\]

By monotonous convergence

\[
v_t(y) \uparrow v(y) \quad \text{as} \quad t \to +\infty.
\]

Taking into account the definition of \( Y \) in (19) and the fact that \( E(N_t) = \lambda t \), we obtain that

\[
v_t(x) = E[Y_t - (\lambda - b)t], \quad Y_0 = x.
\]

Consider the function \( \tilde{v} = \tilde{v}(x) \) defined by

\[
\tilde{v}(x) = x + \frac{1}{\alpha^*} e^{-\alpha^* x},
\]

with \( \alpha^* \) the positive root of the equation (17) that, as \( b < \lambda \) has an unique positive solution. A direct computation shows that the function \( \tilde{v} = \tilde{v}(x) \) satisfies the equation

\[
\lambda \mathcal{L}\tilde{v}(x) - (\lambda - b)t = 0.
\]

In view of (46) the process \( K = (K_t)_{t \geq 0} \) defined by

\[
K_t = \tilde{v}(Y_t) - (\lambda - b)t
\]

is a local martingale.
From the definition of $\tilde{v} = \tilde{v}(x)$ follows that
\begin{equation}
    x \leq \tilde{v}(x) \leq x + \frac{1}{\alpha^*}, \quad x \geq 0,
\end{equation}
and
\[\lim_{x \to +\infty} |\tilde{v}(x) - x| = 0.\]

Notice now, that
\begin{equation}
    \max(0, N_t - bt) \leq Y_t \leq x + N_t
\end{equation}
so the local martingale $K = (K_t)_{t \geq 0}$ is uniformly integrable and in consequence a martingale. Finally, from (51) and $\lambda > b$ follows that
\begin{equation}
    Y_t \to +\infty, \quad t \to +\infty \quad P\text{-a.s.}
\end{equation}
From this fact, and (50), we obtain
\[\lim_{t \to +\infty} |\tilde{v}(Y_t) - Y_t| = 0.\]
So, we have
\[v(x) = \lim_{t \to +\infty} v_t(x) = \lim_{t \to +\infty} E(Y_t - (b - \lambda)t) = \lim_{t \to +\infty} E(\tilde{v}(Y_t) - (b - \lambda)t) = \tilde{v}(x),\]
and the proof of (ii) is concluded.

(iii) Let $c \geq b$. Then, for any $\tau \in \mathcal{M}$
\[E(\max[x, \sup_{0 \leq t \leq \tau} (bt - N_t)] - c\tau) \leq y + (b - c)E(\tau) \leq y,\]
and, if we take $\tau_\ast^s = 0$
\[E(\max[x, \sup_{0 \leq t \leq \tau_\ast^s} (bt - N_t)] - c\tau_\ast^s)] = x,\]
proving (iii).

(iv) Assume $c < b < c + \lambda$. Let us see, that in (2) it is enough to take the supremum over $\mathcal{M}_0$. Take $\delta$ positive, such that $b - \lambda < \delta < c$. Consider an arbitrary stopping time $\tau$ with $E(\tau) = +\infty$. As by (16)
\[E(\sup_{0 \leq t < +\infty} ((b - \delta)t - N_t)) < \infty,\]
we obtain
\[
E(\max\{x, \sup_{0 \leq t \leq \tau} (bt - N_t)\} - c\tau) \leq x + \\
+ E(\sup_{0 \leq t < +\infty} ((b - \delta)t - N_t) - (c - \delta)E(\tau) = -\infty,
\]

See now, that in view of Lemma 2 equation (20) has a unique positive solution \(x^*_v\). Define now the function \(\tilde{v} = \tilde{v}(x)\) by
\[
\tilde{v}(x) = \begin{cases} 
  x, & x \geq x^*_v \\
  x + \frac{\xi}{\lambda}u(x^*_v - x; b, \lambda), & 0 \leq x < x^*_v.
\end{cases}
\]

In order to prove (iv) we verify \(\tilde{v} = v\), with \(v\) in (2). For this, it is enough to verify the following two assertions:

(A) If \(\tau^*_v = \inf\{t \geq 0 : Y_t \geq x^*_v\}\), then
\[
\tilde{v}(y) = E(\max(y, \sup_{0 \leq t \leq \tau^*_v} (bt - N_t)) - c\tau^*_v).
\]

(B) For all \(\tau \in M_0\)
\[
\tilde{v}(x) \geq E(\max\{x, \sup_{0 \leq t \leq \tau} (bt - N_t)\} - c\tau),
\]

We begin by (B). If \(\tau \in M_0\) then \(E(N_\tau) = \lambda E(\tau)\) and in consequence
\[
E(\max\{x, \sup_{0 \leq t \leq \tau} (bt - N_t)\} - c\tau) = E(Y_\tau - (c + \lambda - b)\tau)
\]
\[
\leq E(\tilde{v}(Y_\tau) - (c + \lambda - b)\tau),
\]

because \(\tilde{v}(x) \geq x\), as the function \(u\) is convex (Lemma 2).

Then, in order to conclude the proof of (B) it is enough to see that the process \(V = (V_t)_{t \geq 0}\) given by
\[
V_t = \tilde{v}(Y_{t \wedge \tau}) - (c + \lambda - b)(t \wedge \tau)
\]
is a supermartingale. As \(V_t \geq -(c + \lambda - b)t\), by (46) it is enough to verify
\[
(53) \quad \lambda L\tilde{v}(x) - (c + \lambda - b) \leq 0 \quad \text{for} \quad x \geq 0.
\]
with $\mathcal{L}$ defined in (48). Extend the definition of the function $u$ in a continuous way, as $u(x) = 0$ if $x < 0$. Now, using Lemma 1 we obtain, for $0 \leq x < x^*_v$

$$\lambda \mathcal{L} \tilde{v}(x) = -b I_{\{y > 0\}}[1 - \frac{c}{\lambda} u'(x^*_v - x)] + \lambda + c[u(x^*_v - x - 1) - u(x^*_v - x)]$$

$$= -b + \lambda + cKu(y)|_{y=x^*_v-x} = c + \lambda - b \leq 0.$$  

When $x \geq x^*_v$, then $\tilde{v}(x) = x$ and

$$\lambda \mathcal{L} \tilde{v}(x) - (c + \lambda - b) = -c < 0,$$

and in this way the inequality (51) holds and (B) is proved. 

Let us see (A). Define $V^*_t = (V^*_t)_{t \geq 0}$ by

$$V^*_t = \tilde{v}(Y_t \wedge \tau^*_v) - (c + \lambda - b)(t \wedge \tau^*_v).$$

In view of (46) and (52) $V^*$ is a local martingale. Furthermore, this process is bounded on each interval of the form $[0, T], T < +\infty$, and in consequence is a martingale. So

$$(54) \quad \tilde{v}(x) = E(V^*_t) = E(\tilde{v}(Y_{t \wedge \tau^*_v})) - (c + \lambda - b)E(t \wedge \tau^*_v).$$

As $0 \leq \tilde{v}(Y_{t \wedge \tau^*_v}) \leq \max(x, x^*_v)$, we can take limits as $t \to +\infty$ in (54) obtaining

$$\tilde{v}(x) = E(\tilde{v}(Y^*_\tau - (c + \lambda - b)\tau^*_v)) = E(Y^*_\tau - (c + \lambda - b)\tau^*_v)$$

$$= E(\max[x, \sup_{0 \leq t \leq \tau^*_v} (bt - N_t) - c\tau^*_v]),$$

concluding (A). From (54), we also deduce (21)

$$E(\tau^*_v) = \frac{1}{c + \lambda - b}(\tilde{v}(x^*_v) - \tilde{v}(x)) = \frac{1}{c + \lambda - b}(x^*_v - x - \frac{c}{\lambda}u(x^*_v - x)).$$

concluding the proof of Theorem 2.

\[\square\]

### 3.3. Proof of Theorem 3.

(i) For $T > 0$ equation (24) has an unique positive root $x^*_s(T)$ by Lemma 2. The constant $c(T)$ defined by

$$\frac{a + c - \lambda}{\lambda} u'(x^*_s(T); \frac{a}{\lambda}) = 1,$$

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satisfies
\[ \lambda - a < c(T) < \lambda. \]
So, by (iv) in Theorem 1 the stopping time
\[ \tau_s^*(T) = \inf\{t \geq 0: X_t \geq x_s^*(T)\}, \]
with \( X = (X_t)_{t \geq 0} \) defined by (11) is optimal for the problem (1). By the election of \( c = c(T) \), and (15)
\[ E(\tau_s^*(T)) = T. \]
Also, in view of (1), (11) and (24)
\[ E(\sup_{0 \leq t \leq \tau_s^*(T)} (N_t - at)) = s(0) + c(T)T = x_s^*(T) - (\lambda - a)T, \]
and for any \( \tau \in \mathcal{M}_T \)
\[ E(\sup_{0 \leq t \leq \tau} (N_t - at)) \leq s(0) + c(T)E(\tau) \leq x_s^*(T) - (\lambda - a)T. \]
proving (25). Finally, (26) and (27) are direct consequences of c) in Lemma 2, concluding (i).

(ii) For a positive \( T \), define \( x_v^*(T) \) as the root of the equation
\[ (55) \quad \frac{xu'(x) - u(x)}{1 + (1 - \frac{b}{\lambda})u'(x)} = \lambda T, \]
where \( u = u(x) = u(x; \frac{b}{\lambda}) \). In order to see that the equation (55) has an unique root, we note, that for \( b > \lambda \) the function
\[ f(x) = \frac{xu'(x) - u(x)}{1 + (1 - \frac{b}{\lambda})u'(x)} \]
is increasing (because is the quotient of an increasing function over a decreasing one), \( f(0) = 0 \) and \( f(x) \to +\infty \) if \( u'(x) \to (\frac{b}{\lambda} - 1)^{-1} \).

If \( b \leq \lambda \), L’Hôpital rule gives \( \lim_{x \to +\infty} f(x) = +\infty \), and, as \( f(0) = 0 \), we confirm the existence of only one root, because
\[ f'(x) = \frac{u''(x)[x + u(x)(1 - \frac{b}{\lambda})]}{[1 + (1 - \frac{b}{\lambda})u'(x)]^2} \geq 0. \]
Let us see that the constant \( c(T) \) defined by the relation

\[
\frac{c(T)}{\lambda} u'(x^*_v(T); \frac{b}{\lambda}) = 1
\]  

satisfies \( b - \lambda < c(T) < b \). As, by Lemma 2 \( u'(x^*_v(T)) > u'(0) = \frac{\lambda}{b} \),

\[
c = \frac{\lambda}{u'(x^*_v(T))} < b.
\]

On the other side, if \( b \leq \lambda \) the second inequality is immediate. If \( \lambda < b \), then by b) in Lemma 2,

\[
u'(x^*_v(T)) < \lim_{x \to +\infty} u'(x) = \frac{\lambda}{b - \lambda}
\]

and the second inequality follows.

Then, we are in case (iv) of Theorem 2. The stopping time

\[
\tau^*_v(T) = \inf\{ t \geq 0; Y_t \geq x^*_v(T) \}
\]

is optimal for the problem (2) with \( c = c(T), x = 0 \). Then,

\[
E(\tau^*_v(T)) = \frac{1}{c + \lambda - b} (x^*_v(T) - \frac{c}{\lambda} u(x^*_v)) = T.
\]

Furthermore, (2), (18) and (56) gives

\[
E\left( \sup_{0 \leq t \leq \tau^*_v(T)} (bt - N_t) \right) = v(0) + c(T) E(\tau^*_v(T)) = \frac{c(T)}{\lambda} u(x^*_v(T)) + c(T) T
\]

\[
= \frac{u(x^*_v(T)) + \lambda T}{u'(x^*_v(T))} = x^*_v(T) - (\lambda - b) T,
\]

concluding the proof of (31).

To see (30), take \( b = \lambda \), and denote \( w(x) = xu'(x) - u(x) \). We have \( x^*_v(T) = w^{-1}(T) = \psi(T) \). We know,

\[
\lim_{x \to +\infty} \frac{w(x)}{x^2} = 1,
\]

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this means
\[
\frac{T}{(w^{-1}(T))^2} \rightarrow 1,
\]
and taking square roots, we obtain (30) concluding the proof of Theorem 3.

3.4. Proof of Theorem 4.
(i) Assume \( \lambda > a \). For \( H > 0 \), denote
\[
\tau_H = \{ t \geq 0 : x + N_t - at \geq H \}.
\]
As in the proof of (i) in Theorem 1, we obtain \( P(\tau_H < +\infty = 1) \). Then
\[
c(x) = \sup_{\tau \geq 0} E(x + N_{\tau} - a\tau)^+ \geq E(x + N_{\tau_H} - a\tau_H)^+ \geq H.
\]
As \( H \) is arbitrary, the proof of (i) is concluded.

(ii) Take \( a > \lambda \). Denote \( U_t = x + N_t - at \). The process \( U = (U_t)_{t \geq 0} \) has an infinitesimal operator of the form
\[
Uf(x) = \lambda(f(x + 1) - f(x)) - af'(x).
\]
Itô’s formula in this case reads, with \( c \) in (31)
\[
(57) \quad c(U_t) = c(x) + \int_0^t Uc(U_{r-})dr + M_t, \quad t \geq 0,
\]
with the process \( M = (M_t)_{t \geq 0} \) given by
\[
M_t = \int_0^t [c(U_{r-} + 1) - c(U_{r-})]d(N_r - \lambda r)
\]
a local martingale. Let us now verify that for the function \( c = c(x) \) defined in (31)
\[
(58) \quad Uc(x) = 0 \quad \text{if} \quad x < x^*_c.
\]
\[
(59) \quad Uc(x) < 0 \quad \text{if} \quad x \geq x^*_c.
\]
In fact, (59) is immediate. In order to see (58), taking into account (36)
\[
Uc(x) = \lambda(c(x + 1) - c(x)) - ac'(x)
\]
\[
\lambda - a + (a - \lambda)(u(x_c^* - 1) - u(x_c^* - x) + \frac{a}{\lambda}u'(x_c^* - x)) = 0.
\]

Now, from (57) we obtain that the stopped local martingale \( M^* = \{M_{t \wedge \tau_c}\}_{t \geq 0} \) with \( \tau_c^* \) in (32) is uniformly bounded:

\[-c(x) \leq M_{t \wedge \tau_c^*} \leq c(x_c^*) + 1,\]

so, taking expected values and limits, we obtain \( E(M_{\tau_c^*}) = 0 \) that means, by (31). As on the set \( \{\tau_c^* = +\infty\} \) we have

\[c(x + N_{\tau_c^*} - a\tau_c^*) = (x + N_{\tau_c^*} - a\tau_c^*)^+ = 0.\]

we deduce

(A) \( c(x) = E(c(x + N_{\tau_c^*} - a\tau_c^*)) = E(x + N_{\tau_c^*} - a\tau_c^*)^+. \)

To complete the proof, we will see

(B) For any \( \tau \in \mathcal{M} \)

\[c(x) \geq E(x + N_{\tau} - a\tau)^+. \]

For the process \( M \) defined by (57), as \(-\mathcal{U}c(x) \geq 0\) we have

\[M_t \geq -c(x). \]

This fact, and Fatou’s Lemma gives, that the process \( M \) is a supermartingale. Now, using this fact, and \( c(x) \geq x^+ \) we obtain

\[c(x) \geq E(c(x + N_{\tau} - a\tau)) \geq E(x + N_{\tau} - a\tau)^+, \]

concluding the proof of Theorem 4.

\[\Box\]

3.5. Proof of Theorem 5.

(i) Is analogous to the proof of (i) in Theorem 4.

(ii) Take \( b < \lambda \). Denote \( D_t = x + bt - N_t \). The process \( D = (D_t)_{t \geq 0} \) has in infinitesimal operator of the form

\[\mathcal{D}f(x) = bf'(x) - \lambda(f(x - 1) - f(x)).\]

It is direct to see that

\[\begin{cases} 
\mathcal{D}p(x) = 0, & \text{if } x < x_p^*, \\
\mathcal{D}p(x) = b - \lambda, & \text{if } x > x_p^*.
\end{cases}\]

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Itô’s formula in this case is

\begin{equation}
\label{61}
p(D_t) = p(x) + \int_0^t D(D_r-)dr + M_t, \quad t \geq 0,
\end{equation}

with the process \( M = (M_t)_{t \geq 0} \) given by

\[ M_t = \int_0^t [p(D_r - 1) - p(D_r -))]d(N_r - \lambda r) \]

a local martingale. Now, from (60) we obtain that the stopped local martingale \( M^* = \{M_{t \land \tau^*_p}\}_{t \geq 0} \) with \( \tau^*_p \) defined in (34) is uniformly bounded:

\[ -p(x) \leq M_{t \land \tau^*_p} \leq p(x^*_p) = x^*_p, \]

so, taking expected values and limits, we obtain \( E(M_{\tau^*_p}) = 0 \). As on the set \( \{\tau^*_p = +\infty\} \) we have \( p(D_{\tau^*_p}) = (D_{\tau^*_p})^+ = 0 \), (61) gives

(A) \( p(x) = E(p(x + b\tau^*_p - N_{\tau^*_p})) = E(x + b\tau^*_p - N_{\tau^*_p})^+ \).

To complete the proof, we will see

(B) For any \( \tau \in \hat{\mathcal{M}} \)

\[ p(x) \geq E(x + b\tau - N_{\tau})^+. \]

For the process \( M \) defined by (61), as \( -Dp(x) \geq 0 \) we have

\[ M_t \geq -p(x). \]

This fact, and Fatou’s Lemma gives, that the process \( M \) is a supermartingale. Now, using this fact, and \( p(x) \geq x^+ \) we obtain

\[ p(x) \geq E(p(x + b\tau - N_{\tau})) \geq E(x + b\tau - N_{\tau})^+, \]

concluding the proof of the Theorem \( \square \)

References


