Bounds on option prices for semimartingale market models*

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Abstract

We propose a methodology for the determination of the range of option prices of a European option in a general semimartingale market model, with a convex payoff function. Prices are obtained as expectations along the set of equivalent martingale measures.

Since the set of prices is an interval on the real line, two main questions are considered: (i) how to find upper and lower estimates for the range of prices, and (ii) how to establish the attainability of these estimates. To solve the first question, we introduce a partial ordering in the set of distributions of the discounted stock prices (adapted from the theory of statistical experiments), which allows us to find extremal distributions and, correspondingly, upper and lower bounds for the range of option prices. Weak convergence of probability measures is used to answer the second question, whether the bounds obtained at the first step are exact.

Exploiting stochastic calculus, we give answers to both questions in (the most natural for this problem) terms of predictable characteristics of the stochastic logarithm of the discounted stock price process. Particular attention is given to two examples: discrete time and diffusion with jumps market models.

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1 Introduction

1.1. Consider a mathematical model of a financial market with two assets. The first one is a non-risky asset $B = \{B_t\}_{0\leq t\leq T}$ with a deterministic interest rate $r = r(t)$, such that

$$B_t = B_0 e^{\int_0^t r(s)ds}, \qquad 0 \leq t \leq T.$$

The evolution of the second asset is modeled through a strictly positive semimartingale

$$S = \{S_t\}_{0\leq t\leq T} \text{ with } S_0 \text{ a positive constant,} \tag{1.1}$$

defined on a stochastic basis $\mathcal{B} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, P)$. For simplicity we assume $\mathcal{F} = \mathcal{F}_T$. In this model, we consider a European option with maturity T and pay-off $g(S_T)$, where

$$g\colon (0,\infty) \to \mathbb{R} \text{ is a } \textit{convex} \text{ function.} \tag{1.2}$$

Classical examples are European call and put options with payoff functions $g(x) = (x-K)^+$ and $g(x) = (K-x)^+$ respectively.
Denote by $\mathcal{P}$ the set of equivalent martingale measures, that is

$$\mathcal{P} = \{Q\colon Q \text{ probability measure, } Q \sim P, \left\{\frac{S_t}{B_t}\right\} \text{ is a } Q\text{-martingale}\}.$$

We assume the absence of arbitrage in the sense $\mathcal{P}$ is not empty. It is well known, that if there exists only one martingale measure Q^*, then the market is complete, and the price of the European option introduced above is given by $E_{Q^*}B_T^{-1}g(S_T)$, due to the possibility of perfect replication of the contingent claim $g(S_T)$. In the case of an *incomplete* market, the situation we want to consider, the set $\mathcal{P}$ contains more than one measure. Then, in our view, the problem of pricing options raises two main questions. First, how to choose "good" martingale measures that give satisfactory pricing strategies, and second, a question of more theoretical character, which prices are possible when the set of all martingale measures is considered. In this paper we are concerned with the second question, the determination of the set of possible values for the option, i.e. *admissible* prices. The first remark is that, as $\mathcal{P}$ is a convex set, and the expectation is a linear functional, the set of admissible prices is convex in the real line, in other words, an interval. Based on this remark denoting

$\gamma(Q) = E_Q B_T^{-1} g(S_T)$, we are interested in the computation of the quantities,

$$\mathbb{C}^* = \sup\{\gamma(Q) : Q \in \mathcal{M}\}$$

and

$$\mathbb{C}_* = \inf\{\gamma(Q) : Q \in \mathcal{M}\},$$

where $\mathcal{M}$ is a subset of $\mathcal{P}$, possibly $\mathcal{M} = \mathcal{P}$.

It is important to remark that $\mathbb{C}^*$ is related to the super-replication price of the contingent claim, that is the smallest amount necessary to hold a self-financing strategy whose value at time T is not smaller than $g(S_T)$. Indeed, by the optional decomposition theorem due to Kramkov (1996), see also El Karoui and Quenez (1995) or Föllmer and Kabanov (1998), the super-replication price is equal to $\sup \gamma(Q)$, where the supremum is taken over the set $\mathcal{P}_{\mathrm{loc}}$ of all equivalent *local* martingale measures Q. It is easy to see that, if $\mathcal{P} \neq \emptyset$ and g is nonnegative, then $\sup\{\gamma(Q) : Q \in \mathcal{P}\} = \sup\{\gamma(Q) : Q \in \mathcal{P}_{\mathrm{loc}}\}$, see e.g. Lemma 18 in Jakubenas (1998).

The range of option prices for the simplest one-step model is considered e.g. in Shiryaev (1999), Chapter V, § 1c. A discrete (in time and space) market model with independent returns is studied in Melnikov (1999). Shataev (1998) considers a discrete-time model with conditionally Gaussian returns. See also a recent paper by Rüschendorf (2001) on the discrete-time case. For continuous stock prices related works are Avellaneda et al. (1995), El Karoui et al. (1998), Frey and Sin (1999). The main three references for our work are Bellamy and Jeanblanc (2000) where a mixed jump-diffusion model is considered, and Eberlein and Jacod (1997) and Jakubenas (1998) in the case where S is the exponential of a Lévy process. In this paper we accumulate certain ideas from above works and propose a *general* methodology for the problem under consideration.

1.2. Our approach is based on the following facts adapted from the theory of binary statistical experiments in a form convenient for our purposes. See Section 2 for the details, where some other results used in the paper are given. Here we introduce the notation which will be used throughout the paper.

Let $\mathcal{S}$ be the set of all probability measures μ on $(\mathbb{R}_+, \mathcal{B}(R_+))$ ($\mathcal{B}(\mathcal{X})$ stands for the Borel σ-algebra on $\mathcal{X}$) satisfying the inequality $\int x\, \mu(dx) \leq 1$ (the domain of integration $\mathbb{R}_+ = [0, \infty)$ is omitted for short). When speaking of topological properties

of $\mathcal{S}$, the weak topology on $\mathcal{S}$ is always considered, and the weak convergence is denoted by $\Rightarrow$. By Fatou's lemma, $\mathcal{S}$ is a compact set.
Denote by $\mathcal{C}$ the class of all convex real-valued functions defined on $(0, +\infty)$. Given $f \in \mathcal{C}$, define

$$f(0) = \lim_{x \downarrow 0} f(x), \qquad \frac{f(\infty)}{\infty} = \lim_{x \to \infty} \frac{f(x)}{x}.$$

Due to convexity of f both limits exist and belong to $(-\infty, +\infty]$, and we can define for any $\mu \in \mathcal{S}$ and $f \in \mathcal{C}$ the functional

$$\mathcal{J}_f(\mu) = \int f(x)\,\mu(dx) + \frac{f(\infty)}{\infty}\Big(1 - \int x\,\mu(dx)\Big) \tag{1.3}$$

(with the convention $0 \cdot \infty = 0$). The functional $\mathcal{J}_f(\mu)$ is well defined and takes values in $(-\infty, +\infty]$.

Proposition 1.1 *The mapping $\mathcal{S} \ni \mu \rightsquigarrow \mathcal{J}_f(\mu)$ is lower semi-continuous. Moreover, it is continuous if*

$$f(0) + \frac{f(\infty)}{\infty} < \infty. \tag{1.4}$$

Define a binary relation $\preccurlyeq$ on the set $\mathcal{S}$ by $\mu' \preccurlyeq \mu$ if $\mathcal{J}_f(\mu') \le \mathcal{J}_f(\mu)$ for any $f \in \mathcal{C}$. It can be shown that $\mu' \preccurlyeq \mu$ if and only if

$$\int (a-x)^+ \mu'(dx) \le \int (a-x)^+ \mu(dx) \quad \text{for any } a > 0. \tag{1.5}$$

Proposition 1.2 (a) *The relation $\preccurlyeq$ is a partial ordering on $\mathcal{S}$.*
(b) *$\mu' \preccurlyeq \mu$ if and only if there exist random variables ξ and ξ' on the same probability space with the distributions μ and μ' respectively such that $E(\xi \mid \xi') \le \xi'$ a.s. In particular, $\mu' \preccurlyeq \mu$ implies $\mu'(\{0\}) \le \mu(\{0\})$ and $\int x\,\mu'(dx) \ge \int x\,\mu(dx)$.*
(c) *The partially ordered set $(\mathcal{S}, \preccurlyeq)$ is order complete, i.e. every nonempty subset $\mathcal{T}$ of $\mathcal{S}$ has the least upper bound* $\sup \mathcal{T}$ *and the greatest lower bound* $\inf \mathcal{T}$.
(d) $\sup \mathcal{S} = \delta_{\{0\}}$ *and* $\inf \mathcal{S} = \delta_{\{1\}}$, *where $\delta_{\{x\}}$ is the Dirac measure at x.*

1.3. Let us return to the option pricing problem. We want to show that the preceding results suggest a certain way to find the upper and lower prices.

First, let us remark that considering a discounted stock price one can simplify notationally the problem. Namely, given S as in (1.1) and g as in (1.2), introduce a process $Z = \{Z_t\}_{0\le t\le T}$ and a function $f\colon (0,\infty) \to \mathbb{R}$ by

$$Z_t := \frac{B_0 S_t}{S_0 B_t} \qquad \text{and} \qquad f(x) := B_T^{-1} g(B_0^{-1} S_0 B_T x). \tag{1.6}$$

Then $f \in \mathcal{C}$, $Z_0 = 1$, Z is a strictly positive Q-martingale if (and only if) $Q \in \mathcal{P}$, and

$$\gamma(Q) = E_Q f(Z_T) = \mathcal{J}_f(\mu_Q), \quad Q \in \mathcal{P},$$

where $\mu_Q := \mathcal{L}(Z_T \mid Q) \in \mathcal{S}$ is the distribution of Z_T under Q. Therefore, $\mathbb{C}_*$ and $\mathbb{C}^*$ are the infimum and the supremum, respectively, of the functional $\mathcal{J}_f(\cdot)$ over the set $\mathcal{T}_{\mathcal{M}} := \{\mu_Q\colon Q \in \mathcal{M}\} \subseteq \mathcal{S}$.
When the function g satisfies the finiteness condition (1.4), the same is true for the function f defined in (1.6). In this case, by the relative compactness of the set $\mathcal{T}_{\mathcal{M}}$, it is possible to find a sequence $\{Q^n\}$ in $\mathcal{M}$ and a measure $\mu^* = \mu^*(f) \in \mathcal{S}$ such that $\mu_{Q^n} \Rightarrow \mu^*$ and $\lim_n \mathcal{J}_f(\mu_{Q^n}) = \mathbb{C}^*$. By Proposition 1.1 we have $\mathbb{C}^* = \mathcal{J}_f(\mu^*)$. Similarly, there is a (possibly different) sequence $\{Q^n\}$ in $\mathcal{M}$ and a measure $\mu_* = \mu_*(f) \in \mathcal{S}$ such that $\mu_{Q^n} \Rightarrow \mu_*$ and $\lim_n \mathcal{J}_f(\mu_{Q^n}) = \mathbb{C}_*$. By Proposition 1.1 we have $\mathbb{C}_* = \mathcal{J}_f(\mu_*)$.
Of course, in principle one cannot choose the measures μ^* and μ_* independently of f. However, we restrict our attention to the case where this can be done. As a justification, we mention that many models of interest have this property. In particular, such are the models considered in the works cited above.
The next two propositions describe the situation where the measures μ^* and μ_* can be chosen independently of f.

Proposition 1.3 *Let $\mathcal{M} \neq \emptyset$ and $\mu \in \mathcal{S}$ be such that $\mu_Q \preccurlyeq \mu$ for any $Q \in \mathcal{M}$. There is equivalence between:*

(a) $\sup_{\mathcal{M}} \mathcal{J}_f(\mu_Q) = \mathcal{J}_f(\mu)$ *for all* $f \in \mathcal{C}$;

(b) *there is a strictly convex function $f \in \mathcal{C}$ satisfying* (1.4) *such that* $\sup_{\mathcal{M}} \mathcal{J}_f(\mu_Q) = \mathcal{J}_f(\mu)$;

(c) *there is a sequence $\{Q^n\}$ in $\mathcal{M}$ such that $\mu_{Q^n} \Rightarrow \mu$.*

Moreover, if these conditions are satisfied then $\mu = \sup\{\mu_Q : Q \in \mathcal{M}\}$.

Proposition 1.4 *Let* $\mathcal{M} \neq \emptyset$ *and* $\mu \in \mathcal{S}$ *be such that* $\mu \preccurlyeq \mu_Q$ *for any* $Q \in \mathcal{M}$. *There is equivalence between:*

(a) $\inf_{\mathcal{M}} \mathcal{J}_f(\mu_Q) = \mathcal{J}_f(\mu)$ *for all* $f \in \mathcal{C}$ *satisfying* (1.4);

(b) *there is a strictly convex function* $f \in \mathcal{C}$ *satisfying* (1.4) *such that* $\inf_{\mathcal{M}} \mathcal{J}_f(\mu_Q) = \mathcal{J}_f(\mu)$;

(c) *there is a sequence* $\{Q^n\}$ *in* $\mathcal{M}$ *such that* $\mu_{Q^n} \Rightarrow \mu$.

Moreover, if these conditions are satisfied then $\mu = \inf\{\mu_Q : Q \in \mathcal{M}\}$.

Proof of Proposition 1.3. Since (a)⇒(b) is evident, (c)⇒(a) follows from Proposition 1.1, and the last statement follows from Proposition 1.2(a), it remains to prove (b)⇒(c).

Due to compactness arguments as above, there exist a sequence $\{Q^n\}$ in $\mathcal{M}$ and a measure $\mu' \in \mathcal{S}$ such that $\mu_{Q^n} \Rightarrow \mu'$ and $\mathcal{J}_f(\mu') = \sup_{\mathcal{M}} \mathcal{J}_f(\mu_Q) = \mathcal{J}_f(\mu)$. By Proposition 1.1, we have $\mu' \preccurlyeq \mu$. Using Proposition 1.2(b) and conditional Jensen's inequality, we obtain from the strong convexity of f that $\mu' = \mu$.

Proof of Proposition 1.4 follows the same lines as the previous one.

If the conditions (a)–(c) of Proposition 1.3 are not satisfied, then, by Proposition 1.2(c), there is still the measure $\mu^0 = \sup\{\mu_Q : Q \in \mathcal{M}\}$, but we only have the inequality $\sup_{\mathcal{M}} \mathcal{J}_f(\mu_Q) \leq \mathcal{J}_f(\mu^0)$, and the inequality is strict for strictly convex f satisfying (1.4). In the case of lower bounds the situation is similar.

1.4. Propositions 1.3 and 1.4 suggest what to do in order to find the upper and lower prices:

1) find a measure μ^* (resp. μ_*) which is an upper (resp. lower) bound for the set $\{\mu_Q : Q \in \mathcal{M}\}$, i.e. $\mu_Q \preccurlyeq \mu^*$ (resp. $\mu_* \preccurlyeq \mu_Q$) for all $Q \in \mathcal{M}$;

2) find a sequence $\{Q^n\}$ in $\mathcal{M}$ such that $\mu_{Q^n} \Rightarrow \mu^*$ (resp. $\mu_{Q^n} \Rightarrow \mu_*$).

Then

$$\mathbb{C}^* = \mathcal{J}_f(\mu^*) \qquad \text{(resp. } \mathbb{C}_* = \mathcal{J}_f(\mu_*)\text{)}.$$

If only the first step can be realized then we obtain an estimate from above for the upper price or an estimate from below for the lower price.

Of course, these two steps, in one or another form, are presented at the papers where the upper and lower prices are found.

It is useful to remark that the measure $\mu_* = \inf\{\mu_Q : Q \in \mathcal{M}\}$ always satisfies the properties $\mu_*(\{0\}) = 0$ and $\int x\, \mu_*(dx) = 1$ by Proposition 1.2(b). Thus, it is not surprising that one can often find a probability measure Q_* on $(\Omega, \mathcal{F})$ such that $\mu_* = \mathcal{L}(Z_T \mid Q_*)$, and then we have $\mathbb{C}_* = E_{Q_*} f(Z_T)$. Moreover, it often happens that Z is a martingale under Q_* and there are no other measures equivalent to Q_* for which Z is a martingale. This means that we have a *complete arbitrage-free* market model under Q_* and $\mathbb{C}_*$ is the just the fair price of the option in that model. In the case of the upper price the situation may be different. It may happen that, for $\mu^* = \sup\{\mu_Q : Q \in \mathcal{M}\}$, we have $\mu^*(\{0\}) > 0$ (which does not change the situation drastically) or/and $\int x\, \mu^*(dx) < 1$. For example, a standard situation is $\mu^* = \delta_{\{0\}}$. Even if μ^* can be realized as $\mu^* = \mathcal{L}(Z_T \mid Q^*)$ but if $\int x\, \mu^*(dx) < 1$, we do not have the property $\mathbb{C}^* = E_{Q^*} f(Z_T)$ if $\frac{f(\infty)}{\infty} \neq 0$, it may happen that the market corresponding to Q^* admits arbitrage etc.

A good illustration for the aforesaid is the model $Z = \exp(\overline{X})$, where $\overline{X}$ is a Lévy process, studied in detail by Jakubenas (1998) (we reserve the notation X for the stochastic logarithm of Z, see the subsections 1.5 and 3.1). He considers two cases: $\mathcal{M} = \mathcal{P}$ and $\mathcal{M} = \{Q \in \mathcal{P} : \overline{X}$ is a Lévy process under $Q\}$. As it follows from his Theorem 2, if $\mathcal{P} \neq \emptyset$, in both cases the least upper bounds μ^* coincide, the greatest lower bounds μ_* coincide, and conditions (a)–(c) of Propositions 1.3 and 1.4 are satisfied. Furthermore, μ_* is the distribution of $Z_T = \exp(\overline{X}_T)$, where $\overline{X}$ is one of the following Lévy processes: 1) $\overline{X} \equiv 0$; 2) $\overline{X} = \sigma W$, where W is a standard Brownian motion, $\sigma > 0$; 3) $\overline{X}_t = a\pi_t + bt$, where π is a Poisson process of intensity λ, $a \neq 0$, and b is determined from the condition that $\exp(\overline{X})$ is a martingale. In all the cases the corresponding market model is arbitrage-free and complete.

The description of the upper bound is more complex. First, one possibility is $\mu^* = \delta_{\{0\}}$. Second and third, μ^* is represented through a Wiener process or a Poisson process, as happened to μ_* in the cases 2) and 3) above. Fourth, $\mu^* = \delta_{\{-aT\}}$, $a > 0$, and if one takes $Z = \exp(\overline{X})$, $\overline{X}_t = -at$, then the market model admits arbitrage. Fifth, μ^* can be represented as the distribution of Z_T, where Z is the stochastic exponential $\mathcal{E}(X)$, X is a Lévy process of the form $X_t = -\pi_t + bt$, where π is a Poisson process of intensity λ and $b > 0$ is determined from the condition that X is a martingale; in other words, $Z_t = \exp(bt)$ up to the first jump of π and $Z_t = 0$ after it. The corresponding market model is arbitrage-free and complete, but the price process may take zero value. Note that in all cases, excepting the first one for the upper bound, the process Z under the extremal measures can be also represented as $Z = \mathcal{E}(X)$, where X is a Lévy process (and a martingale excepting the fourth case for the upper measure).

Jakubenas (1998) also founds the quantities $\sup\{\gamma(Q) : Q \in \mathcal{P}_{\text{loc}}\}$ and $\inf\{\gamma(Q) : Q \in \mathcal{P}_{\text{loc}}\}$. He considers only nonnegative f with $f(0) = 0$ and $\frac{f(\infty)}{\infty} = 1$, hence, as it was already mentioned, the upper bound is the same as in the case of equivalent martingale measures, while the lower bound is usually strictly less for $\mathcal{P}_{\text{loc}}$. The explanation is easy: we still have $\mu_* \preccurlyeq \mu_Q$ for any $Q \in \mathcal{P}_{\text{loc}}$ (which is natural in view of our results in Section 5) so that $\inf\{\mathcal{J}_f(\mu_Q) : Q \in \mathcal{P}_{\text{loc}}\} = \mathcal{J}_f(\mu_*)$, but, if $Q \in \mathcal{P}_{\text{loc}} \setminus \mathcal{P}$ and $\frac{f(\infty)}{\infty} > 0$ then $\gamma(Q) < \mathcal{J}_f(\mu_Q)$, see (1.3).

1.5. In the previous subsection we formulated two steps that say *what* to do in order to find the upper and lower prices. The main subject of our paper deals with results concerning *how* to perform these two steps for different market models.

We start explaining the terms in which our main results are formulated. It turns out to be most convenient to take the *stochastic logarithm* X of Z and to express the conditions in terms of the triplet T_Q of *local characteristics* of X under martingale measures Q. There are different reasons for this. A suitable description of the class $\{\mu_Q : Q \in \mathcal{M}\}$ is not available even for simple models (of course, if the market is not complete). What we know is Girsanov's transformation rule connecting T_Q for an arbitrary $Q \sim P$ and the triplet T_P of X corresponding to the original measure P; this also allows us to construct measures Q with desired properties of T_Q. The property of X to be a Q-local martingale (which is equivalent to $Q \in \mathcal{P}_{\text{loc}}$) is easily expressed in terms of T_Q. Next, it is convenient to consider the process Z under $Q \in \mathcal{P}$ as the density process of the probability measure $\overline{Q}$ defined by $d\overline{Q} = Z_T\, dQ$ with respect to Q. Then we can regard the weak convergence of measures in the second step as the weak convergence of the likelihoods and to use corresponding limit theorems, whose assumptions are expressed e.g. in terms of Hellinger processes, which in its turn have a simple representation through T_Q. Finally, our comparison result in Section 5 has a simple form in the same terms.

As a tool for the first step of our approach we prove a comparison result in Section 5, which we call Comparison Lemma. It gives sufficient conditions under which the distributions $\mathcal{L}(Z_T \mid Q)$ and $\mathcal{L}(Z_T^* \mid Q^*)$, where Z is a nonnegative Q-supermartingale and Z^* is a nonnegative Q^*-supermartingale (defined maybe on different stochastic bases) are comparable in the sense of the partial ordering $\preccurlyeq$. In the option pricing context $\mathcal{L}(Z_T^* \mid Q^*)$ plays the role of μ^* or μ_*, while Z and Q are interpreted as before. The conditions on Z^* are restrictive, in particular, it is assumed that Z^* has a Markovian structure. The result generalizes the arguments used in Bellamy and Jeanblanc (2000) and in Jakubenas (1998), see also El Karoui et al. (1998). As far as we know, this method has never been used for comparison of experiments in the statistical literature. In subsection 5.2 we check that our assumptions on Z^* are satisfied if $Z^* = \mathcal{E}(X^*)$, where X^* is a process with independent increments, or Z^* is a diffusion. In subsection 5.3 we indicate some conditions in terms of T_P guaranteeing that the assumptions of the Comparison Lemma are satisfied for any $Q \in \mathcal{P}$.

Tools for the second step are considered in Sections 4 and 6. In Section 4 we deal with the special case $\mu^* = \delta_{\{0\}}$ or $\mu_* = \delta_{\{1\}}$. Then conditions (c) of Propositions 1.3 and 1.4 are equivalent to the convergence of the variation distance $\|Q^n - \overline{Q^n}\|$ to 2 and 0 respectively, where $d\overline{Q^n} = Z_T dQ^n$. Thus we can use predictable criteria for the convergence in variation and for the asymptotic separability of measures, see e.g. Chapter V in Jacod and Shiryaev (1987). At the final stage of the preparation of this paper we got acquainted with an unpublished manuscript by Jacod (1997), where essentially the same results included in our Propositions 4.1 and 4.2 were proved. Similar propositions when Z is a continuous process can be found in Frey and Sin (1999).

If $\mu_* = \delta_{\{0\}}$ or $\mu_* = \delta_{\{1\}}$, under conditions (a)–(c) of Propositions 1.3 and 1.4 we have

$$\mathbb{C}^* = \mathcal{J}_f(\delta_{\{0\}}) = f(0) + \frac{f(\infty)}{\infty} = B_T^{-1} g(0) + B_0^{-1} S_0 \frac{g(\infty)}{\infty}$$

and

$$\mathbb{C}_* = \mathcal{J}_f(\delta_{\{1\}}) = f(1) = B_T^{-1} g(B_0^{-1} S_0 B_T)$$

respectively. Note that, in the general case, by Proposition 1.2(d) we always have these bounds as an upper estimate for $\mathbb{C}^*$ and lower estimate for $\mathbb{C}_*$, i.e.

$$B_T^{-1} g(B_0^{-1} S_0 B_T) \leq \mathbb{C}_* \leq \mathbb{C}^* \leq B_T^{-1} g(0) + B_0^{-1} S_0 \frac{g(\infty)}{\infty}. \tag{1.7}$$

Of course, these inequalities can be obtained directly: the first one follows from Jensen's inequality and the second one follows from the inequality $g(x) \leq g(0) + x\frac{g(\infty)}{\infty}$, $x \in \mathbb{R}_+$. We call these bounds *universal* bounds. The fact that there exist such universal bounds for any market model of the considered type was stated by Eberlein and Jacod (1997).

In Section 6 we are interested how to perform the second step if the measure μ does not coincide with $\delta_{\{0\}}$ or $\delta_{\{1\}}$. Unlike limit theorems for likelihood processes, our task is more simple since the process Z is a strictly positive martingale with respect to any $Q^n \in \mathcal{M}$, which means that it is the density process for equivalent measures. However, the existing results on this subject cannot be completely adapted to our setting, see the discussion in Section 6 and the references therein. We prove two new limit theorems. The first one deals with the case where the limit is the distribution of Z_T^*, $Z^* = \mathcal{E}(X^*)$ is a nonnegative supermartingale, and X^* is a Lévy process. The second theorem corresponds to the case where Z^* is a diffusion.

The rest of the paper is as follows. In Section 2 a short review of comparison and convergence of binary statistical experiments is given. Section 3 contains information about martingale measures and Hellinger processes. In Section 7 we examine the bounds of prices for a general random walk market model, and Section 8 carries the same task when the underlying process is the solution of a stochastic differential equation with jumps. Some concluding remarks are presented in Section 9.

2 Binary statistical experiments

Here we give a brief review of the theory of comparison and convergence of binary statistical experiments. All the statements are known and may be found e.g. in Strasser (1985), Torgersen (1991), Shiryaev and Spokoiny (2000). We also refer to Liese and Vajda (1987) for the notion of f-divergence and its properties. The notation introduced in subsection 1.2 is used.

2.1. By a binary experiment we mean a collection $\mathbb{E} = (\Omega, \mathcal{F}, (P, P'))$, where $(\Omega, \mathcal{F})$ is a measurable space and (P, P') is an ordered pair of probability measures on $(\Omega, \mathcal{F})$. In the sequel we consider only binary statistical experiments, so the word 'binary' will be omitted. The experiment $\mathbb{E}$ is homogeneous if P and P' are equivalent. If $P = P'$ then $\mathbb{E}$ is said to be a totally non informative experiment. If the measures P and P' are singular then $\mathbb{E}$ is said to be a totally informative experiment.
Let Z be a random variable on $(\Omega, \mathcal{F})$ with values in $[0, +\infty]$ such that

$$P'(A) = \int_A Z\, dP + P'(A \cap \{Z = \infty\}) \qquad \text{for any} \quad A \in \mathcal{F}.$$

Given P and P' the variable Z always exists, is unique up to $(P+P')$-null sets, and is called the generalized density of P' with respect to P. We have $P(Z < \infty) = P'(Z > 0) = 1$. If a σ-finite measure λ on $(\Omega, \mathcal{F})$ dominates both P and P', $z = dP/d\lambda$ and $z' = dP'/d\lambda$ are the corresponding Radon–Nikodym densities, then $Z = z'/z$ P- and P'-a.s. (where $0/0$ is interpreted arbitrarily, say, $0/0 = 0$). Of course, if P' is absolutely continuous with respect to P then Z is just the Radon–Nikodym density of P' with respect to P. In general, a (P-a.s. finite) random variable Z is the density of the absolutely continuous part of P' with respect to P.
We associate the following objects with an experiment $\mathbb{E} = (\Omega, \mathcal{F}, (P, P'))$.
1. A probability measure $\mu_{\mathbb{E}}$ belonging to $\mathcal{S}$ defined by

$$\mu_{\mathbb{E}}(B) = P(Z \in B), \qquad B \in \mathcal{B}(\mathbb{R}_+).$$

Conversely, for any probability measure $\mu \in \mathcal{S}$ there is an experiment $\mathbb{E}$ such that $\mu = \mu_{\mathbb{E}}$. For example, take $\Omega = [0, +\infty]$, $\mathcal{F} = \mathcal{B}([0, \infty])$, $P = \mu$, $P'(dx) = x\mu(dx) + \Big(1 - \int_{\mathbb{R}_+} x\mu(dx)\Big)\, \delta_{\{+\infty\}}(dx)$.
2. A probability measure $\nu_{\mathbb{E}}$ on $([0, 2], \mathcal{B}([0, 2]))$ defined by

$$\nu_{\mathbb{E}}(B) = Q(z \in B), \qquad B \in \mathcal{B}([0, 2]),$$

where $Q = (P+P')/2$, $z = dP/dQ$. The measure $\nu_{\mathbb{E}}$ has the property $\int_{[0,2]} x\nu_{\mathbb{E}}(dx) = 1$. Moreover, for any probability measure ν on $([0, 2], \mathcal{B}([0, 2]))$ with $\int_{\mathbb{R}_+} x\nu(dx) = 1$ there is an experiment $\mathbb{E}$ such that $\nu = \nu_{\mathbb{E}}$. For example, take $\Omega = [0, 2]$, $\mathcal{F} = \mathcal{B}([0, 2])$, $P(dx) = x\nu(dx)$, $P'(dx) = (2 - x)\nu(dx)$.

3. A function $\beta_{\mathbb{E}}(\alpha)$, $\alpha \in [0,1]$, where $\beta_{\mathbb{E}}(\alpha)$ is the power of the most powerful level α test for testing 'P' against 'P'', i.e.

$$\beta_{\mathbb{E}}(\alpha) = \sup_{\varphi\in\Phi: E\varphi\le\alpha} E'\varphi,$$

where Φ is the set of all measurable functions φ on $(\Omega, \mathcal{F})$ with values in $[0,1]$ (test functions), E and E' are expectations with respect to P and P' respectively. The function $\beta_{\mathbb{E}}(\alpha)$ is a nondecreasing continuous concave function with values in $[0,1]$ and $\beta_{\mathbb{E}}(1) = 1$. Conversely, if $\beta(\alpha)$ ($\alpha \in [0,1]$) is a nondecreasing continuous concave function with values in $[0,1]$ satisfying $\beta(1) = 1$, then $\beta = \beta_{\mathbb{E}}$, where $(\Omega, \mathcal{F}) = ([0,1], \mathcal{B}([0,1]))$, P is the Lebesgue measure, $P'(d\alpha) = d\beta(\alpha) + \beta(0)\delta_{\{0\}}(d\alpha)$, and $\mathbb{E} = (\Omega, \mathcal{F}, (P, P'))$.

4. A function $b_{\mathbb{E}}(\pi)$, $\pi \in [0,1]$, where $b_{\mathbb{E}}(\pi)$ is the minimum Bayes risk for testing 'P' against 'P'' with a priori probabilities $1-\pi$ and π, i.e.

$$b_{\mathbb{E}}(\pi) = \inf_{\varphi\in\Phi}\{(1-\pi)E\varphi + \pi E'(1-\varphi)\}.$$

The function $b_{\mathbb{E}}(\pi)$ is a concave function and $0 \le b_{\mathbb{E}}(\pi) \le \min\{\pi, 1-\pi\}$, $\pi \in [0,1]$. Moreover, if $b(\pi)$ is a concave function satisfying $0 \le b(\pi) \le \min\{\pi, 1-\pi\}$, $\pi \in [0,1]$, then there is an experiment $\mathbb{E}$ such that $b = b_{\mathbb{E}}$. Note that functions from this class are uniformly equicontinuous. Note also that the distance in variation between P and P' defined as $\|P - P'\| = 2\sup_{A\in\mathcal{F}} |P(A) - P'(A)|$ satisfies

$$\|P - P'\| = 2(1 - 2b_{\mathbb{E}}(1/2)). \tag{2.1}$$

5. A function $H_{\mathbb{E}}(\alpha)$, $\alpha \in (0,1)$, where $H_{\mathbb{E}}(\alpha)$ is the Hellinger integral of order α for P and P':

$$H_{\mathbb{E}}(\alpha) = H(\alpha; P, P') = EZ^{1-\alpha}.$$

2.2. In the following definition λ is a σ-finite measure dominating P and P', z and z' are the densities of P and P' respectively with respect to λ. The conventions $0f(\frac{a}{0}) = a\frac{f(\infty)}{\infty}$ are used.

Definition 1 (Csiszár (1963)) *Let $f \in \mathcal{C}$. The quantity*

$$\mathcal{J}_f(P', P) = \int z f\Big(\frac{z'}{z}\Big) d\lambda$$

is called the f-divergence of P' and P.

It is easy to check that the definition is correct and does not depend on the choice of λ. We shall also write $\mathcal{J}_f(\mathbb{E})$ instead of $\mathcal{J}_f(P', P)$, cf. (2.2) below. If $\mathbb{E}$ is homogeneous then $\mathcal{J}_f(\mathbb{E}) = Ef(Z)$. The same is true if f is nonnegative and decreasing.

Remark 2.1 *If $f(x) = ax + b$, $x > 0$, $a, b \in \mathbb{R}$, then $\mathcal{J}_f(\mathbb{E}) = a + b$ does not depend on $\mathbb{E}$. Due to this fact, one is free to subtract a linear function from an $f \in \mathcal{C}$ to obtain additional desired properties of f, e.g. $f \geq 0$.*

It is easy to express $\mathcal{J}_f(\mathbb{E})$ in terms of $\mu_{\mathbb{E}}$ and $\nu_{\mathbb{E}}$, explaining in particular the notation in (1.3). Namely,

$$\mathcal{J}_f(\mathbb{E}) = \mathcal{J}_f(\mu_{\mathbb{E}}) = \int_{[0,2]} \tilde{f}(x)\nu_{\mathbb{E}}(dx), \tag{2.2}$$

where $\tilde{f}(x)$, $x \in [0, 2]$, is a continuous convex function defined by

$$\tilde{f}(x) = \begin{cases} 2\frac{f(\infty)}{\infty}, & \text{if} \quad x = 0, \\ xf(\frac{2-x}{x}), & \text{if} \quad 0 < x < 2, \\ 2f(0), & \text{if} \quad x = 2. \end{cases}$$

Some of the quantities introduced in **2.1** can be expressed via f-divergences. In particular, put $f_\pi(x) = \max\{1 - \pi - \pi x, 0\}$, $\pi \in [0, 1]$, $g_1(x) = |x - 1|$, $g_2(x) = 2(1 - x)^+$, $g_3(x) = 2(x - 1)^+$, and $\varphi_\alpha(x) = \alpha x - x^{1-\alpha} + 1 - \alpha$, $\alpha \in (0, 1)$. Then

$$b_{\mathbb{E}}(\pi) = 1 - \pi - \mathcal{J}_{f_\pi}(\mathbb{E}), \tag{2.3}$$

$$\|P - P'\| = \mathcal{J}_{g_i}(\mathbb{E}) \qquad (i = 1, 2, 3),$$

$$H_{\mathbb{E}}(\alpha) = 1 - \mathcal{J}_{\varphi_\alpha}(\mathbb{E}). \tag{2.4}$$

2.3. The next definition introduces a partial pre-ordering for experiments, that motivated our partial ordering in Proposition 1.2.

Definition 2 *An experiment $\mathbb{E}$ is said to be more informative than an experiment $\widetilde{\mathbb{E}}$ (denoted $\mathbb{E} \succcurlyeq \widetilde{\mathbb{E}}$ or $\widetilde{\mathbb{E}} \preccurlyeq \mathbb{E}$) if $\beta_{\mathbb{E}}(\alpha) \geq \beta_{\widetilde{\mathbb{E}}}(\alpha)$ for all $\alpha \in [0, 1]$. The experiments $\mathbb{E}$ and $\widetilde{\mathbb{E}}$ are said to be equivalent (denoted $\mathbb{E} \sim \widetilde{\mathbb{E}}$) if $\mathbb{E} \preccurlyeq \widetilde{\mathbb{E}}$ and $\widetilde{\mathbb{E}} \preccurlyeq \mathbb{E}$, i.e. if $\beta_{\mathbb{E}}(\alpha) = \beta_{\widetilde{\mathbb{E}}}(\alpha)$ for all $\alpha \in [0, 1]$. Equivalent experiments are said to have the same type, so a type is a collection of equivalent experiments.*

Simple considerations based on the Neyman–Pearson lemma lead to the following result.

Proposition 2.1 $\mathbb{E} \succcurlyeq \widetilde{\mathbb{E}}$ *if and only if* $b_{\mathbb{E}}(\pi) \le b_{\widetilde{\mathbb{E}}}(\pi)$ *for all* $\pi \in [0,1]$*. In particular,* $\mathbb{E} \sim \widetilde{\mathbb{E}}$ *if and only if* $b_{\mathbb{E}}(\pi) = b_{\widetilde{\mathbb{E}}}(\pi)$ *for all* $\pi \in [0,1]$*.*

The next statement gives a comparison criterion in terms of measures $\mu_{\mathbb{E}}$. In fact, it is an almost immediate consequence of the previous proposition.

Proposition 2.2 $\mathbb{E} \succcurlyeq \widetilde{\mathbb{E}}$ *if and only if*

$$\int_0^a \mu_{\mathbb{E}}([0,x])\,dx \ge \int_0^a \mu_{\widetilde{\mathbb{E}}}([0,x])\,dx \qquad \text{for all} \quad a \ge 0.$$

In particular, $\mathbb{E} \sim \widetilde{\mathbb{E}}$ *if and only if* $\mu_{\mathbb{E}} = \mu_{\widetilde{\mathbb{E}}}$*.*

It follows from Proposition 2.1 and (2.3) that, cf. (1.5),

$$\mathbb{E} \succcurlyeq \widetilde{\mathbb{E}} \qquad \text{if and only if} \qquad \mathcal{J}_{f_\pi}(\mathbb{E}) \ge \mathcal{J}_{f_\pi}(\widetilde{\mathbb{E}}), \quad \pi \in [0,1]. \tag{2.5}$$

Using (2.5) and Remark 2.1, one easily shows that $\mathbb{E} \succcurlyeq \widetilde{\mathbb{E}}$ implies $\mathcal{J}_f(\mathbb{E}) \ge \mathcal{J}_f(\widetilde{\mathbb{E}})$ for any f which is the maximum of a finite set of linear functions on $(0,+\infty)$. This can be extended to all $f \in \mathcal{C}$ by using of monotone approximations.

Proposition 2.3 *If* $\mathbb{E} \succcurlyeq \widetilde{\mathbb{E}}$ *then* $\mathcal{J}_f(\mathbb{E}) \ge \mathcal{J}_f(\widetilde{\mathbb{E}})$ *for any* $f \in \mathcal{C}$*.*

Another proof of this result is based on Proposition 2.4 below.
Let $\mathcal{C}_0^2$ be the subset of $\mathcal{C}$ consisting of all bounded nonnegative decreasing twice continuously differentiable convex functions f with a compact support. Since any function f_π, $\pi \in [0,1]$, can be uniformly approximated by functions from $\mathcal{C}_0^2$, we obtain the following result useful in applications.

Corollary 2.1 $\mathbb{E} \succcurlyeq \widetilde{\mathbb{E}}$ *if and only if* $\int f(x)\,\mu_{\mathbb{E}}(dx) \ge \int f(x)\,\mu_{\widetilde{\mathbb{E}}}(dx)$ *for any* $f \in \mathcal{C}_0^2$*.*

A standard tool for comparison of experiments is the randomization criterion. Proposition 1.2(b) is its corollary.

Proposition 2.4 *Let* $\mathbb{E} = (\Omega, \mathcal{F}, (P, P'))$ *and* $\widetilde{\mathbb{E}} = (\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{P}, \widetilde{P}'))$ *be experiments. If* $(\widetilde{\Omega}, \widetilde{\mathcal{F}})$ *is isomorphic to a Borel subset of a complete separable metric space equipped with the Borel* σ*-algebra, then* $\mathbb{E} \succcurlyeq \widetilde{\mathbb{E}}$ *if and only if there is a Markov kernel* K *from* $(\Omega, \mathcal{F})$ *to* $(\widetilde{\Omega}, \widetilde{\mathcal{F}})$ *such that* $\widetilde{P} = KP$ *and* $\widetilde{P}' = KP'$*, where* $KP(A) := \int_\Omega K(\omega, A)P(d\omega)$*,* $A \in \widetilde{\mathcal{F}}$*, and* KP' *is defined similarly.*

Note that 'if' part of Proposition 2.4 is true without any assumptions on $(\widetilde{\Omega}, \widetilde{\mathcal{F}})$.
We do not give here a comparison criterion in terms of measures $\nu_{\mathbb{E}}$. Let us only mention that $\mathbb{E} \sim \widetilde{\mathbb{E}}$ if and only if $\nu_{\mathbb{E}} = \nu_{\widetilde{\mathbb{E}}}$.
A concluding remark is that $\mathbb{E} \succcurlyeq \widetilde{\mathbb{E}}$ implies $H_{\mathbb{E}}(\alpha) \leq H_{\widetilde{\mathbb{E}}}(\alpha)$ for any $\alpha \in (0, 1)$ (apply Proposition 2.3 and (2.4)). It is important to remark that the converse of this statement is not true. Nevertheless, $H_{\mathbb{E}}(\alpha) = H_{\widetilde{\mathbb{E}}}(\alpha)$ for all $\alpha \in (0, 1)$ implies $\mathbb{E} \sim \widetilde{\mathbb{E}}$.

2.4. Let $(\mathbb{E}_\lambda)_{\lambda \in \Lambda}$ be any collection of experiments. Since $\inf_{\lambda \in \Lambda} b_{\mathbb{E}_\lambda}(\pi)$ is a concave function in π, there is an experiment $\mathbb{E}^*$ such that $b_{\mathbb{E}^*}(\pi) = \inf_{\lambda \in \Lambda} b_{\mathbb{E}_\lambda}(\pi)$, $\pi \in [0, 1]$. Any such experiment $\mathbb{E}^*$ is a least upper bound for $(\mathbb{E}_\lambda)_{\lambda \in \Lambda}$. Note that $\beta_{\mathbb{E}^*}$ is the concave envelope of the pointwise supremum of $\beta_{\mathbb{E}_\lambda}$, $\lambda \in \Lambda$.
Similarly, $\inf_{\lambda \in \Lambda} \beta_{\mathbb{E}_\lambda}(\alpha)$ is a nondecreasing concave function in α. Hence, there is an experiment $\mathbb{E}_*$ such that $\beta_{\mathbb{E}_*}(\alpha) = \inf_{\lambda \in \Lambda} \beta_{\mathbb{E}_\lambda}(\alpha)$, $\alpha \in [0, 1]$. Any such experiment $\mathbb{E}_*$ is a greatest lower bound for $(\mathbb{E}_\lambda)_{\lambda \in \Lambda}$. Note that $b_{\mathbb{E}_*}$ is the concave envelope of the pointwise supremum of $b_{\mathbb{E}_\lambda}$, $\lambda \in \Lambda$.
If $\mathbb{E}$ is an homogeneous experiment and $\widetilde{\mathbb{E}} \preccurlyeq \mathbb{E}$ then $\widetilde{\mathbb{E}}$ is an homogeneous experiment too. Thus, a greatest lower bound for $(\mathbb{E}_\lambda)_{\lambda \in \Lambda}$ is a homogeneous experiment if at least one of $\mathbb{E}_\lambda$ is homogeneous. A least upper bound for $(\mathbb{E}_\lambda)_{\lambda \in \Lambda}$ need not be a homogeneous experiment even if all $\mathbb{E}_\lambda$ are homogeneous.

2.5. To introduce the notion of weak convergence of experiments, we start with a preliminary definition of the deficiency of one experiment with respect to another one. Usually, the deficiency is defined in terms of errors corresponding to test functions. Our definition is equivalent to it, see e.g. Strasser (1985, p. 76).

Definition 3 *A deficiency of an experiment* $\mathbb{E}$ *with respect to an experiment* $\widetilde{\mathbb{E}}$ *is the quantity*

$$\delta_2(\mathbb{E}, \widetilde{\mathbb{E}}) = 2 \sup_{\pi \in [0,1]} [b_{\mathbb{E}}(\pi) - b_{\widetilde{\mathbb{E}}}(\pi)].$$

The deficiency distance between $\mathbb{E}$ *and* $\widetilde{\mathbb{E}}$ *is the quantity*

$$\Delta_2(\mathbb{E}, \widetilde{\mathbb{E}}) = \max\{\delta_2(\mathbb{E}, \widetilde{\mathbb{E}}), \delta_2(\widetilde{\mathbb{E}}, \mathbb{E})\} = 2 \sup_{\pi \in [0,1]} |b_{\mathbb{E}}(\pi) - b_{\widetilde{\mathbb{E}}}(\pi)|.$$

Evidently, Δ_2 satisfies the triangle inequality. Hence, strictly speaking, Δ_2 is a pseudo-distance on the family of all experiments.
Due to Proposition 2.1, $\mathbb{E} \succcurlyeq \widetilde{\mathbb{E}}$ if and only if $\delta_2(\mathbb{E}, \widetilde{\mathbb{E}}) = 0$; $\mathbb{E} \sim \widetilde{\mathbb{E}}$ if and only if $\Delta_2(\mathbb{E}, \widetilde{\mathbb{E}}) = 0$. Hence, Δ_2 defines a metric on the space of all types of experiments. Moreover, this metric space is a compact. The convergence in this space is called the weak convergence.

Definition 4 *A sequence of experiments* $\mathbb{E}_n$ *converges weakly to an experiment* $\mathbb{E}$ (*denoted* $\mathbb{E}_n \xrightarrow{w} \mathbb{E}$) *if* $\Delta_2(\mathbb{E}_n, \mathbb{E}) \to 0$, $n \to \infty$.

Here we do not need expressions for deficiencies and deficiency distances in terms of measures $\mu_{\mathbb{E}}$ or functions $\beta_{\mathbb{E}}$. What is important for us are criteria for weak convergence of experiments.

Proposition 2.5 *Assume that* $\mathbb{E}_n$, $n = 1, 2, \dots,$ *and* $\mathbb{E}$ *are experiments. The following statements are equivalent*:

(i) $\mathbb{E}_n \xrightarrow{w} \mathbb{E}$;

(ii) $\mu_{\mathbb{E}_n} \Rightarrow \mu_{\mathbb{E}}$;

(iii) $\nu_{\mathbb{E}_n} \Rightarrow \nu_{\mathbb{E}}$;

(iv) $\beta_{\mathbb{E}_n}(\alpha)$ *converges pointwise on* $(0, 1]$ *to* $\beta_{\mathbb{E}}(\alpha)$;

(v) $b_{\mathbb{E}_n}(\pi)$ *converges uniformly* (*or pointwise*) *on* $[0, 1]$ *to* $b_{\mathbb{E}}(\pi)$;

(vi) $H_{\mathbb{E}_n}(\alpha)$ *converges pointwise on* $(0, 1)$ *to* $H_{\mathbb{E}}(\alpha)$.

Combining implication (i)⇒(iii), (2.2) and Fatou's lemma or Lebesgue's theorem on dominated convergence (considering nonnegative f due to Remark 2.1), we get the following key result.

Proposition 2.6 *Assume that a sequence of experiments* $\mathbb{E}_n$ *converges weakly to an experiment* $\mathbb{E}$*. If* $f \in \mathcal{C}$ *then*

$$\mathcal{J}_f(\mathbb{E}) \leq \liminf_{n\to\infty} \mathcal{J}_f(\mathbb{E}_n).$$

If, moreover, $f(0) + \frac{f(\infty)}{\infty} < \infty$ *then*

$$\mathcal{J}_f(\mathbb{E}) = \lim_{n\to\infty} \mathcal{J}_f(\mathbb{E}_n).$$

2.6. Let $\mathbb{E}_a$ be a totally informative experiment and $\mathbb{E}_i$ a totally non informative experiment. The following facts are easy to verify:

- $\mu_{\mathbb{E}_a} = \delta_{\{0\}}$, $b_{\mathbb{E}_a}(\pi) \equiv 0$, $H_{\mathbb{E}_a}(\alpha) \equiv 0$ $(0 < \alpha < 1)$, $\mathcal{J}_f(\mathbb{E}_a) = f(0) + \frac{f(\infty)}{\infty}$, $(f \in \mathcal{C})$,
- $\mu_{\mathbb{E}_i} = \delta_{\{1\}}$, $b_{\mathbb{E}_i}(\pi) \equiv \min\{\pi, 1-\pi\}$, $H_{\mathbb{E}_a}(\alpha) \equiv 1$, $(0 < \alpha < 1)$, $\mathcal{J}_f(\mathbb{E}_i) = f(1)$ $(f \in \mathcal{C})$.

Moreover, $H_{\mathbb{E}}(1/2) = 0$ implies $\mathbb{E} \sim \mathbb{E}_a$ and $H_{\mathbb{E}}(1/2) = 1$ implies $\mathbb{E} \sim \mathbb{E}_i$. As a consequence of preceding results, we get the following statements.

Proposition 2.7 *For any experiment* $\mathbb{E}$ *and any* $f \in \mathcal{C}$

$$f(1) \leq \mathcal{J}_f(\mathbb{E}) \leq f(0) + \frac{f(\infty)}{\infty}.$$

Remark 2.2 *Compare the statement of the previous Proposition with the universal bounds in* (1.7)*.*

Proposition 2.8 *Assume that* $\mathbb{E}_n = (\Omega_n, \mathcal{F}_n, (P_n, P_n'))$, $n = 1, 2, \ldots,$ *are experiments. The following statements are equivalent:*

(i) $\mathbb{E}_n \xrightarrow{w} \mathbb{E}_a$;

(ii) $\mu_{\mathbb{E}_n} \Rightarrow \delta_{\{0\}}$;

(iii) $\lim_{n\to\infty} \|P_n - P_n'\| = 2$;

(iv) $\lim_{n\to\infty} H_{\mathbb{E}_n}(1/2) = 0$.

Proposition 2.9 *Assume that* $\mathbb{E}_n = (\Omega_n, \mathcal{F}_n, (P_n, P'_n))$, $n = 1, 2, \ldots,$ *are experiments. The following statements are equivalent*:

(i) $\mathbb{E}_n \xrightarrow{w} \mathbb{E}_i$;

(ii) $\mu_{\mathbb{E}_n} \Rightarrow \delta_{\{1\}}$;

(iii) $\lim_{n\to\infty} \|P_n - P'_n\| = 0$;

(iv) $\lim_{n\to\infty} H_{\mathbb{E}_n}(1/2) = 1$.

3 Martingale measures and Hellinger processes

In this section we recall some useful facts from the theory of martingales for reference purposes. The notation and the details can be found e.g. in Jacod and Shiryaev (1987).

3.1. Let $Z = \{Z_t\}_{0\le t\le T}$ *be a strictly positive* semimartingale on a stochastic basis $\mathcal{B} = (\Omega, \mathcal{F}_T, \{\mathcal{F}_t\}_{0\le t\le T}, P)$ with $Z_0 = 1$.
There are two convenient representations of Z either as the exponential $Z = \exp(\overline{X})$ of $\overline{X} = \{\overline{X}_t\}_{0\le t\le T}$ or as the stochastic exponential $Z = \mathcal{E}(X)$ of $X = \{X_t\}_{0\le t\le T}$ with $X_0 = 0$, the latter means that Z satisfies the stochastic differential equation

$$dZ_t = Z_{t-}dX_t, \qquad 0 \le t \le T, \qquad Z_0 = 1,$$

the solution being written explicitly

$$Z_t = \exp(X_t - \frac{1}{2}\langle X^c, X^c\rangle_t) \prod_{0<s\le t} (1 + \Delta X_s)e^{-\Delta X_s}. \tag{3.1}$$

Of course, X can be easily expressed through $\overline{X}$ and vice versa. In particular, the jumps of X and $\overline{X}$ are connected by the relation

$$\Delta X = \exp(\Delta\overline{X}) - 1. \tag{3.2}$$

Some of the subsequent results have a more simple formulation in terms of X. This is why we use a more simple notation for X rather than $\overline{X}$. See anyway the discussion in Shiryaev (1999).

Let $(B(\phi), C, \nu)$ and $(\overline{B}(\phi), \overline{C}, \overline{\nu})$ be the triplets of local characteristics of X and $\overline{X}$ respectively with respect to a truncation function $\phi(x)$. Then up to a P-null set

$$\begin{cases} \overline{B}(\phi) = B(\phi) - \frac{1}{2}C + \{\phi(\log(1+x)) - \phi(x)\} * \nu, \\ \overline{C} = C, \\ g(\omega, t, x) * \overline{\nu} = g(\omega, t, \log(1+x)) * \nu. \end{cases} \tag{3.3}$$

Note that $\Delta X > -1$, hence ν charges only the set $\{(\omega, t, x) \colon 0 < t \le T,\ x \in (-1, 0) \cup (0, \infty)\}$.

3.2. Let Q be another probability measure on $(\Omega, \mathcal{F}_T)$ equivalent to P. By (3.1) the representation $Z = \mathcal{E}(X)$ also holds with respect to Q. Let $T_Q = (B^Q(\phi), C^Q, \nu^Q)$ be the triplet of local characteristics of X relative to Q. By Girsanov's Theorem for semimartingales, see Proposition III.3.24 in Jacod and Shiryaev (1987), there exist a predictable function $\beta^Q = \{\beta_t^Q\}_{0 \le t \le T}$ and a strictly positive function $Y^Q = Y^Q(\omega, t, x)$ measurable with respect to the predictable σ-algebra on $\Omega \times [0, T] \times \mathbb{R}$ such that P-a.s.

$$|\beta^Q| \cdot C_T < \infty, \quad |\phi(x)(Y^Q - 1)| * \nu_T < \infty, \tag{3.4}$$

and

$$\begin{cases} B^Q(\phi) = B(\phi) + \beta^Q \cdot C + \phi(x)(Y^Q - 1) * \nu, \\ C^Q = C, \\ \nu^Q = Y^Q \cdot \nu. \end{cases} \tag{3.5}$$

Now assume that $Q \in \mathcal{P}$, i.e. $Q \sim P$ and Z is a Q-martingale. Then X is a Q-local martingale. By Proposition II.2.29 in Jacod and Shiryaev (1987), Q-a.s.

$$(x^2 \wedge |x|) * \nu_T^Q < \infty \tag{3.6}$$

and

$$B^Q(\phi) + (x - \phi(x)) * \nu^Q = 0. \tag{3.7}$$

Note that condition (3.7) implies that Z is a Q-local martingale but it is not sufficient for Z to be a Q-martingale. In order to verify that $Z = \mathcal{E}(X)$ is an uniformly integrable martingale a sufficient condition is the existence of a constant H such that

$$\langle X^c, X^c \rangle_T + \frac{x^2}{1 + |x|} * \nu_T^Q \le H. \tag{3.8}$$

See Theorem 12 in Kabanov et al. (1979).

3.3. Let again $Q \in \mathcal{P}$. Define a probability measure $\overline{Q}$ on $\mathcal{F}_T$ by $d\overline{Q} = Z_T dQ$. Then $Z = \{Z_t\}_{0\le t\le T}$ is the density process of $\overline{Q}$ with respect to Q. By Corollary IV.1.37 in Jacod and Shiryaev (1987), the Hellinger process $h(\alpha) = h(\alpha; Q, \overline{Q})$ of order $\alpha \in (0,1)$ for Q and $\overline{Q}$ has the form

$$h(\alpha) = \frac{\alpha(1-\alpha)}{2} C^Q + \{\alpha + (1-\alpha)(1+x) - (1+x)^{1-\alpha}\} * \nu^Q. \tag{3.9}$$

In particular,

$$h\Big(\frac{1}{2}\Big) = \frac{1}{8} C^Q + \frac{1}{2}(1-\sqrt{1+x})^2 * \nu^Q. \tag{3.10}$$

3.4. By reasons that will be apparent later, let us consider a more general situation. Namely, assume that $(\Omega, \mathcal{F}_T, \{\mathcal{F}_t\}_{0\le t\le T}, Q)$ is a stochastic basis, $Z = \{Z_t\}_{0\le t\le T}$ is a *nonnegative* supermartingale defined on this basis, and Z is representable in the form

$$Z = \mathcal{E}(X) \tag{3.11}$$

(note that in general, there are nonnegative supermartingales Z with $Z_0 = 1$ that cannot be represented in this form). The process $X = \{X_t\}_{0\le t\le T}$ is not uniquely determined by (3.11) in general. Nevertheless, one can choose X such that $X_0 = 0$ and X is a Q-local supermartingale with $\Delta X \ge -1$. For example, one can take $X = \frac{\mathbf{1}_{\{Z_->0\}}}{Z_-} \cdot Z$. We shall assume that X satisfies these assumptions. Denote by A the predictable non-increasing process in the Doob–Meyer decomposition of X, i.e. $A_0 = 0$ and $X - A$ is a Q-local martingale. Let $(B(\phi), C, \nu)$ be the triplet of local characteristics of X with respect to Q (we omit Q in the indices here). Of course, ν charges only the set $\{(\omega, t, x)\colon 0 < t \le T,\ x \in [-1,0) \cup (0,\infty)\}$. By Proposition II.2.29 in Jacod and Shiryaev (1987) we have that Q-a.s.

$$(x^2 \wedge |x|) * \nu_T < \infty \tag{3.12}$$

and the processes A and $B(\phi)$ are connected by the relation

$$A = B(\phi) + (x - \phi(x)) * \nu. \tag{3.13}$$

Conversely, if X is a Q-local supermartingale with $X_0 = 0$ and $\Delta X \ge -1$, then $Z = \mathcal{E}(X)$ is a nonnegative Q-local supermartingale, hence a Q-supermartingale.

4 Attainability of universal bounds

In this section we focus on the problem of the attainability of universal bounds, giving, in particular, criteria in terms of Hellinger processes, i.e. *predictable* criteria. Our setting is the same as in Section 1. In particular, B, S and g satisfy the assumptions of subsection 1.1, f and Z are defined as in (1.6). It is always assumed that $\mathcal{M} \neq \emptyset$.

4.1. For any probability measure Q (without or with indices), $Q \in \mathcal{M}$, let $\overline{Q}$ (without or with the same indices) be the probability measure defined by $d\overline{Q} = Z_T dQ$. The expression for the Hellinger process for Q and $\overline{Q}$ in terms of T_Q is given in subsection 3.3.

Proposition 4.1 (a) *The following statements are equivalent*:

(i) *For any $g \in \mathcal{C}$ the upper universal bound is attained, i.e.*

$$\mathbb{C}^* = \sup_{Q\in\mathcal{M}} \gamma(Q) = B_T^{-1} g(0) + S_0 \frac{g(\infty)}{\infty} = f(0) + \frac{f(\infty)}{\infty}. \tag{4.1}$$

(ii) *There exists a sequence $\{Q^n\}_{n\geq 1}$, $Q^n \in \mathcal{M}$, such that*

$$\lim_{n\to\infty} Q^n(Z_T > \varepsilon) = 0 \qquad \textit{for any } \varepsilon > 0.$$

(ii′) *There exists a sequence $\{Q^n\}_{n\geq 1}$, $Q^n \in \mathcal{M}$, such that*

$$\lim_{n\to\infty} \overline{Q^n}(Z_T < N) = 0 \qquad \textit{for any } N > 0.$$

(b) *The following statement implies* (i), (ii) *and* (ii′):

(iii) *There exists a sequence $\{Q^n\}_{n\geq 1}$, $Q^n \in \mathcal{M}$, such that*

$$\lim_{n\to\infty} Q^n(h_T^n > N) = 1 \qquad \textit{for any } N > 0,$$

where $h^n = h(1/2; Q^n, \overline{Q^n})$ is the Hellinger process of order $1/2$ *for Q^n and $\overline{Q^n}$.*

(c) *If $P(\inf_{t\leq T} \frac{Z_t}{Z_{t-}} \leq \beta) = 0$ for some $\beta > 0$ (in particular, if Z is P-a.s. continuous), then* (iii) *is equivalent to* (i), (ii) *and* (ii′).

Proposition 4.2 *The following statements are equivalent*:

(i) *For any* $g \in \mathcal{C}$ *with* $g(0) + \frac{g(\infty)}{\infty} < \infty$ *the lower universal bound is attained, i.e.*

$$\mathbb{C}_* = \inf_{Q \in \mathcal{M}} \gamma(Q) = B_T^{-1} g(S_0 B_T) = f(1). \tag{4.2}$$

(ii) *There exists a sequence* $\{Q^n\}_{n \geq 1}$, $Q^n \in \mathcal{M}$, *such that*

$$\lim_{n \to \infty} Q^n(|Z_T - 1| > \varepsilon) = 0 \qquad \textit{for any } \varepsilon > 0.$$

(iii) *There exists a sequence* $\{Q^n\}_{n \geq 1}$, $Q^n \in \mathcal{M}$, *such that*

$$\lim_{n \to \infty} Q^n(h_T^n > \varepsilon) = 0 \qquad \textit{for any } \varepsilon > 0,$$

where $h^n = h(1/2; Q^n, \overline{Q^n})$ *is the Hellinger process of order* $1/2$ *for* Q^n *and* $\overline{Q^n}$.

Proof of Proposition 4.1. (a) Applying (i) with $f(x) = |x-1|$, one can find a sequence $\{Q^n\}_{n \geq 1}$, $Q^n \in \mathcal{M}$, such that

$$\|Q^n - \overline{Q^n}\| \to 2, \quad n \to \infty. \tag{4.3}$$

Due to Proposition 2.8 the last property is equivalent to (ii) or to

$$\mathbb{E}^n = (\Omega, \mathcal{F}_T, (Q^n, \overline{Q^n})) \xrightarrow{w} \mathbb{E}_a, \quad n \to \infty.$$

Applying Proposition 2.8 to the experiments $\widehat{\mathbb{E}}^n = (\Omega, \mathcal{F}_T, (\overline{Q^n}, Q^n))$, we see that (4.3) is also equivalent to

$$\lim_{n \to \infty} \overline{Q^n}(dQ^n / d\overline{Q^n} > \varepsilon) = \lim_{n \to \infty} \overline{Q^n}(Z_T < \varepsilon^{-1}) = 0 \qquad \text{for any } \varepsilon > 0,$$

which is nothing else than (ii′).
Conversely, if there is a sequence $\{Q^n\}_{n \geq 1}$, $Q^n \in \mathcal{M}$, such that $\mathbb{E}^n \xrightarrow{w} \mathbb{E}_a$, $n \to \infty$, then, by Proposition 2.6,

$$\liminf_{n \to \infty} \gamma(Q^n) = \liminf_{n \to \infty} \mathcal{J}_f(\mathbb{E}^n) \geq f(0) + \frac{f(\infty)}{\infty}$$

for any $f \in \mathcal{C}$. Combining this statement with Proposition 2.7, we get (i).

(b) As $Z_0 = 1$, the restrictions of Q and $\overline{Q}$ onto the σ-algebra $\mathcal{F}_0$ coincide for any $Q \in \mathcal{M}$. By part (ii) of Theorem V.4.32 in Jacod and Shiryaev (1987), the condition (iii) implies (4.3).
(c) It is enough to show that if (iii) is violated, then (4.3) is not valid for any sequence $\{Q^n\}_{n\geq 1}$, $Q^n \in \mathcal{M}$. Assume the converse. Then there is a sequence $\{Q^n\}_{n\geq 1}$, $Q^n \in \mathcal{M}$, satisfying (4.3) and such that

$$\limsup_{n\to\infty} Q^n(h^n_T > N) < 1 \qquad \text{for some } N > 0,$$

where $h^n = h(1/2; Q^n, \overline{Q^n})$. Applying part (a) of Theorem V.2.4 in Jacod and Shiryaev (1987) with $P^n = \overline{Q^n}$ and $P'^n = Q^n$, we arrive at a contradiction. Indeed, since $Q^n \sim P$, we have $Q^n(\inf_{t\leq T} \frac{Z_t}{Z_{t-}} \leq \beta) = 0$, hence the process $i^n(\beta)$ in that Theorem satisfies $i^n(\beta) = 0$, Q^n-a.s. for any n.

Proof of Proposition 4.2. Similar to the preceding proof (use Proposition 2.9 instead of Proposition 2.8), one shows that both (i) and (ii) are equivalent to the existence of a sequence $\{Q^n\}_{n\geq 1}$, $Q^n \in \mathcal{M}$, such that

$$\|Q^n - \overline{Q^n}\| \to 0, \quad n \to \infty. \tag{4.4}$$

The equivalence of (4.4) and (iii) follows from part (i) of Theorem V.4.32 in Jacod and Shiryaev (1987).

Remark 4.1 *It may happen that we have equality in* (4.2) *for a certain function* $g \in \mathcal{C}$, *for example, for* $g(x) = (x-a)^+$ *or* $g(x) = (a-x)^+$, *but not for all* $g \in \mathcal{C}$ *with* $g(0) + \frac{g(\infty)}{\infty} < \infty$. *However, slightly modifying the above proof, one can show that if the corresponding function* f *is strictly convex at point* 1 *in the sense that* f *does not coincide with a linear function on any interval* $(1-\delta, 1+\delta)$, $\delta > 0$, *then equality* (4.2) *for this* f *implies* (4.4) *and hence the statements* (i)–(iii) *of Proposition* 4.2. *An example of such a function* g *is* $g(x) = (x - B_0^{-1} S_0 B_T)^+$, *that according to* (1.6) *gives* $f(x) = B_0^{-1} S_0 (x-1)^+$.
Similarly, if equality (4.1) *takes place for a function* $g \in \mathcal{C}$ *such that* $g(0) + \frac{g(\infty)}{\infty} < \infty$ *and* g *is not identically linear on* $(0, \infty)$, *then the statements* (i), (ii) *and* (ii′) *of Proposition* 4.1 *are valid.*

Remark 4.2 *Propositions* 4.1 *and* 4.2 *with the above remark contain as particular cases Propositions* 2.3 *and* 2.4 *in Frey and Sin* (1999). *In the general case similar results were proved in an unpublished manuscript by Jacod (1997).*

4.2. The preceding results allow us to formulate sufficient conditions for the non-attainability of the universal bounds in the terms of the original measure P. For brevity, we shall say that the upper (resp. lower) universal bound is non-attainable if the statement (i) of Proposition 4.1 (resp. 4.2) does not hold with $\mathcal{M} = \mathcal{P}$ (then it does not hold for any $\mathcal{M} \subset \mathcal{P}$). According to Remark 4.1 this means that

$$\sup_{Q \in \mathcal{P}} \gamma(Q) < B_T^{-1} g(0) + S_0 \frac{g(\infty)}{\infty}$$

for any function $g \in \mathcal{C}$ such that $g(0) + \frac{g(\infty)}{\infty} < \infty$ and g is not identically linear on $(0, \infty)$ (resp.

$$\inf_{Q \in \mathcal{P}} \gamma(Q) > B_T^{-1} g(S_0 B_T)$$

for any function $g \in \mathcal{C}$ which is strictly convex at point $S_0 B_T$).

1) If there is a number $\delta > 0$ such that $P(Z_T < \delta) = 0$ then the upper universal bound is non-attainable. Indeed, then we have $Q(Z_T < \delta) = 0$ for any $Q \in \mathcal{P}$, therefore the statement (ii) of Proposition 4.1 does not hold.

2) Similarly, if there is a number $N > 0$ such that $P(Z_T > N) = 0$ then the upper universal bound is non-attainable (the statement (ii$'$) of Proposition 4.1 does not hold).

3) Similarly, if there is a number $\delta > 0$ such that $P(|Z_T - 1| < \delta) = 0$ then the lower universal bound is non-attainable (use implication (i)$\Rightarrow$(ii) in Proposition 4.2).

4) Now let X be a P-semimartingale such that $Z = \mathcal{E}(X)$, and let $(B(\phi), C, \nu)$ be the triplet of local characteristics of X (relative to P) with respect to a truncation function $\phi(x)$.

4.1) If there is a number $\delta > 0$ such that $P(C_T < \delta) = 0$, then the lower universal bound is non-attainable. Indeed, for any $Q \in \mathcal{P}$ the Hellinger process $h = h(1/2; Q, \overline{Q})$ of order 1/2 for Q and $\overline{Q}$ satisfies Q-a.s.

$$h_T \geq \frac{1}{8} C_T$$

due to (3.10) and (3.5), and the statement follows from the implication (i)⇒(iii) in Proposition 4.2.

4.2) Similarly, if $\nu = 0$ and there is a number $N > 0$ such that $P(C_T > N) = 0$, then the upper universal bound is non-attainable (use part (c) of Proposition 4.1).

5) Assume now that X (or Z, or S) is a process of finite variation, then

$$C = 0 \qquad \text{and} \qquad (|x| \wedge 1) * \nu_T < \infty \qquad P\text{-a.s.} \tag{4.5}$$

Then we can define

$$B' = X - \sum_{s \le \cdot} \Delta X_s.$$

It follows from the canonical representation of semimartingales that

$$B' = B(\phi) - \phi(x) * \nu. \tag{4.6}$$

5.1) If X is a process of finite variation and there is a number $\delta > 0$ such that P-a.s. $\nu([0,T] \times (0,\delta)) = 0$ and $P(B'_T > -\delta) = 0$, then the lower universal bound is non-attainable. Indeed, let $Q \in \mathcal{P}$. Using the notation from subsection 3.2, we obtain from (3.4)–(3.7), (4.5) and (4.6) that Q-a.s.

$$|x| * \nu_T^Q < \infty \tag{4.7}$$

and

$$B' + x * \nu^Q = 0. \tag{4.8}$$

Therefore,

$$x\mathbf{1}_{\{x \ge \delta\}} * \nu_T^Q = |x|\mathbf{1}_{\{|x|<0\}} * \nu_T^Q - B'_T \ge \delta \qquad Q\text{-a.s.},$$

and the claim easily follows from (3.10) and the implication (i)⇒(iii) in Proposition 4.2.

5.2) Similarly, if X is a process of finite variation and there is a number $\delta > 0$ such that P-a.s. $\nu([0,T] \times (-\delta,0)) = 0$ and $P(B'_T < \delta) = 0$, then the lower universal bound is non-attainable.

5 Comparison lemma

Let $\mathcal{T}_{\mathcal{M}} = \{\mu_Q : Q \in \mathcal{M}\}$, $\mu_Q = \mathcal{L}(Z_T \mid Q)$ as in Section 1. In subsection 4.2 we have considered a number of cases where the least upper (resp. the greatest lower) bound for $\mathcal{T}_{\mathcal{M}}$ does not coincide with $\delta_{\{0\}}$ (resp. with $\delta_{\{1\}}$). In such a situation the first step of the method proposed in subsection 1.4 for finding $\mathbb{C}^*$ (resp. $\mathbb{C}_*$) is to show that $\mu_Q \preccurlyeq \mu$ (resp. $\mu_Q \succcurlyeq \mu$) for all $Q \in \mathcal{M}$, where $\mu \in \mathcal{S}$ is a non-trivial candidate to be the least upper (resp. the greatest lower) bound for $\mathcal{T}_{\mathcal{M}}$.

As we have already mentioned, a suitable description of $\mathcal{T}_{\mathcal{M}}$ is usually not available if the market is not complete. The most essential information about Q in our disposal is contained in the relations (3.5) and (3.7) (at least if $\mathcal{M} = \mathcal{P}$). Thus it is desirable to have a comparison tool based on the corresponding local characteristics under Q. In general, this does not give us the possibility of comparing the measure μ_Q with a measure μ if no other specification of μ is given. However, fortunately, in many models the measures μ corresponding to the least upper and the greatest lower bounds can be described by the property $\mu = \mathcal{L}(Z_T^* \mid Q^*)$, where Z^* is either Z or the canonical process on a trajectory space and Q^* is a certain probability measure (maybe on an extension of the original space); moreover, the process Z^* under Q^* has a rather special (e.g. Markovian) structure. In such a case there is an efficient comparison tool in terms of local characteristics presented in this section.

5.1. Let $Z = \{Z_t\}_{0 \le t \le T}$ and $Z^* = \{Z_t^*\}_{0 \le t \le T}$, $Z_0 = Z_0^* = 1$, be two non-negative supermartingales defined on two stochastic basis $\mathcal{B} = (\Omega, \mathcal{F}_T, \{\mathcal{F}_t\}_{0 \le t \le T}, Q)$ and $\mathcal{B}^* = (\Omega^*, \mathcal{F}_T^*, \{\mathcal{F}_t^*\}_{0 \le t \le T}, Q^*)$ respectively. We wish to compare the distributions $\mu = \mathcal{L}(Z_T \mid Q)$ and $\mu^* = \mathcal{L}(Z_T^* \mid Q^*)$ in the sense of the partial ordering $\preccurlyeq$ introduced in subsection 1.2.

In the option pricing context Z is the process defined in Section 1 and $Q \in \mathcal{M}$, thus $\mu = \mu_Q$. Since Q is an arbitrary equivalent martingale measure in general, there are no special assumptions on Z and Q. (In fact, we shall impose some restrictions, see (5.3) below, which exclude, for example, the case of discrete time. This is done for simplicity only. In Section 7 we shall consider an example with discrete time). The measure μ^* plays the role of an upper or a lower bound for $\{\mu_Q \colon Q \in \mathcal{M}\}$. The assumptions on Q^* and Z^* are much more restrictive according to the introduction of this section.

As it was explained in Sections 1 and 2, the least upper bound μ^* for the family $\{\mu_Q \colon Q \in \mathcal{M}\}$ need not satisfy the relations $\mu^*(\{0\}) = 0$ and $\int x \, \mu^*(dx) = 1$. Thus it is reasonable not to assume that Z^* is a Q^*-martingale or that Z^* is strictly positive if

we look for conditions under which $\mu \preccurlyeq \mu^*$. On the other hand, in the option pricing setting $Q(Z_T > 0) = 1$ and $E_Q Z_T = 1$, hence $Q^*(Z_T^* > 0) = 1$ and $E_{Q^*} Z_T^* = 1$ if $\mu^* \preccurlyeq \mu$. Nevertheless, for completeness of the picture in our comparison lemma below both μ and μ^* are allowed to have a positive mass at 0 and to have mean less than 1, i.e. Z and Z^* are not necessarily martingales and may vanish.
We shall assume that Z and Z^* satisfy the stochastic differential equations

$$dZ_t = Z_{t-} dX_t, \qquad 0 \le t \le T, \qquad Z_0 = 1, \tag{5.1}$$

and

$$dZ_t^* = Z_{t-}^* dX_t^*, \qquad 0 \le t \le T, \qquad Z_0^* = 1, \tag{5.2}$$

i.e. $Z = \mathcal{E}(X)$ and $Z^* = \mathcal{E}(X^*)$, where $X = \{X_t\}_{0 \le t \le T}$ and $X^* = \{X_t^*\}_{0 \le t \le T}$ are local supermartingales on $\mathcal{B}$ and $\mathcal{B}^*$ respectively, with characteristics $(B(\phi), C, \nu)$ and $(B^*(\phi), C^*, \nu^*)$, and such that $\Delta X \ge -1$, and $\Delta X^* \ge -1$. Let $A = \{A_t\}_{0 \le t \le T}$ and $A^* = \{A_t^*\}_{0 \le t \le T}$ be predictable non-increasing processes on $\mathcal{B}$ and $\mathcal{B}^*$ respectively such that $A_0 = 0$, $A_0^* = 0$, $X - A$ and $X^* - A^*$ are local martingales on $\mathcal{B}$ and $\mathcal{B}^*$, see subsection 3.4 for the relationship between A and $B(\phi)$, A^* and $B^*(\phi)$.
For simplicity we shall assume that

$$A_t(\omega) = \int_0^t a_s(\omega)\, ds, \quad C_t(\omega) = \int_0^t c_s(\omega)\, ds, \quad \nu(\omega, dt, dx) = K(\omega, t, dx)\, dt, \tag{5.3}$$

$$A_t^*(\omega^*) = \int_0^t a_s^*(\omega^*)\, ds, \quad C_t^*(\omega^*) = \int_0^t c_s^*(\omega^*)\, ds, \quad \nu^*(\omega^*, dt, dx) = K^*(\omega^*, t, dx)\, dt, \tag{5.4}$$

where $a = \{a_t\}_{0 \le t \le T}$, $a^* = \{a_t^*\}_{0 \le t \le T}$ are non-positive and $c = \{c_t\}_{0 \le t \le T}$, $c^* = \{c_t^*\}_{0 \le t \le T}$ are nonnegative predictable processes on the corresponding spaces, and the transition kernels $K = K(\omega, t, dx)$ and $K^* = K^*(\omega, t, dx)$ satisfy the standard assumptions stated in Proposition II.2.9 in Jacod and Shiryaev (1987).
We require an additional Markovian structure on Z^*. Namely, we shall assume the existence of functions $a^* = a^*(t, z)$ and $c^* = c^*(t, z)$ and a kernel $K^* = K^*(t, z, dx)$ such that

$$a_t^*(\omega^*) = a^*(t, Z_{t-}^*(\omega^*)), \qquad c_t^*(\omega^*) = c^*(t, Z_{t-}^*(\omega^*)), \qquad 0 \le t \le T, \tag{5.5}$$

and

$$K^*(\omega^*, t, dx) = K^*(t, Z_{t-}^*(\omega^*), dx), \qquad 0 \le t \le T. \tag{5.6}$$

This makes natural to assume, that the $*$-prices defined by

$$G(t, z) = E_{Q^*}[f(Z_T^*) \mid Z_t^* = z], \tag{5.7}$$

where f belongs to a sufficiently large class $\mathcal{C}' \subseteq \mathcal{C}$, satisfy a pricing equation (i.e. the backward Kolmogorov equation) of the form

$$G_t(t, z) + L_t G(t, z) = 0, \tag{5.8}$$

where

$$L_t G(t, z) = za^*(t, z)G_z(t, z) + \frac{1}{2}z^2 c^*(t, z)G_{zz}(t, z) + \int_{[-1,\infty)} (\Lambda G)(t, z, x)K^*(t, z, dx)$$

and

$$(\Lambda G)(t, z, x) = G(t, z(1+x)) - G(t, z) - zxG_z(t, z), \qquad 0 \le t \le T, \quad z > 0, \quad x \ge -1,$$

and G_t, G_z and G_{zz} are the corresponding partial derivatives. Actually, the formula (5.7) is not part of the assumptions; it serves only for the explanation how the function G appears and what are the boundary conditions for it.
The last assumption is the *propagation of convexity*, stated in our framework as

$$G(t, z) \text{ is convex in } z \in [0, \infty) \text{ for each } t \in [0, T], \tag{5.9}$$

see El Karoui et al. (1998) and Martini (2000).
Here we prefer not to discuss our assumptions. We shall check them later for some particular models.

Lemma 5.1 (Comparison) *Assume that the following hypotheses are satisfied*:

(A) Z *is a nonnegative supermartingale on* $\mathcal{B}$ *satisfying* (5.1), *where* X *is a local supermartingale on* $\mathcal{B}$, $\Delta X \ge -1$, *and the triplet* (A, C, ν) *satisfies* (5.3).

(B) Z^* *is a nonnegative supermartingale on* $\mathcal{B}^*$ *satisfying* (5.2), *where* X^* *is a local supermartingale on* $\mathcal{B}^*$, $\Delta X^* \ge -1$, *and the triplet* (A^*, C^*, ν^*) *satisfies* (5.4)–(5.6).

(C) *For any function* $f \in \mathcal{C}_0^2$ *there is a real-valued function* $G(t, z)$, $0 \le t \le T$, $z \ge 0$, *with the following properties*:

(C1) $G(T,z) = f(z)$, $z \geq 0$;

(C2) $G(0,1) = E_{Q^*} f(Z_T^*)$;

(C3) $G(t,0) = f(0)$ *for* $0 < t \leq T$;

(C4) G *is bounded and continuous on* $[0,T] \times [0,\infty)$, *twice continuously differentiable in* z *and continuously differentiable in* t *on* $[0,T) \times (0,\infty)$;

(C5) *the pricing equation* (5.8) *holds true for* $0 \leq t < T$, $z > 0$;

(C6) *propagation of convexity* (5.9) *is satisfied.*

(D) *Comparison of predictable characteristics.*

$$c_t(\omega) \leq c^*(t, Z_{t-}(\omega)) \qquad dt \times dQ\text{-a.e.} \tag{5.10}$$

For any $w \in (-1,0)$, $dt \times dQ$*-a.e.*

$$\int_{[-1,\infty)} (w-x)^+ K(\omega,t,dx) \leq \int_{[-1,\infty)} (w-x)^+ K^*(t, Z_{t-}(\omega), dx). \tag{5.11}$$

For any $w \in (0,\infty)$, $dt \times dQ$*-a.e.*

$$\int_{[-1,\infty)} (x-w)^+ K(\omega,t,dx) - a_t(\omega)$$

$$\leq \int_{[-1,\infty)} (x-w)^+ K^*(t, Z_{t-}(\omega), dx) - a^*(t, Z_{t-}(\omega)). \tag{5.12}$$

Then $\mathcal{L}(Z_T \mid Q) \preccurlyeq \mathcal{L}(Z_T^* \mid Q^*)$. *If the opposite inequalities hold in* (D) *then* $\mathcal{L}(Z_T^* \mid Q^*) \preccurlyeq \mathcal{L}(Z_T \mid Q)$.

Remark 5.1 *In the case when the processes* Z *and* Z^* *are strictly positive martingales on* $[0,T]$ *satisfying the hypothesis of the previous Lemma, denoting* $\mu_Q = \mathcal{L}(Z_T \mid Q)$ *and* $\mu_{Q^*} = \mathcal{L}(Z_T \mid Q^*)$ *we obtain* $\mathcal{J}_f(\mu_Q) = \int f(x)\,\mu_Q(dx)$ *and* $\mathcal{J}_f(\mu_{Q^*}) = \int f(x)\,\mu_{Q^*}(dx)$. *Then, the conclusion* $\mathcal{L}(Z_T \mid Q) \preccurlyeq \mathcal{L}(Z_T^* \mid Q^*)$ *of the Lemma is equivalent to*

$$E_Q f(Z_T) \leq E_{Q^*} f(Z_T^*) \quad \textit{for all } f \in \mathcal{C}.$$

Remark 5.2 *It follows from the proof that, if the process Z is strictly positive then it is sufficient to assume that the function G is defined on the set $[0,T]\times(0,\infty)$, that it is bounded and continuous only on this set in* (C4)*, and condition* (C3) *is not necessary. On the other hand, if Z and Z^* are local martingales with respect to Q and Q^* respectively, then $a=a^*=0$ and condition* (5.12) *takes a more simple form.*

Proof. We shall prove only the first statement. The proof of the second one is completely similar.
Let us fix a function $f\in\mathcal{C}_0^2$. Due to Corollary 2.1 it is enough to check that

$$E_Q f(Z_T)\le E_{Q^*}f(Z_T^*). \tag{5.13}$$

For $n=1,2,\dots$ define $\tau_n=\inf\{t\colon Z_t<1/n\}$ ($\inf\emptyset=T$), $\tau=\lim_{n\to\infty}\tau_n$. It is well known that Q-a.s. $\{Z=0\}=[\![\tau,T]\!]$. Moreover, in view of (5.1), Q-a.s. $Z_{\tau-}>0$ and $[\![0,\tau]\!]=\bigcup_n[\![0,\tau_n]\!]$. Take also an increasing sequence $\{t_n\}$, $t_n\to T$. Applying Itô's formula to $\{G(t,Z_t)\}$ on $\mathcal{B}$, we obtain for $t\in[0,t_n]$

$$\begin{aligned} G(t\wedge\tau_n,Z_{t\wedge\tau_n}) &= G(0,1)+\int_0^{t\wedge\tau_n}G_t(s,Z_{s-})\,ds+\int_0^{t\wedge\tau_n}G_z(s,Z_{s-})Z_{s-}dX_s\\ &+\frac12\int_0^{t\wedge\tau_n}G_{zz}(s,Z_{s-})Z_{s-}^2c_s\,ds+\int_0^{t\wedge\tau_n}\int_{[-1,\infty)}(\Lambda G)(s,Z_{s-},x)\mu(ds,dx), \end{aligned}\tag{5.14}$$

where μ is the jump measure of X. Due to the properties of G, all the terms in (5.14) excluding the last one are special semimartingales in t. Hence, the last term is also a special semimartingale with the compensator

$$\int_0^{t\wedge\tau_n}\int_{[-1,\infty)}(\Lambda G)(s,Z_{s-},x)K(s,dx)\,ds,$$

and we may rewrite (5.14) in the form

$$\begin{aligned} G(t\wedge\tau_n,Z_{t\wedge\tau_n}) &= G(0,1)+\int_0^{t\wedge\tau_n}G_t(s,Z_{s-})\,ds+\int_0^{t\wedge\tau_n}G_z(s,Z_{s-})Z_{s-}a_s ds\\ &+\frac12\int_0^{t\wedge\tau_n}G_{zz}(s,Z_{s-})Z_{s-}^2c_s\,ds+\int_0^{t\wedge\tau_n}\int_{[-1,\infty)}(\Lambda G)(s,Z_{s-},x)K(s,dx)\,ds+m_t^n, \end{aligned}$$

where m^n is a local martingale on $\mathcal{B}$. Now replace $G_t(s,Z_{s-})$ according to (5.8), to obtain

$$G(t\wedge\tau_n,Z_{t\wedge\tau_n})=G(0,1)+\frac12\int_0^{t\wedge\tau_n}G_{zz}(s,Z_{s-})Z_{s-}^2(c_s-c^*(s,Z_{s-}))\,ds+$$

$$
+\int_0^{t\wedge\tau_n}\Big[\int_{[-1,\infty)}(\Lambda G)(s,Z_{s-},x)K(s,dx)+a_sG_z(s,Z_{s-})Z_{s-}
$$

$$
-\Big\{\int_{[-1,\infty)}(\Lambda G)(s,Z_{s-},x)K^*(s,Z_{s-},dx))+a^*(s,Z_{s-})G_z(s,Z_{s-})Z_{s-}\Big\}\Big]\,ds+m_t^n
$$

$$
=G(0,1)+D_t^{n,1}+D_t^{n,2}+m_t^n=G(0,1)+D_t^n+m_t^n.
$$

Assume for the moment that the process $D^n=\{D_t^n\}_{0\le t\le T}$ is decreasing Q-a.s. Then $\{G(t\wedge\tau_n,Z_{t\wedge\tau_n})\}_{0\le t\le t_n}$ is a Q-local supermartingale, hence a Q-supermartingale since it is bounded. Observe also, that as $n\to\infty$,

$$
G(t_n\wedge\tau_n,Z_{t_n\wedge\tau_n})\to G(T,Z_T)\mathbf{1}_{\{\tau=T\}}+G(\tau,0)\mathbf{1}_{\{\tau<T\}}=f(Z_T),
$$

because the process Z is stochastically continuous, and boundary conditions in (C). Then, as G is bounded,

$$
E_Qf(Z_T)=\lim_n E_QG(t_n\wedge\tau_n,Z_{t_n\wedge\tau_n})\le G(0,1)=E_{Q^*}f(Z_T^*),
$$

that yields (5.13).

It remains to show that D^n is decreasing Q-a.s. In view of (5.9) and (5.10) it is clear that $D^{n,1}=\{D_t^{n,1}\}_{0\le t\le T}$ is decreasing Q-a.s.

Let Γ be the set of all $(\omega,t)\in\Omega\times[0,T]$ for which (5.11) and (5.12) hold simultaneously for all rational w; we have $\lambda\times Q((\Omega\times[0,T])\setminus\Gamma)=0$, where λ is the Lebesgue measure on $[0,T]$. Denote by $\widehat{\mathcal{C}}$ the set of all nonnegative continuous convex functions g on $[-1,\infty)$ such that $g(0)=0$ and $\frac{g(\infty)}{\infty}:=\lim_{x\uparrow\infty}\frac{g(x)}{x}<\infty$. Then for all $(\omega,t)\in\Gamma$

$$
\int_{[-1,\infty)}g(x)K(\omega,t,dx)-\frac{g(\infty)}{\infty}a_t(\omega)
\le\int_{[-1,\infty)}g(x)K^*(t,Z_{t-}(\omega),dx)-\frac{g(\infty)}{\infty}a^*(t,Z_{t-}(\omega)). \tag{5.15}
$$

Indeed, let $\widehat{\mathcal{C}}_n$ be the subset of $\widehat{\mathcal{C}}$ consisting of piecewise linear functions with break points in the set $\{k/n\colon k=\pm1,\pm2,\dots,\pm(n-1),n,n+1,\dots,n^2\}$. In view of (5.11) and (5.12), (5.15) is valid for any $g\in\bigcup_n\widehat{\mathcal{C}}_n$. On the other hand, it is clear that for any function $g\in\widehat{\mathcal{C}}$ one can find a sequence $\{g_n\}_{n\ge1}$ such that $g_n\in\widehat{\mathcal{C}}_n$, $g_n(x)\le g_{n+1}(x)$, $n\ge1$, $\lim_{n\to\infty}g_n(x)=g(x)$ for all $x\in[-1,\infty)$, and $\frac{g(\infty)}{\infty}=\lim_{n\to\infty}\frac{g_n(\infty)}{\infty}$.

Since $G(t,z)$ is bounded, continuous and convex in z, the function $x \rightsquigarrow (\Lambda G)(t,z,x)$ belongs to $\widehat{\mathcal{C}}$ for any t and z. In particular, $x \rightsquigarrow (\Lambda G)(t, Z_{t-}(\omega), x)$ belongs to $\widehat{\mathcal{C}}$ for any t and ω and $\frac{\Lambda G(t,z,\infty)}{\infty} = -zG_z(t,z)$. So, (5.15) gives

$$\int_{[-1,\infty)} (\Lambda G)(t, Z_{t-}(\omega), x) K(\omega, t, dx) + Z_{t-}(\omega) G_z(t, Z_{t-}(\omega)) a_t(\omega)$$
$$\leq \int_{[-1,\infty)} (\Lambda G)(t, Z_{t-}(\omega), x) K^*(t, Z_{t-}(\omega), dx) + Z_{t-}(\omega) G_z(t, Z_{t-}(\omega)) a^*(t, Z_{t-}(\omega))$$

for all $(t,\omega) \in \Gamma$. It follows that $D^{n,2}$ is decreasing Q-a.s. The lemma has been proved.

Remark 5.3 *It follows from the proof that in condition* (C) *one can replace the class $\mathcal{C}_0^2$ by any subclass $\mathcal{C}' \subset \mathcal{C}_0^2$ with the property that the inequality $\int f(x)\,\mu_{\mathbb{E}}(dx) \geq \int f(x)\,\mu_{\widetilde{\mathbb{E}}}(dx)$ for any $f \in \mathcal{C}'$ always implies $\mathbb{E} \succcurlyeq \widetilde{\mathbb{E}}$.*

Remark 5.4 *Of course, the inequalities* (5.10)–(5.12) *in* (D) *are only sufficient for $\mathcal{L}(Z_T \mid Q) \preccurlyeq \mathcal{L}(Z_T^* \mid Q^*)$. We prove, in fact, much more. Informally speaking, for any $t \in [0,T]$ we construct a new process Z^t which coincides with Z on $[0,t]$ and evolves according to* (5.2), (5.4)–(5.6) *on $[t,T]$ starting from Z_t at the moment t. The proof shows that the family $\mathcal{L}(Z_T^t)$ is monotone in t with respect to the partial ordering $\preccurlyeq$.*

Remark 5.5 *To our knowledge, this type of arguments was introduced in Bellamy and Jeanblanc* (2000), *where Z^* is a diffusion and Z is a diffusion with jumps, and used afterwards in Jakubenas* (1998) *in the context of Lévy processes, see also El Karoui et al.* (1998).

5.2. We want now to examine some examples where the hypotheses (B) and (C) on Z^* in Comparison Lemma hold. For notational simplicity we drop the $*$ in this subsection.

5.2.1. Processes with independent increments

Assume that X is a process with independent increments (from now on PII), with deterministic characteristics $(B(\phi), C, \nu)$, ν is concentrated on $[0,T] \times [-1,\infty)$ and the process A defined by (3.13) is non-increasing to ensure that $Z = \mathcal{E}(X)$ is a nonnegative supermartingale. We suppose that X is locally homogeneous in the sense of § 3.4 in Skorokhod (1991). This amounts to say that there exist a non-positive function $a = a(t)$, a nonnegative function $c = c(t)$ and a transition kernel $K = K(t, dx)$ such that

$$A_t = \int_0^t a(s)\, ds, \qquad C_t = \int_0^t c(s)\, ds, \qquad \nu(dt, dx) = K(t, dx)\, dt.$$

Given $t \in [0, T]$ denote by $\tilde{X} = \{\tilde{X}_u\}_{0 \le u \le T}$ the PII given by

$$\tilde{X}_u = \begin{cases} 0, & \text{if} \quad 0 \le u \le t, \\ X_u - X_t, & \text{if} \quad t < u \le T. \end{cases} \tag{5.16}$$

In accordance with (5.7) define

$$G(t, z) = E(f(Z_T) \mid Z_t = z) = E[f(z\mathcal{E}(\tilde{X})_T)], \qquad f \in \mathcal{C}_0^2. \tag{5.17}$$

The most part of the properties of G required in (C) including convexity in z are immediate from (5.17); only the pricing equation (5.8) needs to be justified. First assume that Z is strictly positive. Then put $\overline{X}_t = \log Z_t$. The process $\overline{X} = \{\overline{X}_t\}_{0 \le t \le T}$ is a PII with the characteristics given by (3.3). Since $G(t, z) = Ef\left(ze^{\overline{X}_T - \overline{X}_t}\right) = Eh(\log z + \overline{X}_T - \overline{X}_t)$, where $h(x) = f(e^x)$, $x \in \mathbb{R}$, one can apply Theorem 25 in Skorokhod (1991, p. 161) to get the equation (3.55) ibid-em. After the corresponding change of variables and using (3.3) and (3.13), one obtains

$$G_t(t, z) + za(t)G_z(t, z) + \frac{1}{2} z^2 c(t) G_{zz}(t, z) + \int_{[-1,\infty)} (\Lambda G)(t, z, x) K(t, dx) = 0. \tag{5.18}$$

In the general case, when Z can vanish, let us take a truncation function $\phi(x)$ with $\phi(-1) = 0$ and define the processes $X^{(1)} = \{X_t^{(1)}\}_{0 \le t \le T}$ and $X^{(2)} = \{X_t^{(2)}\}_{0 \le t \le T}$ as independent PII with characteristics $(B(\phi), C, \mathbf{1}_{[0,T]\times(-1,\infty)}\nu)$ and $(0, 0, \mathbf{1}_{[0,T]\times\{-1\}}\nu)$ respectively defined on some probability space. It is clear that $X \overset{d}{=} X^{(1)} + X^{(2)}$; moreover, $\tilde{X} \overset{d}{=} \tilde{X}^{(1)} + \tilde{X}^{(2)}$ and $\mathcal{E}(\tilde{X}) \overset{d}{=} \mathcal{E}(\tilde{X}^{(1)} + \tilde{X}^{(2)}) = \mathcal{E}(\tilde{X}^{(1)})\mathcal{E}(\tilde{X}^{(2)})$ defining $\tilde{X}^{(i)}$ as in (5.16) for $i = 1, 2$. The process $\mathcal{E}(\tilde{X}^{(2)})$ takes only values 1 and 0, and $P(\mathcal{E}(\tilde{X}^{(2)})_T = 1) = \exp\left(-\int_t^T K(s, \{-1\})\, ds\right)$. Therefore,

$$\begin{aligned} G(t, z) \quad &= \quad G^{(1)}(t, z) \exp\left(-\int_t^T K(s, \{-1\})\, ds\right) \\ &\qquad + f(0)\left\{1 - \exp\left(-\int_t^T K(s, \{-1\})\, ds\right)\right\}, \end{aligned}$$

where $G^{(1)}(t,z) = Ef(z\mathcal{E}(\tilde{X}^{(1)})_T)$. Since $X^{(1)}$ is strictly positive, we already know the pricing equation for $G^{(1)}$: replace $a(t)$ by $a(t) - K(t, \{-1\})$ and $\int_{[-1,\infty)}$ by $\int_{(-1,\infty)}$ in (5.18). Now the pricing equation (5.18) for G follows from the previous formula.

5.2.2. Diffusions

Assume now for Z a dynamics given by

$$dZ_t = Z_t \sigma(t, Z_t) dW_t, \quad 0 \le t \le T, \quad Z_0 = 1,$$

where W is a standard Wiener process on a stochastic basis $\mathcal{B}$. Assume that the function $\sigma(t,z)$ is continuous in (t,z), and bounded from above, i.e.

$$\sigma(t,z) \le \sigma_1 < \infty.$$

Assume that the function $\frac{\partial}{\partial z}(z\sigma(t,z))$ is continuous for all (t,z), and Lipschitz continuous and bounded in $z \in (0,\infty)$ uniformly in $t \in [0,T]$. Under these conditions, propagation of convexity (5.9) holds (see El Karoui et al. (1998)).
We give a proof of this fact for the case of time-homogeneous diffusions, i.e. when σ does not depend on t. Assume then that

$$dZ_t = Z_t \sigma(Z_t) dW_t, \quad 0 \le t \le T, \quad Z_0 = 1, \tag{5.19}$$

with $\sigma = \sigma(z)$ is continuous, and the function $z\sigma(z)$ is Lipschitz (or satisfies a local Lipschitz condition). Under this hypothesis, there exists a unique strong solution to (5.19), see Theorem IV.3.1 in Ikeda and Watanabe (1981). Let $P(x,t,\Gamma)$ be the corresponding transition probability function. Define G according to (5.7):

$$G(t,z) = \int f(x) P(z, T-t, dx), \qquad f \in \mathcal{C}_0^2.$$

Then the backward Kolmogorov equation is

$$G_t(t,z) + \frac{1}{2} z^2 \sigma^2(z) G_{zz}(t,z) = 0, \tag{5.20}$$

see e.g. Shiryaev (1999, Chapter III, § 3f).
According to (5.20), in order to check the convexity of G in z for a fixed t, it is sufficient to show that G is decreasing in t for each fixed z. Consider $0 \le t \le$

$t+h \le T$. Using the Kolmogorov–Chapman equation, Jensen's inequality and the martingale property of Z, we get

$$\begin{aligned} G(t,z) &= \int f(x)P(z,T-t,dx) = \int\int f(x)P(y,h,dx)P(z,T-t-h,dy) \\ &\ge \int f\left(\int xP(y,h,dx)\right) P(z,T-t-h,dy) = \int f(y)P(z,T-t-h,dy) \\ &= G(t+h,z). \end{aligned}$$

5.3. In this subsection we return to the initial setting of Section 1. We then assume that $Z = \{Z_t\}_{0\le t\le T}$, $Z_0 = 1$, is a strictly positive semimartingale on a stochastic basis $\mathcal{B} = (\Omega, \mathcal{F}_T, \{\mathcal{F}_t\}_{0\le t\le T}, P)$, $\mathcal{P} \neq \emptyset$ is the set of all equivalent martingale measures for Z. The process $Z^* = \{Z^*_t\}_{0\le t\le T}$, $Z^*_0 = 1$, defined on a stochastic basis $\mathcal{B}^* = (\Omega^*, \mathcal{F}^*_T, \{\mathcal{F}^*_t\}_{0\le t\le T}, Q^*)$ is the same as in subsection 5.1, in particular, Z^* is a nonnegative supermartingale satisfying (5.2), and (5.4)–(5.6) hold.
Here $(B(\phi), C, \nu)$ is the triplet of local characteristics of X, where $Z = \mathcal{E}(X)$, relative to the measure P, while the triplet of X with respect to $Q \in \mathcal{P}$ is denoted by $(B^Q(\phi), C^Q, \nu^Q)$. We assume that

$$C_t(\omega) = \int_0^t c_s(\omega)\, ds, \qquad \nu(\omega, dt, dx) = K(\omega, t, dx)\, dt, \tag{5.21}$$

where $c = \{c_t\}_{0\le t\le T}$ is a predictable nonnegative process and $K = K(\omega, t, dx)$ is a transition kernel satisfying the standard assumptions. Then there is a predictable process $b = \{b_t\}_{0\le t\le T}$ such that

$$B_t(\phi)(\omega) = \int_0^t b_s(\omega)\, ds \tag{5.22}$$

(otherwise $\mathcal{P} = \emptyset$ as can be easily seen from (3.5) and (3.7)). Using the notation as in (3.4) and (3.5), we see that for any $Q \in \mathcal{P}$

$$C^Q_t(\omega) = \int_0^t c^Q_s(\omega)\, ds, \qquad \nu^Q(\omega, dt, dx) = K^Q(\omega, t, dx)\, dt, \qquad P\text{-a.s.}, \tag{5.23}$$

where

$$c^Q_t(\omega) = c_t(\omega), \qquad K^Q(\omega, t, dx) = Y^Q(\omega, t, x)K(\omega, t, dx). \tag{5.24}$$

Note also that X is a Q-local martingale for any $Q \in \mathcal{P}$, hence the corresponding process A^Q is equal to 0.

Our goal here is to describe some particular cases where the inequalities (5.10)–(5.12) (or the opposite inequalities) with c, K, a replaced by c^Q, K^Q, 0 respectively hold for any $Q \in \mathcal{P}$. More precisely, we are interested in the situation when, for every $Q \in \mathcal{P}$,

$$c_t^Q(\omega) \le c^*(t, Z_{t-}(\omega)) \qquad dt \times dQ\text{-a.e.,} \tag{5.25}$$

$$\int_{[-1,\infty)} (w-x)^+ K^Q(\omega, t, dx) \le \int_{[-1,\infty)} (w-x)^+ K^*(t, Z_{t-}(\omega), dx) \qquad dt \times dQ\text{-a.e.} \tag{5.26}$$

for any $w \in (-1, 0)$,

$$\int_{[-1,\infty)} (x-w)^+ K(\omega, t, dx) \\ \le \int_{[-1,\infty)} (x-w)^+ K^*(t, Z_{t-}(\omega), dx) - a^*(t, Z_{t-}(\omega)) \quad dt \times dQ\text{-a.e.} \tag{5.27}$$

for any $w \in (0, \infty)$, or

$$c_t^Q(\omega) \ge c^*(t, Z_{t-}(\omega)) \qquad dt \times dQ\text{-a.e.,} \tag{5.28}$$

$$\int_{[-1,\infty)} (w-x)^+ K^Q(\omega, t, dx) \ge \int_{[-1,\infty)} (w-x)^+ K^*(t, Z_{t-}(\omega), dx) \qquad dt \times dQ\text{-a.e.} \tag{5.29}$$

for any $w \in (-1, 0)$,

$$\int_{[-1,\infty)} (x-w)^+ K(\omega, t, dx) \\ \ge \int_{[-1,\infty)} (x-w)^+ K^*(t, Z_{t-}(\omega), dx) - a^*(t, Z_{t-}(\omega)) \quad dt \times dQ\text{-a.e.} \tag{5.30}$$

for any $w \in (0, \infty)$. The attainability of the corresponding bounds is not discussed. The first two cases are easy consequences of (5.24).

1) Let X^* be a continuous Q^*-local martingale (i.e. $\nu^* = 0$ and $A^* = 0$). If

$$c_t(\omega) \ge c^*(t, Z_{t-}(\omega)) \qquad dt \times dP\text{-a.e.,}$$

then we have (5.28)–(5.30) for every $Q \in \mathcal{P}$.

2) Let X be a continuous process (i.e. $\nu = 0$). If

$$c_t(\omega) \le c^*(t, Z_{t-}(\omega)) \qquad dt \times dP\text{-a.e.,}$$

then we have (5.25)–(5.27) for every $Q \in \mathcal{P}$.
Assume now that X is a process of finite variation. Then (cf. subsection 4.2)

$$c = 0 \qquad \text{and} \qquad \int (|x| \wedge 1) K(\omega, t, dx) < \infty \qquad dt \times dP\text{-a.e.}$$

Define $B' = X - \sum_{s \leq \cdot} \Delta X_s$. From (4.6), we obtain

$$B'_t(\omega) = \int_0^t b'_s(\omega) ds, \qquad b'_t(\omega) = b_t(\omega) - \int \phi(x) K(\omega, t, dx).$$

Moreover, for any $Q \in \mathcal{P}$ we have (4.7) and (4.8), hence

$$\int |x| K^Q(\omega, t, dx) < \infty \qquad \text{and} \qquad \int x K^Q(\omega, t, dx) = -b'_t(\omega) \qquad dt \times dP\text{-a.e.}$$

The last equality is crucial for the remaining cases.
3) Let X be a process of finite variation and P-a.s. $\Delta X \geq 0$. If

$$-b'_t(\omega) \leq a^*(t, Z_{t-}(\omega)) \qquad dt \times dP\text{-a.e.},$$

then we have (5.25)–(5.27) for every $Q \in \mathcal{P}$. Indeed, (5.25) and (5.26) are trivial and, for $w > 0$,

$$\begin{aligned} \int_{[-1,\infty)} (x - w)^+ K^Q(\omega, t, dx) &= \int_{(0,\infty)} (x - w)^+ K^Q(\omega, t, dx) \\ &\leq \int_{(0,\infty)} x K^Q(\omega, t, dx) = -b'_t(\omega) \leq a^*(t, Z_{t-}(\omega)) \end{aligned}$$

$dt \times dP$-a.e. and $dt \times dQ$-a.e.
4) Let X be a process of finite variation and P-a.s. $0 \leq \Delta X \leq L$ for some $L < \infty$. If

$$-b'_t(\omega) \leq L K^*(t, Z_{t-}(\omega), \{L\}) \qquad dt \times dP\text{-a.e.},$$

then we have (5.25)–(5.27) for every $Q \in \mathcal{P}$. Indeed, only (5.27) for $w \in (0, L)$ has to be checked. Since $(x - w)^+ \leq L^{-1}(L - w)x$, $x \in [0, L]$, we obtain

$$\begin{aligned} \int_{[-1,\infty)} (x - w)^+ K^Q(\omega, t, dx) &= \int_{(0,L]} (x - w)^+ K^Q(\omega, t, dx) \\ &\leq \frac{L - w}{L} \int_{(0,L]} x K^Q(\omega, t, dx) = -\frac{L - w}{L} b'_t(\omega) \leq (L - w) K^*(t, Z_{t-}(\omega), \{L\}) \\ &\leq \int_{[-1,\infty)} (x - w)^+ K^*(t, Z_{t-}(\omega), dx) \end{aligned}$$

$dt \times dQ$-a.e.

5) Let X be a process of finite variation and P-a.s. $-L \le \Delta X \le 0$ for some $L \in (0,1]$. If

$$b'_t(\omega) \le LK^*(t, Z_{t-}(\omega), \{-L\}) \qquad dt \times dP\text{-a.e.},$$

then we have (5.25)–(5.27) for every $Q \in \mathcal{P}$. The proof is similar to the previous case.

6) Assume that X is a process of finite variation, there is a number $\delta > 0$ such that $P\{\exists t\colon \Delta X_t \in (0,\delta)\} = 0$, and X^* is a local martingale of finite variation with jumps of a fixed size δ. If

$$-b'_t(\omega) \ge \delta K^*(t, Z_{t-}(\omega), \{\delta\}) \qquad dt \times dP\text{-a.e.},$$

then we have (5.28)–(5.30) for every $Q \in \mathcal{P}$. Indeed, (5.28) and (5.29) are trivial, and, for $w \in (0,\delta)$,

$$\begin{aligned}
\int_{[-1,\infty)} (x-w)^+ K^Q(\omega,t,dx) &= \int_{[\delta,\infty)} (x-w)^+ K^Q(\omega,t,dx) \\
&\ge \int_{[\delta,\infty)} (x-w) K^Q(\omega,t,dx) \ge (1-\frac{w}{\delta}) \int_{[\delta,\infty)} x K^Q(\omega,t,dx) \\
&\ge (1-\frac{w}{\delta}) \int_{[-1,\infty)} x K^Q(\omega,t,dx) = -(1-\frac{w}{\delta}) b'_t(\omega) \\
&\ge (\delta - w) K^*(t, Z_{t-}(\omega), \{\delta\}) = \int_{[-1,\infty)} (x-w)^+ K^*(t, Z_{t-}(\omega), dx)
\end{aligned}$$

$dt \times dQ$-a.e.

7) Assume that X is a process of finite variation, there is a number $\delta \in (0,1)$ such that $P\{\exists t\colon \Delta X_t \in (-\delta,0)\} = 0$, and X^* is a local martingale of finite variation with jumps of a fixed size $-\delta$. If

$$b'_t(\omega) \ge \delta K^*(t, Z_{t-}(\omega), \{-\delta\}) \qquad dt \times dP\text{-a.e.},$$

then we have (5.28)–(5.30) for every $Q \in \mathcal{P}$. The proof is similar to the previous case.

5.4. In this subsection we show that the conditions (5.10)–(5.12) in the Comparison Lemma are only sufficient even if the local characteristic of X and X^* are homogeneous in time and, moreover, deterministic, i.e. if X and X^* are Lévy processes.

Assume then that $X = \{X_t\}_{0\le t<\infty}$ is a Lévy process on a stochastic basis $\mathcal{B} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t<\infty}, P)$; moreover, X is a supermartingale and $\Delta X \ge -1$. Then its characteristics $(B(\phi), C, \nu)$ can be chosen in the form

$$B_t(\phi) = b(\phi)t, \qquad C_t = ct, \qquad \nu(dt, dx) = dt\, F(dx), \tag{5.31}$$

where $b(\phi) \in \mathbb{R}$, $c \in \mathbb{R}_+$, F is a σ-finite measure on $[-1,\infty) \setminus \{0\}$, which integrates $x^2 \wedge 1$, and $a := -b(\phi) - \int(x - \phi(x))F(dx) \ge 0$, cf. (3.13). Conversely, if the triplet $(b(\phi), c, F)$ satisfies these conditions, then there is a Lévy process X with the characteristics as in (5.31), which is a supermartingale with $\Delta X \ge 1$.
Let $Z = \mathcal{E}(X)$ and define a statistical experiment $\mathbb{E}_t$, $t \ge 0$, by $\mu_{\mathbb{E}_t} = \mu_t := \mathcal{L}(Z_t \mid P)$. The statistical experiment $\mathbb{E}_t$ is infinitely divisible (see e.g. Janssen et al. (1985)). It is not difficult to show that if $\mathbb{E}$ is an infinitely divisible binary experiment and $\mathbb{E}$ is not totally informative, then one can find the triplet $(b(\phi), c, F)$ with the above properties such that $\mathbb{E} = \mathbb{E}_1$.
Similarly, we assume that $X^* = \{X^*_t\}_{0\le t<\infty}$ is a Lévy process and a supermartingale with $\Delta X^* \ge -1$ (without loss of generality, we may suppose that X^* is given on the same stochastic basis). The corresponding triplet is denoted by $(b^*(\phi), c^*, F^*)$, $a^* := b^*(\phi) + \int(x - \phi(x))F^*(dx)$. Put $Z^* = \mathcal{E}(X^*)$ and define $\mu^*_t = \mathcal{L}(Z^*_t \mid P)$.
It follows from Comparison Lemma and considerations in subsection 5.2 that $\mu_t \preccurlyeq \mu^*_t$ for every $t > 0$ if

$$c \le c^*, \tag{5.32}$$

$$\int_{[-1,\infty)} (w - x)^+ F(dx) \le \int_{[-1,\infty)} (w - x)^+ F^*(dx) \qquad \text{for any} \quad w \in (-1, 0), \tag{5.33}$$

$$\int_{[-1,\infty)} (x - w)^+ F(dx) - a \le \int_{[-1,\infty)} (x - w)^+ F^*(dx) - a^* \qquad \text{for any} \quad w \in (0, \infty). \tag{5.34}$$

We now prove that the reciprocal of this last statement is true in the context of Lévy processes. Assume $\mu_t \preccurlyeq \mu^*_t$ for all $t > 0$ (or for a sequence $\{t_n\}$ with $t_n \to 0$). Then (5.32)–(5.34) are satisfied. Indeed, let $f \in \mathcal{C}^2_0$. Using arguments based on Itô's formula, or the ones that we used deducing the pricing equation, one can show that

$$\lim_{t\downarrow 0} \frac{Ef(Z_t) - 1}{t} = af'(1) + \frac{1}{2}cf''(1) + \int_{[-1,\infty)} (f(1 + x) - f(1) - f'(1)x)\, F(dx).$$

A similar formula holds for Z^*. Therefore, if $\mu_{t_n} \preccurlyeq \mu^*_{t_n}$ for a sequence $\{t_n\}$ converging to 0 then

$$af'(1) + \frac{1}{2}cf''(1) + \int_{[-1,\infty)} (f(1+x) - f(1) - f'(1)x)\, F(dx) \leq$$
$$\leq a^* f'(1) + \frac{1}{2}c^* f''(1) + \int_{[-1,\infty)} (f(1+x) - f(1) - f'(1)x)\, F^*(dx) \quad (5.35)$$

for any $f \in \mathcal{C}_0^2$. Smoothing $f(x) = (1 + w - x)^+$ near the break point, we come to the inequalities (5.33) and (5.34). Applying (5.35) for

$$f_n(x) = \begin{cases} \frac{2}{n}(1-x), & \text{if} \quad 0 \leq x < 1 - \frac{\pi}{2n}, \\ \frac{1}{n}(1 - x + \frac{\pi}{2n}) - \frac{1}{n^2}\cos n(x-1), & \text{if} \quad 1 - \frac{\pi}{2n} \leq x \leq 1 + \frac{\pi}{2n}, \\ 0, & \text{if} \quad x > 1 + \frac{\pi}{2n}, \end{cases}$$

and passing to the limit as $n \to \infty$, we get (5.32).
However, it may happen that $\mu_t \preccurlyeq \mu^*_t$ for a fixed $t > 0$ but the conditions (5.32)–(5.34) are not satisfied. Let us take a truncation function ϕ such that $\phi(x) = x$ on $[-1, e-1]$ and put $b(\phi) = 0$, $c = \sigma^2$, $F = 0$, $b^*(\phi) = c^* = 0$, $F^* = \delta_{\{e^{-1}-1\}} + \delta_{\{e-1\}}$, where $\sigma > 0$ is not fixed yet. Then $a = a^* = 0$, and for any $\sigma > 0$ inequality (5.32) does not hold. To emphasize the dependence of the first model on σ, we write μ^σ_t and Z^σ_t instead of μ_t and Z_t.
Let us fix t, say, $t = 1$. We assert that $\mu^\sigma_1 \preccurlyeq \mu^*_1$ if σ is small enough. To prove it, we first note, cf. (3.2) and (3.3), that

$$Z^\sigma_t = \exp(\sigma W_t - \frac{\sigma^2}{2}t) \qquad \text{and} \qquad Z^*_t = \exp(\pi^{(1)}_t - \pi^{(2)}_t - \gamma t), \quad \gamma = e + e^{-1} - 2,$$

where W is a standard Wiener process, $\pi^{(1)}$ and $\pi^{(2)}$ are independent Poisson processes with intensity 1. Let $G_\sigma(x) = P(Z^\sigma_1 \leq x)$ and $G^*(x) = P(Z^*_1 \leq x)$. We have

$$G_1(x) = P(W_1 \leq \log x + 1/2) = P(W_1 \geq -\log x - 1/2),$$
$$1 - G_1(x) = P(W_1 > \log x + 1/2),$$

$$\begin{aligned} G^*(x) &= P(\pi^{(1)}_1 - \pi^{(2)}_1 \leq \log x + \gamma) \geq P(\pi^{(1)}_1 = 0)P(\pi^{(2)}_1 \geq -\log x - \gamma) \\ &= e^{-1}P(\pi^{(2)}_1 \geq -\log x - \gamma), \end{aligned}$$

$$1 - G^*(x) = P(\pi_1^{(1)} - \pi_1^{(2)} > \log x + \gamma) \geq e^{-1} P(\pi_1^{(1)} > \log x + \gamma).$$

Since Gaussian tails decrease faster than Poisson tails,

$$\lim_{x \downarrow 0} \frac{G_1(x)}{G^*(x)} = \lim_{x \uparrow \infty} \frac{1 - G_1(x)}{1 - G^*(x)} = 0.$$

Therefore,

$$\lim_{y \downarrow 0} \frac{\int_0^y G_1(x)\, dx}{\int_0^y G^*(x)\, dx} = \lim_{y \uparrow \infty} \frac{\int_y^\infty [1 - G_1(x)]\, dx}{\int_y^\infty [1 - G^*(x)]\, dx} = 0, \tag{5.36}$$

since

$$\int_0^\infty [1 - G_1(x)]\, dx = \int_0^\infty [1 - G^*(x)]\, dx = 1. \tag{5.37}$$

In view of (5.36) and (5.37), there are numbers $0 < \delta < L < \infty$ such that

$$\int_0^y G_1(x)\, dx \leq \int_0^y G^*(x)\, dx \qquad \text{for any} \quad y \in [0, \delta) \cup (L, \infty). \tag{5.38}$$

On the other hand, it is easy to see that $\mu_1^\sigma \preccurlyeq \mu_1^{\sigma'}$ if $\sigma \leq \sigma'$ and $\mu_1^\sigma \Rightarrow \delta_{\{1\}}$ as $\sigma \downarrow 0$. Hence $\lim_{\sigma \downarrow 0} G_\sigma(x) = \begin{cases} 0, & x < 1, \\ 1, & x > 1, \end{cases}$ and $\int_0^y G_\sigma(x)\, dx$ converges to $(y-1)^+$ uniformly in y on any compact interval as $\sigma \downarrow 0$. But $\int_0^y G^*(x)\, dx \geq (y-1)^+$ for any $y > 0$ by Proposition 2.2 and it is easy to see that the equality is not possible. Therefore, there is $\sigma_0 \leq 1$ such that

$$\int_0^y G_\sigma(x)\, dx \leq \int_0^y G^*(x)\, dx \qquad \text{for any} \quad y \in [\delta, L]$$

if $0 < \sigma \leq \sigma_0$. In view of Proposition 2.2 and (5.38) we have

$$\int_0^y G_\sigma(x)\, dx \leq \int_0^y G_1(x)\, dx \leq \int_0^y G^*(x)\, dx \qquad \text{for any} \quad y \in [0, \delta) \cup (L, \infty)$$

if $\sigma \leq 1$. Combining two last inequalities and using Proposition 2.2, we obtain $\mu_1^\sigma \preccurlyeq \mu_1^*$ if $0 < \sigma \leq \sigma_0$.

6 Attainability of non-trivial bounds

In Section 5 we have considered a certain machinery which can be useful for realizing the first step of our method in concrete models. Roughly speaking, the corresponding distributions are comparable if the triplets of local characteristics are comparable in a certain sense. The second step of the method proposed consists in finding a sequence of measures Q^n in $\mathcal{M}$ such that $\mu_{Q^n} \Rightarrow \mu^*$, where μ^* is a candidate to be the greatest lower or the least upper bound for the family $\{\mu_Q : Q \in \mathcal{M}\}$. Of course, it is natural to expect this convergence if we have a convergence of the corresponding triplets in the right sense. The aim of this section is to adapt some of known theorems on the weak convergence of likelihood processes, see e.g. Jacod and Shiryaev (1987), Jacod (1989), Coquet and Jacod (1990), Kramkov (1993), to the option pricing context in accordance with this point of view.

Throughout the section we assume that $Z = \mathcal{E}(X)$, $X_0 = 0$, is a strictly positive semimartingale on a stochastic basis $(\Omega, \mathcal{F}_T, \{\mathcal{F}_t\}_{0\le t\le T}, P)$, $\{Q^n\}_{n\ge 1}$ is a sequence of equivalent probability measures on $(\Omega, \mathcal{F}_T)$, i.e. $Q^n \sim P$ and Z is a Q^n-martingale for every $n \ge 1$. The triplets of X with respect to Q^n are denoted by $(B^n(\phi), C, \nu^n)$ (the second characteristics do not depend on n in view of (3.5)); they satisfy (3.6) and (3.7). The truncation function ϕ is assumed to be continuous.

The limiting measure μ^* is described in the spirit of the previous section by $\mu^* = \mathcal{L}(Z_T^* \mid Q^*)$, where Z^* is a nonnegative supermartingale on a stochastic basis $\mathcal{B}^* = (\Omega^*, \mathcal{F}_T^*, \{\mathcal{F}_t^*\}_{0\le t\le T}, Q^*)$, representable as $Z^* = \mathcal{E}(X^*)$, where X^* is a local supermartingale with $\Delta X^* \ge -1$. We shall consider separately two cases: X^* is a PII and Z^* is a diffusion.

6.1. We assume here that X^* is a stochastically continuous process with independent increments with a deterministic triplet $(B^*(\phi), C^*, \nu^*)$; $A^* := -B^*(\phi) - (x - \phi(x)) * \nu^*$.

Proposition 6.1 *Under the above assumptions, let*

$$\sup_{t\le T} \int_{\{|x|>\varepsilon\}} |x| \nu^n(\{t\} \times dx) \xrightarrow{Q^n} 0 \qquad \textit{for all} \quad \varepsilon > 0, \tag{6.1}$$

$$C_T + \phi^2(x) * \nu_T^n \xrightarrow{Q^n} C_T^* + \phi^2(x) * \nu_T^*, \tag{6.2}$$

$$(x - \phi(x)) * \nu_T^n \xrightarrow{Q^n} (x - \phi(x)) * \nu_T^* + A_T^*, \tag{6.3}$$

$$g(x) * \nu_T^n \xrightarrow{Q^n} g(x) * \nu_T^* \tag{6.4}$$

for any continuous bounded function $g: [-1, \infty) \to \mathbb{R}$ *vanishing in a neighborhood of* 0. *Then* $\mathcal{L}(Z_T \mid Q^n) \Rightarrow \mathcal{L}(Z_T^* \mid Q^*)$.

Remark 6.1 *Under* (6.4), *conditions* (6.2) *and* (6.3) *do not depend on the choice of a continuous truncation function* ϕ.

Remark 6.2 *If* Z^* *is a strictly positive martingale, this proposition is an immediate consequence of Theorem* X.2.12 *in Jacod and Shiryaev* (1987). *The method of the proof of that theorem* (*to check the weak convergence of the terminal value of* $\log Z$ *under* Q^n *to the terminal value of the PII* $\log Z^*$ *under* Q^*) *can be applied also in the case where* Z^* *is a strictly positive supermartingale. The limiting process* Z^* *under our hypotheses was considered in Coquet and Jacod* (1990, *Theorem* 3.6 *and Lemma* 3.11), *where only the functional convergence was proved under more restrictive assumptions.*

Proof. We give only a sketch of the proof. It is easy to check that under (6.2) and (6.3), (6.4) implies $g(x) * \nu_T^n \xrightarrow{Q^n} g(x) * \nu_T^*$ for any continuous bounded function $g\colon [-1,\infty) \to \mathbb{R}$ satisfying $g(x) = o(x^2)$ as $x \to 0$ and $g(x) = o(x)$ as $x \to +\infty$. Hence,

$$h_T^n(\alpha) \xrightarrow{Q^n} h_T^*(\alpha), \tag{6.5}$$

where $\alpha \in (0,1)$, $h^n(\alpha) = \frac{\alpha(1-\alpha)}{2}C + \{\alpha + (1-\alpha)(1+x) - (1+x)^{1-\alpha}\} * \nu^n$ is the Hellinger process of order α for Q^n and $\overline{Q^n}$ (see subsection 3.3), $d\overline{Q^n} = Z_T dQ^n$, $h^*(\alpha) = (1-\alpha)A^* + \frac{\alpha(1-\alpha)}{2}C^* + \{\alpha + (1-\alpha)(1+x) - (1+x)^{1-\alpha}\} * \nu^*$. It is easy to show using Itô's formula that the Hellinger integrals $H(\alpha; Q^*, \overline{Q^*})$, where $d\overline{Q^*} = Z_T^* dQ^*$, satisfy

$$H(\alpha; Q^*, \overline{Q^*}) = \mathcal{E}(-h^*(\alpha))_T = \exp\left(-h_T^*(\alpha)\right), \tag{6.6}$$

the last equality follows from stochastic continuity of X^*.
On the other hand, since $\nu(\{t\} \times \mathbb{R}) \le 1$, it follows from (6.1) that

$$\sup_{t \le T} |\Delta h_t^n(\alpha)| \xrightarrow{Q^n} 0. \tag{6.7}$$

Combining (6.5)–(6.7), we get

$$\mathcal{E}(-h^n(\alpha))_T \xrightarrow{Q^n} \mathcal{E}(-h^*(\alpha))_T = H(\alpha; Q^*, \overline{Q^*})$$

for any $\alpha \in (0,1)$. Following the same lines as in the proof of Theorem 3.1 in Jacod (1989), one can show that the last relation implies

$$\lim_{n\to\infty} H(\alpha; Q^n, \overline{Q^n}) = H(\alpha; Q^*, \overline{Q^*}), \qquad \alpha \in (0,1),$$

and the claim follows from Proposition 2.5.

6.2. Let us fix some notation. We denote by $\mathbb{D}(\mathbb{R}^d)$ the Skorokhod space $\mathbb{D}([0,T],\mathbb{R}^d)$ of càdlàg functions $[0,T] \ni t \rightsquigarrow \alpha(t) \in \mathbb{R}^d$, equipped with the Skorokhod topology, $\mathcal{D}(\mathbb{R}^d)$ is the Borel σ-algebra in this space, $\{\mathcal{D}_t(\mathbb{R}^d)\}_{0\le t\le T}$ is the filtration generated by the canonical process. The weak convergence in this space is denoted by $\xrightarrow{\mathcal{L}(\mathbb{D}^d)}$. If $d=1$ then the index d is omitted in this notation. A sequence $\{P^n\}$ of probability measures on $(\mathbb{D}(\mathbb{R}^d),\mathcal{D}(\mathbb{R}^d))$ is said to be C-tight if it is tight and any cluster point P of $\{P^n\}$ is concentrated on the subspace of all continuous functions. Given a càdlàg stochastic process $V=\{V_t\}_{0\le t\le T}$ with values in $\mathbb{R}^d$, we denote by $\mathcal{L}(V \mid P)$ its distribution in $\mathbb{D}(\mathbb{R}^d)$ relative to a measure P.

Here we assume that Z^* is a solution to the stochastic differential equation

$$dZ^*_t = Z^*_t\sigma(t,Z^*_t)dW_t, \quad 0\le t\le T, \qquad Z^*_0=1,$$

where W is a standard Wiener process. More precisely, we suppose that σ is a continuous bounded function, $\Omega^*=\mathbb{D}(\mathbb{R})$, $\mathcal{F}^*_T=\mathcal{D}(\mathbb{R})$, $\mathcal{F}^*_t=\mathcal{D}_t(\mathbb{R})$, $Z^*_T(\alpha)=\alpha(t)$ is the canonical process, and Q^* is the *unique* probability measure on $(\Omega^*,\mathcal{F}^*_T)$ under which $Q^*(Z^*_0=1)=1$, Z is an a.s. continuous strictly positive martingale with the quadratic characteristic $\int_0^t (Z^*_{s-})^2 c(s,Z_{s-})\,ds$, where $c(t,x)=\sigma^2(t,x)$.

There are general theorems on the weak convergence of likelihood processes to the limit of our type, see Theorems X.1.59 and X.1.64 in Jacod and Shiryaev (1987) and Theorem 1 in Kramkov (1993), the latter deals even with a more general limiting model. But their assumptions are not well adapted to our case. We propose another result of this type.

Proposition 6.2 *Under the above assumptions, let*

$$\sup_{t\le T}\left|\frac{1}{8}C_T+\frac{1}{2}(1-\sqrt{1+x})^2*\nu^n-\frac{1}{8}\int_0^t c(s,Z_{s-})\,ds\right| \xrightarrow{Q^n} 0, \tag{6.8}$$

and

$$|x|\mathbf{1}_{\{|x|>\varepsilon\}}*\nu^n_T \xrightarrow{Q^n} 0 \qquad \textit{for all} \quad \varepsilon>0. \tag{6.9}$$

Then $\mathcal{L}(Z_T \mid Q^n) \Rightarrow \mathcal{L}(Z^*_T \mid Q^*)$.

Proof. Again, we give only an outline of the proof.

Since the function c is bounded, it follows from (6.8) that the sequences $\{\mathcal{L}(C \mid Q^n)\}$ and $\{\mathcal{L}((x^2 \wedge |x|) * \nu^n \mid Q^n)\}$ are C-tight. Taking into account (3.7) and (6.9), we obtain from Theorem VI.4.18 in Jacod and Shiryaev (1987) that the sequence $\{\mathcal{L}(X \mid Q^n)\}$ is tight in $\mathbb{D}(\mathbb{R})$. Moreover, (6.9) implies that

$$\sup_{t \le T} |\Delta X_t| \xrightarrow{Q^n} 0, \tag{6.10}$$

hence the sequence $\{\mathcal{L}(X \mid Q^n)\}$ is C-tight. Furthermore, Jacod's condition

$$\text{the sequence} \quad \{\mathcal{L}(\mathrm{Var}(B^n(\phi))_T \mid Q^n)\} \quad \text{is tight in } \mathbb{R} \tag{6.11}$$

is satisfied due to (3.7) and (6.8). Let $\mathcal{L}(X \mid Q^{n_k}) \xrightarrow{\mathcal{L}(\mathbb{D})} \mathcal{L}(\hat{X} \mid \hat{Q})$ for a subsequence $\{n_k\}$, where $\hat{Q}$ is a probability measure on $(\mathbb{D}(\mathbb{R}), \mathcal{D}(\mathbb{R}))$ and $\hat{X}$ is the canonical process on $\mathbb{D}(\mathbb{R})$. By Corollary 2.3 and Lemma 3.1 in Jakubowski et al. (1989), $\hat{X}$ is a semimartingale on $(\mathbb{D}(\mathbb{R}), \mathcal{D}(\mathbb{R}), \{\mathcal{D}_t(\mathbb{R})\}_{0 \le t \le T}, \hat{Q})$. Put $\hat{Z} = \mathcal{E}(\hat{X})$. Then $\hat{Z}$ is a $\hat{Q}$-a.s. strictly positive continuous process. In view of (6.11), we have

$$\mathcal{L}(X, [X, X], Z \mid Q^{n_k}) \xrightarrow{\mathcal{L}(\mathbb{D}^3)} \mathcal{L}(\hat{X}, [\hat{X}, \hat{X}], \hat{Z} \mid \hat{Q}), \tag{6.12}$$

see Theorem VI.6.1 in Jacod and Shiryaev (1987) and Proposition 6.1 in Mémin (1985), cf. also Jakubowski et al. (1989). Thus, the sequence $\{\mathcal{L}(Z \mid Q^n)\}$ is tight in $\mathbb{D}(\mathbb{R})$, and we only need to prove that $\mathcal{L}(\hat{Z} \mid \hat{Q}) = Q^*$ every time (6.12) holds. To simplify the notation we shall assume without loss of generality that the subsequence $\{n_k\}$ coincides with the original sequence $\{n\}$. Denote $Q' = \mathcal{L}(\hat{Z} \mid \hat{Q})$. Let us recall that the canonical process on $\mathbb{D}(\mathbb{R})$ is denoted also by Z^* so that

$$\mathcal{L}(Z \mid Q^n) \xrightarrow{\mathcal{L}(\mathbb{D})} \mathcal{L}(Z^* \mid Q') = Q'.$$

Put $d\overline{Q^n} = Z_T dQ^n$ and let h^n be the Hellinger process of order 1/2 for Q^n and $\overline{Q^n}$. In view of (3.10), (6.8) can be rewritten in the form

$$\sup_{t \le T} \left| h^n_T - \frac{1}{8} \int_0^t c(s, Z_{s-})\, ds \right| \xrightarrow{Q^n} 0. \tag{6.13}$$

Hence, the sequence $\{\mathcal{L}(h^n \mid Q^n)\}$ is C-tight, in particular,

$$\sup_{t \le T} |\Delta h^n_t| \xrightarrow{Q^n} 0. \tag{6.14}$$

It follows from (6.13) and (6.14) that

$$\sup_{t\le T} |\,\mathcal{E}(-h^n)_t - \exp(-h^n_t)| \xrightarrow{Q^n} 0. \tag{6.15}$$

Put $Y = \sqrt{Z}$ and $y = \frac{1}{Y_-}\cdot Y$. Then

$$\Delta y = \sqrt{1+\Delta X} - 1, \tag{6.16}$$

in particular,

$$\sup_{t\le T} |\Delta y_t| \xrightarrow{Q^n} 0. \tag{6.17}$$

Since $\mathcal{E}(X) = Z = Y^2 = \mathcal{E}(y)^2$, by Yor's formula

$$X = 2y + [y, y]. \tag{6.18}$$

Put $L^n = Y/\mathcal{E}(-h^n)$. By Proposition V.4.16 in Jacod and Shiryaev (1987), L^n is a Q^n-local martingale. Since c is bounded, it follows from (6.13) and (6.14) that there is a $\delta > 0$ such that $\lim_{n\to\infty} Q^n(\mathcal{E}(-h^n)_T \le \delta) = 0$. Then, similarly to the proof of Theorem 3.1 in Jacod (1989), one can construct a sequence $\{\tau_n\}$ of $\{\mathcal{F}_t\}$-stopping times such that

$$\lim_{n\to\infty} Q^n(\tau_n < T) = 0 \qquad \text{and} \qquad \mathcal{E}(-h^n)_{\tau_n} \ge \delta. \tag{6.19}$$

Since the family $\{\mathcal{L}(Y_{\tau_n\wedge t} \mid Q^n)\}_{0\le t\le T,\, n\ge 1}$ is uniformly integrable, the same is true for the family $\left\{\mathcal{L}(L^n_{\tau_n\wedge t} \mid Q^n)\right\}_{0\le t\le T,\, n\ge 1}$; the same argument with an arbitrary stopping time τ instead of t shows that the stopped process $\left\{L^n_{\tau_n\wedge t}\right\}_{0\le t\le T}$ is a Q^n-martingale. Since

$$L^n_{\tau_n\wedge t} - \sqrt{Z_t}\exp\left(\frac{1}{8}\int_0^t c(s, Z_{s-})\, ds\right) \xrightarrow{Q^n} 0$$

for all $t \in [0, T]$ due to (6.13), (6.15) and (6.19), we may apply Proposition IX.1.12 in Jacod and Shiryaev (1987) to obtain that $L^* := \sqrt{Z^*}\exp\left(\frac{1}{8}\int_0^t c(s, Z^*_{s-})\, ds\right)$ is a martingale on $(\Omega^*, \mathcal{F}^*_T, \{\mathcal{F}^*_t\}, Q')$. In particular, Z^* is a Q'-semimartingale. Put $X^* = \frac{1}{Z^*_-}\cdot Z^*$.

The next step of the proof is to determine the quadratic characteristic $[X^*, X^*]$ of X^* under Q'. To proceed, put

$$m^n = y + h^n. \tag{6.20}$$

It follows from the general definition of the Hellinger process, see Chapter IV, § 1b in Jacod and Shiryaev (1987), that m^n is a Q^n-local martingale. Since X is a Q^n-local martingale, it is easy to show replacing y in (6.18) by its expression from (6.20), that m^n is a Q^n-locally square integrable martingale with the quadratic characteristic (relative to Q^n)

$$\langle m^n, m^n \rangle = 2h^n - [h^n, h^n], \tag{6.21}$$

see also Mémin (1985).

Let $\tilde{\nu}^n$ be the Q^n-compensator of the jump measure of m^n. It follows from (6.16) and (6.9) that the Lindeberg-type condition

$$x^2 \mathbf{1}_{\{|x|>\varepsilon\}} * \tilde{\nu}^n_T \xrightarrow{Q^n} 0 \qquad \text{for all} \quad \varepsilon > 0 \tag{6.22}$$

holds. Using (6.13), (6.21), (6.22) and some standard arguments, see e.g. the proof of Lemma 5 in § 5 of Chapter 5 in Liptser and Shiryaev (1989), one obtains

$$\sup_{t \le T} |[m^n, m^n]_t - \langle m^n, m^n \rangle_t| \xrightarrow{Q^n} 0.$$

Combining this relation with (6.13), (6.14), (6.20) and (6.21), we get

$$\sup_{t \le T} \left| [y, y]_t - \frac{1}{4} \int_0^t c(s, Z_{s-})\, ds \right| \xrightarrow{Q^n} 0. \tag{6.23}$$

Moreover, it can be easily seen from (6.18), (6.17) and (6.23) that

$$\sup_{t \le T} |[X, X]_t - 4[y, y]_t| \xrightarrow{Q^n} 0,$$

hence

$$\sup_{t \le T} \left| [X, X]_t - \int_0^t c(s, Z_{s-})\, ds \right| \xrightarrow{Q^n} 0. \tag{6.24}$$

Now we remark that, evidently,

$$\mathcal{L}(\hat{X}, [\hat{X}, \hat{X}], \hat{Z} \mid \hat{Q}) = \mathcal{L}(X^*, [X^*, X^*], Z^* \mid Q').$$

Combining this statement with (6.12) and (6.24), we obtain

$$[X^*, X^*] = C^* \quad Q'\text{-a.s., where} \quad C^*_t = \int_0^t c(s, Z^*_{s-})\, ds.$$

The rest is easy. The processes Z^*, L^* and C^* are Q'-a.s. continuous, and it follows from the previous formula that

$$[Z^*, Z^*] = (Z^*)^2 \cdot C^*, \tag{6.25}$$

thus it remains to prove that Z^* is a Q'-martingale. By Itô's formula

$$Z^* = (L^*)^2 e^{-C^*/4} = 2\frac{Z^*}{L^*} \cdot L^* - \frac{1}{4} Z^* \cdot C^* + \frac{Z^*}{(L^*)^2} \cdot [L^*, L^*]. \tag{6.26}$$

Taking the quadratic variations in (6.26), we obtain

$$[Z^*, Z^*] = 4\left(\frac{Z^*}{L^*}\right)^2 \cdot [L^*, L^*]. \tag{6.27}$$

Combining (6.25)–(6.27), we obtain

$$Z^* = 2\frac{Z^*}{L^*} \cdot L^*.$$

Since L^* is a Q'-martingale, Z^* is a Q'-local martingale. The proper martingale property follows now from Novikov's criterion, as σ is bounded.

7 Discrete time models

Let us consider a discrete time market model. For notational simplicity, we take $B_n = 1$ for $n = 0, 1, \dots, N$, but the arguments can be applied for any deterministic sequence $\{B_n\}_{0 \le n \le N}$. We assume that the risky asset is modeled by

$$S_0 = 1, \qquad S_n = \prod_{k=1}^{n} (1 + Y_k), \quad n = 1, \dots, N,$$

for $\{Y_k\}_{1 \le k \le N}$ an adapted sequence of random variables on $(\Omega, \mathcal{F}, \mathcal{F} = \{\mathcal{F}_t\}_{0 \le t \le T}, P)$, with Y_{k+1} independent from $\mathcal{F}_k$ for each k, and such that $Y_k > -1$ for all k. Also for notational simplicity we assume $P \in \mathcal{P}$. By independence, denoting μ_k the distribution of Y_k under P, we have

$$\int (1 + x)\, \mu_k(dx) = 1, \quad k = 1, \dots, N.$$

In order to describe the upper and lower distributions, denote $I_k = \operatorname{supp}(\mu_k)$ the support of μ_k and consider also for each k

$$-a_k = \inf I_k, \quad b_k = \sup I_k, \quad -\alpha_k = \sup I_k \cap [-1,0], \quad \beta_k = \inf I_k \cap [0,\infty].$$

Observe that some b_k can take the value ∞. Consider now an auxiliary probability space $(\Omega^0, \mathcal{F}^0, P^0)$ and define on it independent random variables $Y_1^*, \ldots, Y_N^*$ such that $Y_k^* = -a_k$ with probability one if $a_k = b_k = 0$ or $b_k = \infty$, and, otherwise,

$$Y_k^* = \begin{cases} b_k, & \text{with probability } \frac{a_k}{a_k+b_k}, \\ -a_k, & \text{with probability } \frac{b_k}{a_k+b_k}. \end{cases}$$

Similarly, let $Y_{*1}, \ldots, Y_{*N}$ be independent random variables on $(\Omega^0, \mathcal{F}^0, P^0)$ such that $Y_{*k} = 0$ with probability one if $\alpha_k = \beta_k = 0$, and, otherwise,

$$Y_{*k} = \begin{cases} \beta_k, & \text{with probability } \frac{\alpha_k}{\alpha_k+\beta_k}, \\ -\alpha_k, & \text{with probability } \frac{\beta_k}{\alpha_k+\beta_k}. \end{cases}$$

Put $S_0^* = S_{*0} = 1$, $S_n^* = \prod_{k=1}^n (1+Y_k^*)$, $S_{*n} = \prod_{k=1}^n (1+Y_{*k})$, $n = 1, \ldots, N$, and denote by Q^* (resp. Q_*) the law of S_N^* (resp. S_{*N}) under P^0. It is clear that $E_{P^0} S_N^* \le E_{P^0} S_{*N} = 1$.

Proposition 7.1 *Consider the discrete market model described above and a convex pay-off function $f \in \mathcal{C}$ such that $f(0) + \frac{f(\infty)}{\infty} < \infty$. Then*
(a) *The upper price is given by*

$$\mathbb{C}^* = \sup_{Q \in \mathcal{P}} E_Q f(S_N) = \mathcal{J}_f(Q^*) = E_{P^0} f(S_N^*) + \frac{f(\infty)}{\infty}(1 - E_{P^0} S_N^*).$$

(b) *The lower price is given by*

$$\mathbb{C}_* = \inf_{Q \in \mathcal{P}} E_Q f(S_N) = \mathcal{J}_f(Q_*) = E_{P_0} f(S_{*N}).$$

Remark 7.1 *If the σ-algebras $\mathcal{F}_n^0$ are generated by the process $S_* = \{S_{*n}\}_{0 \le n \le N}$, i.e. $\mathcal{F}_n^0 = \sigma\{S_{*0}, \ldots, S_{*n}\}$, the corresponding market is no-arbitrage and complete and $\mathbb{C}_*$ is the fair price of the option $f(S_{*N})$. Similarly, if $\mathcal{F}_n^0$ are generated by the price process $S^* = \{S_n^*\}_{0 \le n \le N}$ and all b_k are finite, then the corresponding market is no-arbitrage and complete and $\mathbb{C}^*$ is the fair price of the option $f(S_N^*)$.*

Proof. First we verify that the bounds hold, i.e. for any $Q \in \mathcal{P}$

$$\mathcal{J}_f(Q_*) \le E_Q f(S_N) \le \mathcal{J}_f(Q^*). \tag{7.1}$$

The arguments are similar to those used in the proof of Comparison Lemma. According to Corollary 2.1 it is sufficient to check (7.1) for a nonnegative non-increasing convex function f.

Let μ be a probability measure on $[-1, \infty)$ such that $\int (1+x)\,\mu(dx) = 1$ and $\operatorname{supp}(\mu) \subseteq I_n = [-a_n, -\alpha_n] \cup [\beta_n, b_n]$. Denote also by μ_n^* and μ_{*n} the distributions of Y_n^* and Y_{*n} respectively under P^0. Let us first show that

$$\int g(1+x)\,\mu_{*n}(dx) \le \int g(1+x)\,\mu(dx) \le \int g(1+x)\,\mu_n^*(dx) \tag{7.2}$$

for any nonnegative non-increasing $g \in \mathcal{C}$. The first inequality follows from Jensen's inequality if $\alpha_n = \beta_n = 0$, the second one is trivial if $a_n = b_n = 0$. In other cases denote by $m(x)$ and $M(x)$ the functions corresponding to the straight lines passing through the points $(-\alpha_n, g(1-\alpha_n))$, $(\beta_n, g(1+\beta_n))$ and $(-a_n, g(1-a_n))$, $(b_n, g(1+b_n))$ respectively (if $b_n = \infty$ put $M(x) \equiv g(-a_n)$). Then

$$m(x) \le g(1+x) \le M(x) \qquad \mu\text{-a.s.}$$

and

$$m(x) = g(1+x) \quad \mu_{*n}\text{-a.s.}, \qquad M(x) = g(1+x) \quad \mu_n^*\text{-a.s.},$$

and (7.2) follows.

Put

$$G^*(n,z) = E_{P^0} f\Big(z \prod_{k=n+1}^{N} (1+Y_k^*)\Big), \quad G_*(n,z) = E_{P^0} f\Big(z \prod_{k=n+1}^{N} (1+Y_{*k})\Big).$$

Let $\nu_n(\omega, dy)$ be a regular conditional probability of Y_n given $\mathcal{F}_{n-1}$ under Q. Then, Q-a.s., $\int (1+y)\,\nu_n(\omega, dy) = 1$ (since $E_Q(1+Y_n \mid \mathcal{F}_{n-1}) = 1$) and $\operatorname{supp}(\nu_n(\omega, dy)) \subseteq I_n$ (since $Q \sim P$). Since $G^*(n,z)$ and $G_*(n,z)$ are convex and non-increasing in z, we obtain from (7.2) that Q-a.s.

$$\begin{aligned} E_Q(G^*(n,S_n) \mid \mathcal{F}_{n-1}) &= \int G^*(n, S_{n-1}(1+z))\nu_n(dz) \\ &\le \int G^*(n, S_{n-1}(1+z))\mu_n^*(dz) = G^*(n-1, S_{n-1}), \end{aligned}$$

where the last equality follows from the independence of Y_n^*:

$$G^*(n-1,z) = E_{P^0} E_{P^0} \Big\{ f\Big(z \prod_{k=n}^{N} (1+Y_k^*)\Big) \;\Big|\; Y_n^* \Big\} = E_{P^0} G(n, z(1+Y_n^*)).$$

Hence $G^*(n, S_n)$ is a Q-supermartingale. Similarly, $G_*(n, S_n)$ is a Q-submartingale. It remains to note that $G^*(0,S_0) = E_{P^0} f(S_N^*) = \mathcal{J}_f(Q^*)$, $G_*(0,S_0) = E_{P^0} f(S_{*N}) = \mathcal{J}_f(Q_*)$, $E_Q G^*(N,S_N) = E_Q G_*(N,S_N) = E_Q f(S_N)$ completing the first step.
Now we check that the obtained bounds are attained. This will be done by finding sequences of measures in the set

$$\mathcal{M} = \{Q \in \mathcal{P} \colon Y_k \text{ and } \mathcal{F}_{k-1} \text{ are independent with respect to } Q,\; k = 1, \dots, n\}. \tag{7.3}$$

Namely, assume that for all k and small $\varepsilon > 0$ we have constructed strictly positive functions $h_{k,\varepsilon}(x)$ defined on the support of μ_k such that

$$\int h_{k,\varepsilon}(x)\, \mu_k(dx) = 1 \qquad \text{and} \qquad \int (1+x) h_{k,\varepsilon}(x)\, \mu_k(dx) = 1. \tag{7.4}$$

Define Q_ε by

$$dQ_\varepsilon = \prod_{k=1}^{N} h_{k,\varepsilon}(Y_k)\, dP.$$

It is clear that $Q_\varepsilon \in \mathcal{M}$. Now, if the distributions $\mu_{k,\varepsilon}$, defined by $\mu_{k,\varepsilon}(dx) = h_{k,\varepsilon}(x)\, \mu_k(dx)$, weakly converge to μ_k^* (resp. μ_{*k}) as $\varepsilon \to 0$, then $\mathcal{L}(S_N \mid Q_\varepsilon)$ weakly converge to Q^* (resp. Q_*), and we have $\lim_{\varepsilon\to 0} E_{Q_\varepsilon} f(S_N) = \mathcal{J}_f(Q^*)$ (resp. $= \mathcal{J}_f(Q_*)$) by Propositions 2.5 and 2.6.
The required functions $h_{k,\varepsilon}$ are constructed as follows (we omit the index k everywhere). In the upper bound case it is sufficient to consider the case $a > 0$ and $b > 0$. Put

$$h_\varepsilon(x) = A(\varepsilon)\mathbf{1}_{[-a,-a+\varepsilon]}(x) + \varepsilon \mathbf{1}_{(-a+\varepsilon, b-\varepsilon)}(x) + B(\varepsilon)\mathbf{1}_{[b-\varepsilon, b]}(x),$$

where $b-\varepsilon$ is replaced by ε^{-1} if $b = \infty$. The positive constants $A(\varepsilon)$ and $B(\varepsilon)$ are chosen in order to (7.4) be satisfied. This can be done if ε is small enough. It is clear that the family $\mu_{k,\varepsilon}$ is asymptotically tight as $\varepsilon \to 0$ and the support of any cluster point is $\{-a\}$ if $b = \infty$ or $\{-a, b\}$ if $b < \infty$; moreover, in the second case ($b < \infty$) the mean of a cluster point is 0. Hence, $\mu_{k,\varepsilon} \Rightarrow \mu_k^*$ as $\varepsilon \to 0$.

In the lower bound case the construction is similar if $\alpha_k > 0$ and $\beta_k > 0$:

$$h_\varepsilon(x) = A(\varepsilon)\mathbf{1}_{[-\alpha-\varepsilon,\alpha]}(x) + \varepsilon\mathbf{1}_{[-a,-\alpha-\varepsilon)\cup(\beta+\varepsilon,b]}(x) + B(\varepsilon)\mathbf{1}_{[\beta,\beta+\varepsilon]}(x),$$

where $A(\varepsilon)$ and $B(\varepsilon)$ are chosen in such a way that (7.4) holds. Finally, if $\alpha = \beta = 0$ (but $a > 0$ and $b > 0$), we use the functions

$$h_\varepsilon(x) = A(\varepsilon)\mathbf{1}_{[-a,-\varepsilon)}(x) + \frac{1-\varepsilon}{\mu([-\varepsilon,\varepsilon])}\mathbf{1}_{[-\varepsilon,\varepsilon]}(x) + B(\varepsilon)\mathbf{1}_{(\varepsilon,b]}(x).$$

If ε is small enough, then one can find positive $A(\varepsilon)$ and $B(\varepsilon)$ so that (7.4) is satisfied, concluding the proof.

A final remark, concerning more general discrete time market models (not necessarily satisfying our independence assumptions), is that if there exists a martingale measure equivalent to the initial measure P belonging to the set $\mathcal{M}$ in (7.3), then the results in Proposition 7.1 are still in force.
Let us illustrate this remark with the conditionally Gaussian model considered in Shataev (1998). In this model, $B_n = 1$ for all $n = 0, \dots, N$, and a vector of independent random variables ε_n $(n = 1, \dots, N)$ with standard Gaussian distribution is considered, $\mathcal{F}_n = \sigma\{\varepsilon_1, \dots, \varepsilon_n\}$. The stock prices evolves according to $S_n = S_0 e^{h_1+\dots+h_n}$, for each $n = 1, \dots, N$, where the conditionally Gaussian random variables h_n satisfy $h_n = \sigma_n\varepsilon_n$, with $\sigma_n = \sigma_n(\varepsilon_1, \dots, \varepsilon_{n-1})$, and σ_n are positive and continuous functions. The joint density of the vector $(h_1, \dots, h_N)$ is given by

$$f(x_1, \dots, x_N) = \prod_{n=1}^{N} \frac{1}{\sqrt{2\pi}\tilde{\sigma}_n(x_1, \dots, x_{n-1})} \exp\Big(\frac{-x_n^2}{2\tilde{\sigma}_n^2(x_1, \dots, x_{n-1})}\Big),$$

where $\tilde{\sigma}_n(x_1, \dots, x_{n-1})$ is defined by $\tilde{\sigma}_n(h_1, \dots, h_{n-1}) = \sigma_n(\varepsilon_1, \dots, \varepsilon_{n-1})$. A martingale measure Q in the set $\mathcal{M}$ can be then constructed with the density of the random vector $(h_1, \dots, h_N)$ given by

$$f^Q(x_1, \dots, x_N) = \prod_{n=1}^{N} \frac{1}{\sqrt{2\pi}} \exp\Big(-\frac{1}{2}(x_n^2 - 1/2)^2\Big).$$

Namely, put $\frac{dQ}{dP} = \frac{f^Q(h_1,\dots,h_N)}{f(h_1,\dots,h_N)}$. Therefore, conclusions of Proposition 7.1 hold. As the support of the distribution of the random variables $Y_n = e^{h_n} - 1$ is the set $(-1, \infty)$, both the upper and lower universal bounds are attained.

8 Diffusion with jumps model

8.1. We now consider a continuous time model. For simplicity we take $B = \{B_t\}_{0\le t\le T}$ with $B_t = 1$. In what respects the risky asset, denoted by $Z = \{Z_t\}_{0\le t\le T}$ we assume that is the solution of the stochastic differential equation (s.d.e.)

$$dZ_t = Z_{t-}\Big[a(t, Z_{t-})dt + \sigma(t, Z_{t-})dW_t + \delta(t, Z_{t-}, x) * (p(dt, dx) - q(dt, dx))\Big], \quad (8.1)$$

and $Z_0 = 1$. The driving terms are

- $W = \{W_t\}_{0\le t\le T}$ a standard Wiener process on $\mathcal{B} = (\Omega, \mathcal{F}_T, \{\mathcal{F}_t\}_{0\le t\le T}, P)$, a stochastic basis;
- p a Poisson random measure on $[0, T] \times \mathbb{R}$ independent of W with intensity $q(dt, dx) = dt\, K(dx)$, on the same stochastic basis $\mathcal{B}$, K is a σ-finite measure on $\mathbb{R}$.

The coefficients $a\colon [0, T] \times (0, \infty) \to \mathbb{R}$, $\sigma\colon [0, T] \times (0, \infty) \to (0, \infty)$, and $\delta\colon [0, T] \times (0, \infty) \times \mathbb{R} \to (-1, \infty)$ are Borel functions, that satisfy:

(i) Local Lipschitz conditions: For each $n \in \mathbb{N}$ there exist $\theta_n > 0$ and $\rho_n\colon \mathbb{R} \to [0, \infty)$ satisfying $\int_{\mathbb{R}} \rho_n^2(x)\, K(dx) < \infty$, such that for each $t \in [0, T]$ and $0 < z, z' \le n$,

$$|za(t, z) - z'a(t, z')| \le \theta_n |z - z'|, \quad (8.2)$$

$$|z\sigma(t, z) - z'\sigma(t, z')| \le \theta_n |z - z'|, \quad (8.3)$$

$$|z\delta(t, z, x) - z'\delta(t, z', x)| \le \rho_n(x)|z - z'|. \quad (8.4)$$

(ii) Boundness conditions: There exist positive constants a_1, σ_0, σ_1 and $\rho\colon \mathbb{R} \to [0, \infty)$ satisfying $\int_{\mathbb{R}} \rho^2(x)\, K(dx) < \infty$, such that for each $t \in [0, T]$ and all $z > 0$,

$$|a(t, z)| \le a_1, \quad (8.5)$$

$$0 < \sigma_0 \le |\sigma(t, z)| \le \sigma_1, \quad (8.6)$$

$$|\delta(t, z, x)| \le \rho(x). \quad (8.7)$$

According to classical results on existence and uniqueness of solutions of s.d.e. (see for instance Jacod (1979), Theorem 14.23), conditions (8.2) to (8.7) ensure the existence of strong solutions of (8.1). Conditions (8.5), (8.6) and (8.7) are in fact slightly more stringent than usual conditions, and allow some useful developments.

We will need some additional assumptions:

(iii) Conditions on volatility: The function $\sigma(t,z)$ is continuous in (t,z). The function $\frac{\partial}{\partial z}(z\sigma(t,z))$ is continuous for all (t,z), and Lipschitz continuous and bounded in $z \in (0,\infty)$ uniformly in $t \in [0,T]$.

(iv) Conditions on jumps: In order to ensure the presence of jumps in the model, assume the existence of a positive constant c_0 such that

$$\int_{[0,T]\times\mathbb{R}} |\delta(t,z,x)|dt\,K(dx) \geq c_0 > 0. \tag{8.8}$$

According to our previous framework, if we introduce $X = \{X_t\}_{0\leq t\leq T}$ given by

$$X_t = \int_0^t a(t,Z_{t-})dt + \int_0^t \sigma(t,Z_{t-})dW_t + \int_{[0,T]\times\mathbb{R}} \delta(t,Z_{t-},x) * (p(dt,dx) - q(dt,dx)), \tag{8.9}$$

then we can write $Z = \mathcal{E}(X)$. Observe that if p jumps at point (t,x), then $\Delta X_t = \delta(t,Z_{t-},x) > -1$, so Z is strictly positive under P.

Equation (8.9) is the canonical decomposition of X as a special semimartingale, and (as in Theorem III.2.26 in Jacod and Shiryaev (1987)) the triplet of predictable characteristics of X under P is

$$\begin{cases} A_t = \int_0^t a(s,Z_{s-})ds, \\ C_t = \int_0^t \sigma(s,Z_{s-})^2 ds, \\ \nu(\omega,dt,B) = \int_{\mathbb{R}} \mathbf{1}_{B\setminus\{0\}}(\delta(t,Z_{t-},x))K(dx)\,dt. \end{cases} \tag{8.10}$$

where B is a Borel set in $\mathbb{R}$. (Here and in (8.14) and (8.17) below it is more convenient to consider the predictable process A of bounded variation in the canonical decomposition of the special semimartingale X instead of the first characteristics $B(\phi)$; A and $B(\phi)$ are connected according to (3.13).) From this it follows that, for $f\colon\mathbb{R}\to\mathbb{R}$, with $f(0)=0$,

$$f(x) * \nu_T = \int_{[0,T]\times\mathbb{R}} f(\delta(t,Z_{t-},x))dt\,K(dx). \tag{8.11}$$

8.2. Now we construct a subset $\mathcal{M}$ of martingale measures ($\mathcal{M} \subset \mathcal{P}$) in the following way. Given constants $H > -1$ and $\varepsilon \in (0,1)$, consider a function $Y = Y_{H,\varepsilon}\colon \mathbb{R} \to (0,\infty)$ given by

$$Y(x) = 1 + H\mathbf{1}_{\{|x|>\varepsilon\}}. \tag{8.12}$$

Note that by condition (8.7) and (8.11),

$$|x(Y(x)-1)| * \nu_T = H|x|\mathbf{1}_{\{|x|>\varepsilon\}} * \nu_T = H \int_{[0,T]\times\mathbb{R}} |\delta(t, Z_{t-}, x)|\mathbf{1}_{\{|\delta(t,Z_{t-},x)|>\varepsilon\}} dt\, K(dx)$$

$$\leq HT \int_{\mathbb{R}} \rho(x)\mathbf{1}_{\{\rho(x)>\varepsilon\}} K(dx) \leq \varepsilon^{-1} HT \int_{\mathbb{R}} \rho^2(x) K(dx) < \infty.$$

Define $\beta = \beta(t, z)$ by the equation

$$a(t, z) + \beta(t, z)\sigma^2(t, z) + \int_{\mathbb{R}} \delta(t, z, x)[Y(\delta(t, z, x)) - 1]K(dx) = 0. \tag{8.13}$$

Then

$$\beta(t, z) = \frac{-1}{\sigma^2(t, z)} \Big[a(t, z) + H \int_{\mathbb{R}} \delta(t, z, x)\mathbf{1}_{\{|\delta(t,z,x)|>\varepsilon\}} K(dx)\Big].$$

Therefore, given H and ε, the function β is uniformly bounded,

$$|\beta(t, z)| \leq \frac{1}{\sigma_0^2} \Big[a_1 + H \int_{\mathbb{R}} \rho(x)\mathbf{1}_{\{\rho(x)>\varepsilon\}} K(dx)\Big] =: \beta_1.$$

Now we can define the local martingale $N = \{N_t\}_{0\leq t\leq T}$ by

$$N_t = \int_0^t \beta(s, Z_{s-})\sigma(s, Z_{s-})\, dW_s + \int_{[0,t]\times\mathbb{R}} (Y(x) - 1)\, d(\mu^X - \nu),$$

with μ^X the jump measure of the process X, and consider $D = \mathcal{E}(N)$. As $\Delta N = H\mathbf{1}_{\{|\Delta X|>\varepsilon\}}$, for $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$, and ν^N the compensator under P of the jump measure of the process N, we have

$$f(x) * \nu_T^N = f(H) \int_{[0,T]\times\mathbb{R}} \mathbf{1}_{\{|\delta(t,Z_{t-},x)|>\varepsilon\}} dt\, K(dx).$$

Then we can estimate

$$\langle N^c, N^c\rangle_T + \frac{x^2}{1+|x|} * \nu_T^N \leq T\beta_1^2\sigma_1^2 + T\frac{H^2}{1+H} K(\{x\colon \rho(x) > \varepsilon\}) < \infty,$$

and condition (3.8) holds, giving that D is in fact a martingale. This allows us to introduce the probability measure Q on $(\Omega, \mathcal{F}_T)$ by the relation

$$\frac{dQ}{dP} = D_T.$$

Now we calculate the triplet of predictable characteristics of X under Q. It follows from the construction of D and from Girsanov's Theorem, see Proposition III.3.24

in Jacod and Shiryaev (1987), that the Girsanov coefficients β^Q and Y^Q for X, see (3.5), satisfy $\beta_t^Q = \beta(t, Z_{t-})$ and $Y^Q = Y$. Using (3.5), (3.13), (8.10) and (8.13), we obtain that the predictable characteristics of X under Q are

$$\begin{cases} A_t^Q = 0, \\ C^Q = C, \\ \nu^Q = Y\nu. \end{cases} \tag{8.14}$$

Therefore, X is a local martingale under Q. By (8.11) and (8.14)

$$\langle X^c, X^c \rangle_T + \frac{x^2}{1+|x|} * \nu_T^Q \leq T\sigma_1^2 + (1+H)T \int_{\mathbb{R}} \rho^2(x)\, K(dx) < \infty,$$

and condition (3.8) holds, so $Z = \mathcal{E}(X)$ is a uniformly integrable Q-martingale. This shows that the set $\mathcal{P}$ is not empty.

8.3. Consider a convex function f that satisfies the finiteness condition (1.4). In order to determine a lower bound for option prices in the model just introduced, consider the process $Z^* = \{Z_t^*\}_{0 \leq t \leq T}$ defined as the solution of the s.d.e.

$$dZ_t^* = Z_t^* \sigma(t, Z_t^*) dW^*, \qquad 0 \leq t \leq T, \qquad Z_0^* = 1, \tag{8.15}$$

with σ introduced in (8.1), and W^* a standard Wiener process on a stochastic basis $\mathcal{B}^* = (\Omega^*, \mathcal{F}_T^*, \{\mathcal{F}_t^*\}_{0 \leq t \leq T}, Q^*)$. Define $X^* = \{X_t^*\}_{0 \leq t \leq T}$ by

$$X_t^* = \int_0^t \sigma(s, Z_s^*) dW_s^*. \tag{8.16}$$

For each $f \in \mathcal{C}_0^2$ consider

$$G(t, z) = E_{Q^*}(f(Z_T^*) | Z_t^* = z).$$

Observe that, conditions (B) and (C) in the comparison Lemma hold, as discussed in **5.2.2**.

8.4. Let now Q be an arbitrary martingale measure i.e. $Q \in \mathcal{P}$. The process Z is a positive Q-martingale, and the representation $Z = \mathcal{E}(X)$ holds under Q. Using (3.5) and the fact that X is a local martingale under Q, we obtain that the triplet of predictable characteristics of X under Q has the form

$$\begin{cases} A^Q = 0, \\ C_t^Q = C_t = \int_0^t \sigma^2(s, Z_{s-})ds, \\ \nu^Q(\omega, dt, B) = \int_B Y^Q(\omega, t, x)\nu(\omega, dt, dx) \\ \quad = \int_{\mathbb{R}} \mathbf{1}_{B\setminus\{0\}}(\delta(t, Z_{t-}, x))Y^Q(\omega, t, \delta(t, Z_{t-}, x))K(dx)dt, \end{cases} \tag{8.17}$$

where Y^Q is a nonnegative predictable function. This shows that decomposition (5.3) and condition (A) in the comparison Lemma hold. Observe finally, that the triplet for X^* in (8.16) under Q^* is $(0, C^*, 0)$, where C^* satisfies (5.4) and (5.5) with $c^* = \sigma^2$. So conditions (5.10), (5.11) and (5.12) hold (with the opposite inequalities). We are then in position to apply the Comparison Lemma 5.1 for the process Z on $\mathcal{B} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, Q)$ and Z^* on $\mathcal{B}^* = (\Omega^*, \mathcal{F}_T^*, \{\mathcal{F}_t^*\}_{0\le t\le T}, Q^*)$, obtaining

$$E_Q f(Z_T) \ge E_{Q^*} f(Z_T^*).$$

In order to conclude that $\mathbb{C}_* = E_{Q^*} f(Z_T^*)$, we consider the question of attainability of lower bounds. The idea is to construct a sequence of martingale measures Q^n in $\mathcal{M} \subset \mathcal{P}$ as in subsection 8.2 that verify conditions (6.8) and (6.9) in Proposition 6.2. Take two sequences $\{\varepsilon_n\}$, $\{H_n\}$, with $0 < \varepsilon_n < 1$, $H_n > -1$, and $\varepsilon_n \to 0$, $1+H_n \to 0$. Define, for each n, $Y^n = Y_{H^n,\varepsilon^n}$ as in (8.12). Define, also for each n, β^n by formula (8.13), with Y^n instead of Y. As done in subsection 8.2, we construct Q^n. Triplets of X under Q^n are denoted by $(0, C, \nu^n)$, as in (8.14).

We verify (6.8). As $C_t = \int_0^t \sigma^2(s, Z_{s-})ds$, this condition reduces to the verification of

$$(1 - \sqrt{1+x})^2 * \nu_T^n \xrightarrow{Q^n} 0.$$

Now,

$$\begin{aligned} (1 - \sqrt{1+x})^2 * \nu_T^n \le (1 - \sqrt{1+x})^2 \mathbf{1}_{\{|x|\le\varepsilon_n\}} * \nu_T \\ + (1 + H_n)(1 - \sqrt{1+x})^2 \mathbf{1}_{\{|x|>\varepsilon_n\}} * \nu_T. \end{aligned} \tag{8.18}$$

The second term in (8.18) vanishes by the dominated convergence theorem, as on the one hand, the integrand goes to zero, and on the other hand,

$$\begin{aligned} (1 - \sqrt{1+x})^2 \mathbf{1}_{\{|x|\le 1\}} * \nu_T &\le x^2 * \nu_T = \int_{[0,T]\times\mathbb{R}} \delta^2(t, Z_{t-}, x)dt\, K(dx) \\ &\le T \int_{\mathbb{R}} \rho^2(x)\, K(dx). \end{aligned}$$

Similar computations give

$$(1+H_n)(1-\sqrt{1+x})^2\mathbf{1}_{\{|x|>\varepsilon_n\}} * \nu_T \le (1+H_n)x^2 * \nu_T$$

$$= (1+H_n)\int_{[0,T]\times\mathbb{R}} \delta^2(t, Z_{t-}, x)dt\, K(dx) \le (1+H_n)T\int_{\mathbb{R}} \rho^2(x)\, K(dx) \to 0,$$

if $n \to \infty$, and (6.8) holds. Observe that in this particular case, (6.9) follows from (6.8). In this way, we conclude that the lower bound given by the Comparison Lemma is attained, and in consequence

$$\mathbb{C}_* = E_{Q^*} f(Z_T^*).$$

8.5. We finally verify that the upper bound is trivial, i.e. the upper universal bound is attained. Consider the same sequence of functions Y^n as above, but take H_n arbitrary by the moment, and $\varepsilon_n \to 0$ as formerly. We verify that the corresponding Hellinger processes satisfy condition (iii) in Proposition 4.1. Denote $\varphi(x) = \frac{1}{2}(1-\sqrt{1+x})^2$, and note that for each ε_n there exists $a_n > 0$ such that

$$\varphi(x)\mathbf{1}_{\{|x|>\varepsilon_n\}} \ge a_n|x|\mathbf{1}_{\{|x|>\varepsilon_n\}}.$$

The Hellinger process of order $\frac{1}{2}$ in (3.10), in this case satisfy

$$h^n\Big(\frac{1}{2}\Big)_T = \frac{1}{8}C_T + \frac{1}{2}(1-\sqrt{1+x})^2 * \nu_T^n \ge a_n|x|\mathbf{1}_{\{|x|>\varepsilon_n\}} * \nu_T^n$$

$$= a_n(1+H_n)|x|\mathbf{1}_{\{|x|>\varepsilon_n\}} * \nu_T = a_n(1+H_n)\int_{[0,T]\times\mathbb{R}} |\delta(t, Z_{t-}, x)|\mathbf{1}_{\{|\delta(t,Z_{t-},x)|>\varepsilon_n\}}dt\, K(dx).$$

If we choose $\{H_n\}$ such that $a_n(1+H_n) \to \infty$, as

$$\lim_n \int_{[0,T]\times\mathbb{R}} |\delta(t, Z_{t-}, x)|\mathbf{1}_{\{|\delta(t,Z_{t-},x)|>\varepsilon_n\}}dt\, K(dx) = \int_{[0,T]\times\mathbb{R}} |\delta(t, Z_{t-}, x)|dt\, K(dx) \ge c_0$$

by condition (8.8), we obtain that for any $N > 0$

$$\lim_{n\to\infty} Q^n\Big(h^n\Big(\frac{1}{2}\Big)_T > N\Big) = 1$$

and the proof of the upper bound is complete. In conclusion we formulate our result.

Proposition 8.1 *Consider a model of a financial market with* $B = \{B_t\}_{0 \le t \le T}$ *such that* $B_t = 1$, *and* $Z = \{Z_t\}_{0 \le t \le T}$ *the solution of the s.d.e. given in* (8.1). *Assume that conditions* (i), (ii), (iii) *and* (iv) *in subsection* 8.1 *are fulfilled. Then for a convex payoff function* f *satisfying the finiteness condition* (1.4) *we have the following bounds on option prices.*
(a) *The upper price is given by the universal bound, i.e.*

$$\mathbb{C}^* = \sup_{Q \in \mathcal{P}} E_Q f(Z_T) = f(0) + \frac{f(\infty)}{\infty}.$$

(b) *The lower price is given by a pure diffusion model with the same volatility structure, i.e.*

$$\mathbb{C}_* = \inf_{Q \in \mathcal{P}} E_Q f(Z_T) = E_{Q^*} f(Z_T^*),$$

where Z^* *is the solution of the s.d.e.* (8.15).

9 Conclusions

Consider a general semimartingale model of a security market with finite horizon T, risk-less asset B and risky asset S. Given a convex function $g = g(x)$ consider a European option with payoff $g(S_T)$. In the presented paper, we study the problem of the determination of all possible prices for this option, obtained as expectations of the discounted payoff with respect to a generic probability measure that ranges along the set of all equivalent martingale measures. In other words, our purpose is the determination of the set of *admissible prices* for this option.
As this set is an interval, our task is fulfilled in two steps: (i) the comparison of option prices under different martingale measures, more precisely, we give criteria to know whether the prices of European options with the same payoff function g and different stocks can be compared, and (ii) the convergence of option prices corresponding to a sequence of martingale measures, in order to verify optimality of the bounds obtained using (i).
The main idea in (i) is to introduce a partial order in the set of (one-dimensional) probability distributions of discounted stock prices at the exercise time T, i.e. the law of the random variable $Z_T := S_T / B_T$ under an arbitrary equivalent martingale measure. This partial order (adapted from the theory of statistical experiments)

allows to find certain extremal measures, not necessarily in the original set (in many cases they are *singular* martingale measures). These measures give upper and lower bounds for the range of options prices. Weak convergence of probability measures is used in (ii), in order to determine whether the mentioned bounds are optimal, i.e. whether the obtained bounds are the supremum and infimum of the set of admissible option prices.

Furthermore, the presence of the filtration makes possible to appeal to the theory of stochastic calculus for semimartingales. In reference to (i) we obtain *predictable* criteria to compare option prices (more precisely we represent $Z^i = \mathcal{E}(X^i)$ for $i = 1, 2$, and give a comparison result that involves the comparison (in certain sense) of the predictable local characteristics of X^1 and X^2 under the corresponding martingale measures). Particular attention is given to the necessity of the hypothesis in this comparison result. This question is answered, constructing a counterexample of two Lévy driven stocks (one driven by a standard Wiener processes, and the other by the compensated difference of two Poisson processes), with comparable prices for all convex functions for some fixed exercise time, but predictable characteristics not comparable in the mentioned sense.

In what respects (ii) two types of results are discussed. First, the case were the *universal* (i.e. that hold *a priori* for any martingale measure) bounds are attained, obtaining predictable criteria in terms of the Hellinger process corresponding to the discounted stock price. In second place results about weak convergence of the discounted stock prices in terms of the predictable characteristics of their stochastic logarithms are given in two situations: when the process X in the limit is a process with independent increments, and when the limit is a stochastic volatility model.

Special attention is given to random walk models, and models where the stock satisfies a stochastic differential equation with jumps. In both cases complete solution to the problem of determination of the range of option prices is given, under mild assumptions.

Finally, it is interesting to remark that the partial ordering introduced in (i) is strongly related to the comparison of binary statistical experiments, where the discounted stock process Z plays the role of the density process of the statistical experiment. Part (ii) can be seen as weak convergence of the corresponding density process. This analogy, from one side, helped to understand the above mentioned problems, and, on the other, gives the possibility of interpreting our results in terms of the theory of statistical experiments.

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