LYAPUNOV EXPONENTS, SYMMETRIC SPACES, AND A MULTIPLICATIVE ERGODIC THEOREM FOR SEMISIMPLE LIE GROUPS

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The classical theory of Lyapunov characteristic exponents is reformulated in invariant geometric terms and carried over to arbitrary noncompact semisimple Lie groups with finite center. A multiplicative ergodic theorem (a generalization of a theorem of Oseledets) and the global law of large numbers are proved for semisimple Lie groups, as well as a criterion for Lyapunov regularity of linear systems of ordinary differential equations with subexponential growth of coefficients.

The concepts of characteristic exponents and regularity of a one-parameter family of matrices were introduced by Lyapunov in his fundamental work [1] and were originally used to describe solutions of systems of ordinary differential equations with variable coefficients [2]. Applications of these concepts to the theory of dynamical systems are based on the "multiplicative ergodic theorem" of Oseledets [3], which establishes regularity conditions for products of random matrices. Despite much work devoted to Lyapunov exponents in various situations, judging overall, the simple fact that Lyapunov regularity of a sequence of matrices is equivalent to its asymptotic proximity to the sequence of powers of some fixed matrix has remained unnoticed up to now. This observation makes it possible to carry the classical theory of Lyapunov characteristic exponents over to arbitrary noncompact semisimple Lie grous with finite center, reformulating it in invariant geometric terms without using matrix representations of these groups. Application of the apparatus of Riemannian geometry and the theory of symmetric spaces allows us to get both generalizations of already facts, and results which apparently have not been previously formulated in matrix form, in particular a "global" version of the law of large numbers for semisimple Lie groups, simply. Our proofs are new even for matrix groups.

The structure of the paper is the following. In Sec. 1 we introduce the definitions and notation needed from the theory of semisimple Lie groups and symmetric spaces, and also discuss the connection between different compactifications of symmetric spaces. Section 2 is basic; in it we define regular sequences in a symmetric spaces as sequences which are asymptotically close to geodesics, and we give some criteria for regularity (in polar and horospherical coordinates, and also in terms of finite-dimensional representations). In Section 3 these criteria are used to prove the multiplicative ergodic theorem (a generalization of a theorem of Oseledets) and the global law of large numbers for semisimple Lie groups. In Section 4 we show that for sequences of matrices, regularity in our sense is naturally connected with Lyapunov regularity, and as a consequence we get a criterion for Lyapunov regularity of linear systems of ordinary differential equations with subexponential growth of coefficients.

Theorem 2.1, which is key in this paper, was announced in [4], where it was used to describe harmonic functions on discrete subgroups of semisimple groups. The author thanks Yu. D. Burago, A. M. Vershik, M. I. Zakharevich, G. A. Margulis, and M. A. Ol'shanetskii for helpful discussions and comments.

1. Preliminary Information and Notation

The concepts and notation introduced here will be used below without further mention.

1.1. Let G be a noncompact semisimple real Lie group with finite center, K be a maximal compact subgroup of it, A be a principal vector subgroup, $\mathfrak{g}, \mathfrak{k}, \mathfrak{a}$ be the corresponding Lie algebras $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition, $\mathfrak{h}$ be a principal Cartan subalgebra such that $\mathfrak{a} = \mathfrak{h} \cap \mathfrak{p}$. By $\tilde{\Delta}$ we denote a system of roots of the complexification $\tilde{\mathfrak{g}}$ with respect to

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the complexification $\tilde{\mathfrak{h}}$ by $\mathfrak{g}^{\alpha}$ ($\alpha \in \tilde{\Delta}$), the root subspaces of $\tilde{\mathfrak{g}}$. We fix a basis $\tilde{\Pi} \subset \tilde{\Delta}$ so $\Delta = \{\alpha|_{\mathfrak{a}} : \alpha \in \tilde{\Delta}\}$ and $\Pi = \{\alpha|_{\mathfrak{a}} : \alpha \in \tilde{\Pi}\}$ are respectively a system of restricted roots and a basis of it [5]. We denote the subsets of positive roots of $\tilde{\Delta}$ and Δ by $\tilde{\Delta}_+$ and Δ_+. We set $\tilde{\mathfrak{n}} = \sum \mathfrak{g}^{\alpha}$ were the sum is taken over all $\alpha \in \tilde{\Delta}_+$. By W we denote the restricted Weyl group, acting on $\mathfrak{a}$ by $\mathfrak{a}^+ = \{x \in \mathfrak{a} : \langle x, \alpha\rangle \geqslant 0\ \forall \alpha \in \Pi\}$ the closure of a dominant Weyl chamber. We denote the norm in $\mathfrak{p}$ induced by the Killing form $\langle \cdot , \cdot \rangle$ by $\|\cdot\|$. We set $\mathfrak{a}_1^+ = \{x \in \mathfrak{a}^+ : \|x\| = 1\}$. In addition, we shall use the notation $X^{\perp} = \{p \in \mathfrak{p} : \langle p, x\rangle = 0\ \forall x \in X\}$, $X^{\circ} = X^{\perp} \cap \Pi$, where X is a subset of $\mathfrak{p}$.

1.2. For any element $g \in G$ for a Cartan decomposition $g = \kappa_1(\exp\alpha)\kappa_2$, $\kappa_{1,2} \in K$, $\alpha \in \mathfrak{a}^+$ there is uniquely defined the "complex radius" $r(g) = \alpha \in \mathfrak{a}^+$. We set $\delta(g) = \|r(g)\|$. Then $\delta(g_1 g_2) \leqslant \delta(g_1) + \delta(g_2) + C$, and if $M \subset G$ is a compactum, whose interior contains the identity of the group G, and $\delta_M(g) = \min\{n : g \in M^n\}$ then $a\delta_M + b \leqslant \delta \leqslant A\delta_M + B$ (a, b, A, B, C are constants, where a and A are positive). Thus, δ is a principal gauge on G [6].

1.3. If $E \subset \Pi$ we set

$$\tilde{\Delta}_0(E) = \left\{\alpha \in \tilde{\Delta} : \alpha|_{\mathfrak{a}} = \sum \kappa_{\beta}\beta\ (\beta \in E)\right\}$$

and

$$\tilde{\Delta}_+(E) = \tilde{\Delta}_+ \setminus \tilde{\Delta}_0(E).$$

We define subalgebras

$$\tilde{\mathfrak{a}}'(E) = \sum \mathbb{C}\alpha \quad (\alpha \in \tilde{\Delta}_0(E)),$$
$$\tilde{\mathfrak{g}}'(E) = \tilde{\mathfrak{a}}'(E) + \sum \mathfrak{g}^{\alpha} \quad (\alpha \in \tilde{\Delta}_0(E)),$$
$$\tilde{\mathfrak{n}}(E) = \sum \mathfrak{g}^{\alpha} \quad (\alpha \in \tilde{\Delta}_+(E)),$$
$$\tilde{\mathfrak{a}}(E) = \tilde{\mathfrak{h}} \ominus \tilde{\mathfrak{a}}'(E),$$
$$\tilde{\mathfrak{l}}(E) = \tilde{\mathfrak{a}}(E) + \tilde{\mathfrak{n}}(E),$$
$$\tilde{\mathfrak{p}}(E) = \tilde{\mathfrak{g}}'(E) + \tilde{\mathfrak{l}}(E)$$

(here $\ominus$ denotes the orthogonal complement with respect to the Killing form). The real parts $\mathfrak{a}'(E) = \tilde{\mathfrak{a}}'(E) \cap \mathfrak{g}$, etc., are subalgebras of $\mathfrak{g}$ while the algebra $\mathfrak{g}'(E)$ is semisimple, $\mathfrak{a}'(E)$ is a principal vector subalgebra of it $\mathfrak{k}'(E) = \mathfrak{g}'(E) \cap \mathfrak{k}$, is a maximal compact subalgebra of it, $\mathfrak{n}'(E) = \mathfrak{g}'(E) \cap \mathfrak{n}$ is a nilpotent subalgebra, and $\mathfrak{g}'(E)$ has Iwasawa decomposition $\mathfrak{k}'(E) + \mathfrak{a}'(E) + \mathfrak{n}'(E)$. Further, the subalgebra $\mathfrak{n}(E)$ is nilpotent, while $\mathfrak{n}$ splits into a direct sum $\mathfrak{n}(E) + \mathfrak{n}'(E)$, and the subalgebra $\mathfrak{l}(E)$ is solvable. The subalgebra $\mathfrak{p}(E)$ is called the standard parabolic subalgebra (of type E), and the decomposition $\mathfrak{g} = \mathfrak{n}(E) + \mathfrak{a}(E) + \mathfrak{g}'(E) + \mathfrak{k}$ (nonunique by $\mathfrak{k}'(E)$) is a generalized Iwasawa decomposition.

1.4. Let $A(E)$ etc. be the analytic subgroup of G corresponding to the subalgebras $\mathfrak{a}(E)$ etc. Then $N(E)$ and $N'(E)$ are simply connected nilpotent subgroups, where any element $n_0 \in N$ can be represented uniquely in the form $n_0 = nn'$ where $n \in N(E)$, $n' \in N'(E)$ and the group $L(E) = N(E)A(E)$ is simply connected and solvable. The group $G'(E)$ is semisimple with finite center, $G'(E) = N'(E)\ A'(E)\ K'(E)$ is its Iwasawa decomposition

$A = A(E) \oplus A'(E)$ and one has the global decomposition $G = L(E)\,G'(E)\,K$ where $L(E) \cap G'(E) = 1$ and $L(E)\,G'(E) \cap K = K'(E)$. The group $P(E) = L(E)\,G'(E)\,Z$, where Z is the centralizer of A in K, is called the standard parabolic subgroup of G (of type E) [7-10].

1.5. By S we denote the Riemannin symmetric space $S = G/K = \{gK\}$ with distinghished point $x_0 = \{K\}$ in it and canonical invariant metric d. If $x = g x_0$, then there is uniquely defined the "complex radius" $\tau(x) = \tau(g) \in \mathfrak{a}^+$ and $d(x, x_0) = \|\tau(x)\|$.

We denote by S^∞ the set of asymptotic pencils of geodesics in S. Each pencil contains a unique geodesic $\gamma(\tau) = k(\exp \alpha\tau)\,x_0$, $\alpha \in \mathfrak{a}_1^+$, $k \in K$, issuing from the point x_0 so S^∞ can be identified with the unit sphere of the tangent space $\mathfrak{p}$ to S at the point x_0. Here, any point $\gamma \in S^\infty$ corresponds to a uniquely defined vector $\alpha = \alpha(\gamma) \in \mathfrak{a}_1^+$. The set $\bar{S} = S \cup S^\infty$ is the compactification of S in the conical topology — convergence of sequences of points $\{x_t\} \subset S$ going to infinity in $\bar{S}$ is equivalent to convergence of the directing vectors of the geodesics $(x_0 x_t)$ [11]. The action of G extends naturally from S to S^∞. With respect to this action S^∞ splits into orbitals $S_\alpha^\infty = \{\gamma : \alpha(\gamma) = \alpha\}$, $\alpha \in \mathfrak{a}_1^+$, while the group K acts transitively on each orbit. The stabilizer of the vector $\alpha \in \mathfrak{a}_1^+$ as an element of S^∞ is a parabolic subgroup $P(\alpha^\circ)$ and the whole orbit S_α^∞ is canonically isomorphic to the space $\partial_E S = G/P(E)$, where $E = \alpha^\circ$. If the vector $\alpha \in \mathfrak{a}_1^+$ is fixed, then any point $x \in S$ can be represented as $x = n a g' x_0$ where $n \in N(\alpha^\circ)$, $a \in A(\alpha^\circ)$, $g' \in G'(\alpha^\circ)$, while n and a are uniquely defined, and g' is unique up to a factor from $K'(\alpha^\circ)$. Thus, the points $x \in S$ are in one-one correspondence with triples $(n, a, \mathcal{T}_\alpha(x))$ where $\mathcal{T}_\alpha(x) = g' K'(\alpha^\circ)$ is a point of the symmetric space $\mathcal{T}_\alpha(S) = G'(\alpha^\circ)/K'(\alpha^\circ)$. We shall call the coordinates $(n, a, \mathcal{T}_\alpha(x))$ standard horospheric coordinates (of type α^0) on S. Since any point $\gamma \in S^\infty$ can be represented in the form $\gamma = (\mathrm{Ad}\, k)(\alpha)$ where $k \in K$, $\alpha \in \mathfrak{a}_1^+$, then the horospheric coordinates with respect to γ are obtained from the standard horospheric coordinates of type α° by rotation with the help of the group K [9]. The orbits of the group $P(E)$ in S_α^∞ are in one-to-one correspondence with the double cosets $P(E)\, g\, P(\alpha^\circ)$, i.e., with the double cosets $W(E)\, w\, W(\alpha^\circ)$, where $W(E)$ is the subgroup of W, generated by reflections with respect to elements of E [10]. Thus, any point $\gamma \in S_\alpha^\infty$ can be represented as $\gamma = \mathrm{Ad}(n n')(w\alpha)$ where $n \in N(E)$, $n' \in N'(E)$, $w \in W$ (Bruhat decomposition).

We note that the conical compactification is incomparable with the Satake–Furstenberg–Moore compactification in which the boundary of S is $\partial_\emptyset S$ or $\partial_E S$ for suitable choice of E [7-9]. The Martin compactification and the stronger Karpelevich compactification are gotten from the conical compactification by closure of the components $\mathcal{T}_\gamma(S)$. Both of them are stronger than both the conical compactification and the Satake–Furstenberg–Moore compactification [9, 12].

1.6. Let ψ be a representation of G in a finite-dimensional complex space V. By restricted weights of ψ we mean the restrictions of the weights of the representation $\widetilde{d\psi} : \tilde{\mathfrak{g}} \longrightarrow \mathrm{End}(V)$ on $\mathfrak{a}$. By ψ_λ we shall denote the irreducible representation of G in the finite-dimensional space V_λ corresponding to the integer-valued restricted highest weight $\lambda \in \mathfrak{a}^+$ [5]. Then if $\|\cdot\|$ is a norm on $\mathrm{End}(V_\lambda)$ then $|\log\|\psi_\lambda(g)\| - \langle \tau(g), \lambda\rangle| < C$

for some constant C. For any finite-dimensional representation Ψ of the group G one can choose a Hermitian form in the space of the representation so that the matrices $\Psi(\kappa)$, $\kappa \in K$ lie in $SU(V)$, and the matrices $\Psi(a)$, $a \in A$ are diagonal with positive diagonal elements. Then the representation Ψ induces a map $s\Psi : S \to S_\Psi = SL(V)/SU(V)$, where the image of S is a completely geodesic submanifold of the symmetric space S_Ψ. If the representation $d\Psi$ is faithful, then $s\Psi$ is an isometric imbedding [7].

2. Regular Sequences in Symmetric Spaces

Definition. We call a sequence of points $\{x_t\}_{t=1}^{\infty}$ of the symmetric space S regular, if there exists a geodesic $\gamma : [0,\infty) \to S$ and a number $\theta \geq 0$ such that $d(x_t, \gamma(\theta t)) = o(t)$. If $\theta = 0$, i.e., if $d(x_t, x) = o(t)$ for $x \in S$, then we shall say that $\{x_t\}$ is a trivial regular sequence. We call a sequence $\{g_t\}$ of elements of the group G regular if the sequence $\{g_t x\}$ is regular in S for some (or equivalently all) $x \in S$.

As follows from point 1.5, regularity of a sequence $\{x_t\}$ is equivalent to the existence of a $\kappa \in K$ and an $\alpha \in \mathfrak{a}^+$ such that $d(x_t, \kappa(\exp t\alpha)x_0) = o(t)$ where the vector α is uniquely determined by the sequence $\{x_t\}$ ($\alpha = \lim r(x_t)/t$). We shall call the vector $\alpha \in \mathfrak{a}^+$ the vector of exponents of the sequence $\{x_t\}$.

THEOREM 2.1. For the sequence $\{x_t\}$ of points of the symmetric space S to be regular, it is necessary and sufficient that $d(x_t, x_{t+1}) = o(t)$ and the limit $\alpha = \lim r(x_t)/t$ exist.

LEMMA. Let $\{\gamma_t\}$ be a sequence of geodesics issuing from one point in the Lobachevskii plane $H^2_\varkappa$ of curvature $\varkappa < 0$. If

$$d_t = d(\gamma_t(\theta t), \gamma_{t+1}(\theta t)) = o(t)$$

for some $\theta > 0$ then γ_t converges pointwise to a limit geodesic γ_∞ and

$$d(\gamma_t(\theta t), \gamma_\infty(\theta t)) = o(t).$$

Proof. Let φ_t be the angle between the directing vectors of the geodesics γ_t and γ_{t+1} so by the sine theorem [13]

$$\sin \varphi_t/2 = \operatorname{sh}(\sqrt{-\varkappa}\, d_t/2)/\operatorname{sh}(\sqrt{-\varkappa}\,\theta t),$$

from which $\log \varphi_t = \sqrt{-\varkappa}\,\theta t + o(t)$. Consequently, for $\Psi_t = \sum_{i\geq t} \varphi_i$ one has $\log \Psi_t = \sqrt{-\varkappa}\,\theta t + o(t)$ also. Since for any $i > t$ the angle between the directing vectors of the geodesics γ_t and γ_i does not exceed Ψ_t applying the sine theorem again we get the assertion of the lemma.

Proof of the Theorem. The necessity of the condition of the theorem is obvious. We establish its sufficiency. Without loss of generality one can assume that $x_t = \kappa_t(\exp t\alpha)\, x_0$ and $\alpha \neq 0$. We must prove that from the condition $d(x_t, x_{t+1}) = o(t)$ the regularity of the sequence $\{x_t\}$ follows. Let $\beta(\tau) = \kappa(\tau)\exp\alpha(\tau)\, x_0$ be a geodesic joining the points x_t and x_{t+1}. By $\dot\beta(\tau) \in \mathfrak{p}$ we denote the tangent vector to β at the point $\beta(\tau)$ translated along the geodesic $(\beta(\tau), x_0)$ to the point x_0. The components of the vector $\dot\beta(\tau)$ in $\mathfrak{a}$ and $\mathfrak{p} \ominus \mathfrak{a}$ are, respectively, $\dot\beta_1(\tau) = \dot\alpha(\tau)$ and $\dot\beta_2(\tau) = \Pr(\exp(-\operatorname{ad}\alpha(\tau))\,\dot\kappa(\tau))$

where $\dot{\alpha}(\tau) \in \mathfrak{a}$ and $\dot{\kappa}(\tau) \in \mathfrak{k}$ are the tangent vectors to the curves α and κ, carried to the identity of the group G by left translation, $P\mathfrak{r}$ is the orthogonal projection of the algebra $\mathfrak{g}$ to $\mathfrak{p}$. Obviously,

$$\|\dot{\beta}(\tau)\|^2 = \|\dot{\beta}_1(\tau)\|^2 + \|\dot{\beta}_2(\tau)\|^2$$

where the vector $\dot{\beta}_2(\tau)$ lies in $\alpha(\tau)^{\perp\perp}$. Since $d(x_t, x_{t+1}) = o(t)$, one can find an $s = s(t) = t + o(t)$, such that $\alpha(\tau) - s\alpha \in \mathfrak{a}^+$ for all τ.

Now we consider the curve $\beta'(\tau) = \kappa(\tau)(\exp s\alpha)x_0$, joining the points $x'_t = \kappa_t(\exp s\alpha)x_0$ and $x'_{t+1} = \kappa_{t+1}(\exp s\alpha)x_0$. Since $\dot{\beta}_1(\tau) = 0$ and $a\|\dot{\beta}'_2(\tau)\| < \|\dot{\beta}_2(\tau)\|$ for any τ with respect to the choice of s, the length of the curve β' does not exceed the length of β i.e., the distance between the points x_t and x_{t+1}. Thus the distance between the points x_t and x_{t+1} in the intrinsic metric of the surface

$$R = K(\exp \mathbb{R}_+\alpha)x_0 = \{\kappa(\exp s\alpha)x_0 : \kappa \in K, s \geqslant 0\}$$

is also $o(t)$. We join the points x_t and x_{t+1} by the geodesic $\delta(\tau) = \kappa(\tau)\exp\alpha(\tau)x_0$ on the surface R. We set

$$R_t = \{\kappa(\tau)(\exp s\alpha)x_0 : s \geqslant 0\}$$

and we denote by d_t the intrinsic metric in R_t. It is easy to see that either x_t and x_{t+1} lie on the geodesic ray, issuing from x_0 in which case R_t degenerates into a ray, or $\dot{\delta}_2(\tau) \neq 0$ for all τ in which case R_t is the infinite sector included between the rays $(x_0 x_t)$ and $(x_0 x_{t+1})$ and is geodesically convex (in the metric of the surface R). In the latter case the curvature of R_t at the point $\kappa(\tau)(\exp s\alpha)x_0$ is (cf. [13])

$$K = -\|[\dot{\delta}_2(\tau), \alpha]\| / \|\dot{\delta}_2(\tau)\|\,\|\alpha\|.$$

Since $\dot{\delta}_2(\tau) \in \alpha^{\perp\perp}$ for all τ, we get that $K \leqslant \varkappa < 0$, where

$$\varkappa = \varkappa(\alpha) = -\min\{\langle\lambda, \alpha\rangle / \|\lambda\|\|\alpha\| : \lambda \in \Pi \setminus \alpha^0\}.$$

Now we set $\gamma_t(\tau) = \kappa_t \exp(\tau\alpha/\|\alpha\|)x_0$ and we consider the sequence of geodesics $\tilde{\gamma}_t$ on the Lobachevskii plane $H^2_\varkappa$ of curvature $\varkappa$ with metric $\tilde{d}$, constructed as follows: all $\tilde{\gamma}_t$ issue from one point, and the directing vector of the geodesic $\tilde{\gamma}_{t+1}$ is laid off clockwise from the directing vector of the geodesic $\tilde{\gamma}_t$ so that

$$\tilde{d}(\tilde{\gamma}_t(t\|\alpha\|), \tilde{\gamma}_{t+1}((t+1)\|\alpha\|)) = d_t(x_t, x_{t+1}).$$

The geodesic ε in $H^2_\varkappa$, joining the points $\tilde{\gamma}_t(t\|\alpha\|)$ and $\tilde{\gamma}_{t+\kappa}(t\|\alpha\|)$ for large t intersects all the intermediate geodesics $\tilde{\gamma}_{t+i}$, $1 \leqslant i \leqslant \kappa-1$. We denote by r_i the lengths of the segments cut off by ε on $\tilde{\gamma}_{t+i}$ (i.e., $\tilde{\gamma}_{t+i}(r_i)$ lies on ε). Then it follows from the Aleksandrov theorem on comparison of triangles [14] that

$$d(\gamma_{t+i}(r_i), \gamma_{t+i+1}(r_{i+1})) \leq$$

$$\leqslant d_{t+i}(\gamma_{t+i}(r_i), \gamma_{t+i+1}(r_{i+1})) \leqslant \tilde{d}(\tilde{\gamma}_{t+i}(r_i), \tilde{\gamma}_{t+i+1}(r_{i+1})),$$

so

$$d(\gamma_t(t\|\alpha\|), \gamma_{t+\kappa}(t\|\alpha\|)) \leqslant \tilde{d}(\tilde{\gamma}_t(t\|\alpha\|), \tilde{\gamma}_{t+\kappa}(t\|\alpha\|))$$

for all κ. Using the lemma, we now get that the deodesic γ_t converges point-wise in S to the limit geodesic γ_∞ and

$$d(\gamma_t(t\|\alpha\|), \gamma_\infty(t\|\alpha\|)) = o(t).$$

The theorem is proved.

Remark. For the symmetric space $SL(n,\mathbb{R})/SO(n)$ our theorem is similar to Raghunathan's lemma [15] which gives an estimate of the rate of convergence of the eigenspaces of positive definite matrices. Our proof, however, is based on different considerations.

COROLLARY. A sequence of points $\{x_t\}$ of a symmetric space S is regular if and only if $d(x_t, x_{t+1}) = o(t)$, the limit $\lim d(x_0, x_t)/t = d$ exists, and, if d is positive, the directing vectors of the geodesics $(x_0 x_t)$ converge.

From Theorem 2.1 we now get:

THEOREM 2.2. The following conditions are equivalent:

1) The sequence of points $\{x_t\}$ of the symmetric space S is regular;

2) The sequence of points $\{s\psi(x_t)\}$ is regular in the symmetric space S_ψ for any finite-dimensional representation ψ of the group G;

3) $d(x_t, x_{t+1}) = o(t)$ and for any finite-dimensional representation ψ the limit $\lim \|\tau(s\psi(x_t))\|/t$ exists. Moreover, the coordinates of the vector of exponents of the sequence $\{s\psi(x_t)\}$ (in the standard basis, as an element of $\mathbb{R}^n$, $n = \dim V_\psi$) are $\langle\alpha, \lambda\rangle$, where α is the vector of exponents of the sequence $\{x_t\}$ and λ runs through the set of all restricted weights of the representation ψ (taken with their multiplicities considered). If $s\psi$ is an imbedding, then the regularity of the sequence $\{s\psi(x_t)\}$ in the symmetric space S_ψ is equivalent with the regularity of the sequence $\{x_t\}$ in the symmetric space S.

THEOREM 2.3. The sequence of elements $\{g_t\}$ of the group G is regular if and only if $\delta(g_t^{-1} g_{t+1}) = o(t)$ and for any finite-dimesional representation ψ of the group G, the limit $\lim \log\|\psi(g_t)\|/t$ exists (which is equivalent to the existence of this limit for fundamental representations of G only).

Now we establish a criterion for the regularity of a sequence $\{x_t\}$ in the horospherical coordinates of the symmetric space. As follows from point 1.5, it suffices to restrict oneself to considering only standard horospherical coordnates.

THEOREM 2.4. Let $(n_t, a_t, \pi_E(x_t))$ be the standard horospherical coordinates of type E of the points of the sequence $\{x_t\} \subset S$ i.e., $n_t \in N(E)$, $a_t \in A(E)$. The sequence $\{x_t\}$ is regular in S if and only if $d(x_t, x_{t+1}) = o(t)$, the sequence of points $\{\pi_E(x_t)\}$ is regular in the symmetric space $\pi_E(S)$ and the limit $\alpha = \lim \log a_t/t \in \mathfrak{a}(E)$ exists. If $\alpha' \in \mathfrak{a}'(E)$ is the vector of exponents of the sequence $\{\pi_E(x_t)\}$ then the vector of exponents $\{x_t\}$ of the sequence $\tilde{\alpha} \in \mathfrak{a}$ has the form $\tilde{\alpha} = \omega(\alpha + \alpha')$, where ω is an element of the Weyl group W.

LEMMA (cf. [16], Lemma 9.4). In the space $\mathrm{Mat}(n, \mathbb{C})$ of matrices of size $n \times n$ with complex coefficients, we fix a norm $\|\cdot\|$ and a block partition into blocks of size $n_i \times n_j$ $(\sum n_i = n)$. Let $\{A_t\} \subset \mathrm{Mat}(n, \mathbb{C})$ be a sequence of upper quasitriangular matrices with diagonal blocks $A_t^i \in \mathrm{Mat}(n_i, \mathbb{C})$. If $|\det A_t| = 1$, $\log \|A_t^{-1} A_{t+1}\| = o(t)$ and for all i the limit $a^i = \lim \log \|A_t^i\|/t$, also exists and $\lim \log \|A_t\|/t = a$ $a = \max a^i$.

Proof of the Theorem. Let the sequence $\{x_t\}$ be regular $\tilde{\alpha} \in \mathfrak{a}^+$ be its vector of exponents; then as is clear from point 1.4, one can find a $\tilde{w} \in W$ and an $\tilde{n} \in N$ such that $d(x_t, n\tilde{w}(\exp t\tilde{\alpha})x_0) = o(t)$. We decompose $\tilde{n} = nn'$ and $\tilde{w}\tilde{\alpha} = \alpha + \alpha'$, where $n \in N(E)$, $n' \in N'(E)$ and $\alpha \in \mathfrak{a}(E)$, $\alpha' \in A'(E)$. Then $g_t = n\tilde{w}\exp t\tilde{\alpha} = (n \exp t\alpha)\cdot(n' \exp t\alpha')$, since $A(E)$ centralizes $N'(E)$. Since $d(x_t, g_t x_0) = o(t)$, the first part of the theorem is proved. The converse assertion is easily proved, using Theorem 2.2 and the lemma.

The next theorem follows from Theorem 2.4.

THEOREM 2.5. The sequence of elements $g_t = n_t a_t g_t' \in P(E)$ where $n_t \in N(E)$, $a_t \in A(E)$, $g_t' \in G(E)$, is regular in the group G if and only if $\delta(g_t^{-1} g_{t+1}) = o(t)$, the limit $\alpha = \lim \log a_t/t$ exists, and the sequence $\{g_t'\}$ is regular in the group $G'(E)$.

When $E = \emptyset$, i.e., $P(\emptyset) = P$ is a minimal parabolic subgroup of G, we get

COROLLARY. The sequence of elements $g_t = n_t a_t \in P$, where $n_t \in N$, $a_t \in A$ is regular in G if and only if $\delta(g_{t^{-1}} g_{t+1}) = o(t)$ and the limit $\alpha = \lim \log a_t/t$ exists. In this case the vector of exponents of the sequence $\{g_t\}$ is $\tilde{w}\alpha$ where $\tilde{w}$ is an element of the Weyl group W, such that $\tilde{w}\alpha \in \mathfrak{a}^+$.

We note that the element $\tilde{w}$ of the Weyl group participating in the formulation of the corollary determines uniquely on which of the orbits of the group P in $S^\infty_{\tilde{w}\alpha}$ the limit of the sequence $x_t = g_t x_0$ can lie (cf. point 1.5). In particular, if $\alpha \in \mathfrak{a}^+$, i.e., $\tilde{w} = e$ then the limit of the sequence $\{x_t\}$ in S^∞ is defined by the vector α and does not depend on the nilpotent factors $\{n_t\}$.

Remark. Theorems 2.1-2.5 carry over naturally to the case when the parameter t assumes continuous values. Here it is necessary to replace the condition $d(x_t, x_{t+1}) = o(t)$ by the condition

$$\sup\{d(x_t, x_{t+\tau}) : 0 \leq \tau \leq 1\} = o(t).$$

3. Multiplicative Ergodic Theorem and Law of Large Numbers for Semisimple Lie Groups

We shall say that the probability distribution μ on the group G has finite first moment, if

$$\int \delta(g)\, d\mu(g) < \infty.$$

THEOREM 3.1. If $\{h_t\}$ is a stationary sequence of random variables with values in G and finite first moment, then the sequence of products $g_t = h_1 \dots h_t$ is a.s. regular.

Proof. Since the distribution h_t has finite first moment, by virtue of the ergodic theorem a.s. $\delta(h_t) = o(t)$. The use of the Furstenberg–Kesten theorem or the subadditive ergodic theorem of Kingman (cf. [15, 17]) now lets us apply Theorem 2.3.

As we shall see in Sec. 4, for the case when $G = SL(n, \mathbb{R})$, Theorem 3.1 coincides with the multiplicative ergodic theorem of Oseledets, so it is natural to call it the multiplicative ergodic theorem for semisimple Lie groups. The next theorem follows quickly from Theorem 3.1.

THEOREM 3.2. If $\{h_t\}$ is a stationary sequence of random variables with values in G and finite first moment, then for a.a. trajectories of the random walk $g_t = h_1 \dots h_t$ on the group G with increments $\{h_t\}$ there exists a "mean" $g = g(\{g_t\})$ such that $\delta(g^{-t} g_t) = o(t)$. One can always choose the mean to have the form $g = \kappa(\exp \alpha)\kappa^{-1}$ where $\alpha \in \mathfrak{a}^+$ and $\kappa \in K$ under this condition $g(\{g_t\})$ defines $\{g_t\}$ uniquely and is measurable with respect to the tail σ-algebra of the sequence $\{g_t\}$.

Remarks. 1. The classical strong law of large numbers for stationary random sequences (the ergodic theorem) can be written in two forms: $(X_1 + \dots + X_n)/n \to a$ and $X_1 + \dots + X_n = na + o(n)$. Carrying the law of large numbers in the first form over to noncommutative groups in general is impossible due to the absence of a normalizing operation, so as a rule one considers either the law of large numbers for some numerical functionals on the group, or the law of large numbers for infinitesimal systems [6, 16, 18]. We give a law of large numbers for noncommutative groups, starting from the second way of writing the classical law of large numbers. Despite the complete naturality of such a formulation, it has apparently not occurred in the literature previously. The arbitrariness one has in the choice of values of the "mean" is explained by the fact that the classes of the relation of asymptotic equivalence in G

$$g_1 \sim g_2 \iff \delta(g_1^{-t} g_2^{t}) = o(t)$$

generally contain more than one element. It is clear from point 1.5 and Theorem 2.3 that the elements of the form $\kappa(\exp \alpha)\kappa^{-1}$ form a complete system of representatives of the classes. The measurable dependence of the mean g on the tail behavior of the trajectory $\{g_t\}$ has the same character as in the ordinary ergodic theorem for nonergodic stationary sequences (we stress that the tail σ-algebra of the sequence $\{g_t\}$ is nontrivial as a rule, despite the ergodicity of $\{h_t\}$).

2. The theorem proved shows the equivalence (as in the classical case of the additive group $\mathbb{R}$) of the law of large numbers ("in the global formulation") and the multiplicative ergodic theorem for semisimple Lie groups with finite center. For the group of matrices our theorem can be obtained from Oseledets' theorem if one notes that the regularity of a sequence

of matrices in the sense of Lyapunov is equivalent to its proximity to the sequence of powers of a symmetric matrix (cf. Sec. 4).

3. It is interesting to clarify when the tail σ -algebra of the sequence $\{g_t\}$ is completely determined by themeans $q(\{g_t\})$ (this is so for the case when $\{h_t\}$ are independent and their distribution is not a singular Haar measure [16], or, on the contrary, is concentrated on some discrete subgroup of G [4]). We note that if the first moment of the increments h_t is infinite, then it is not entirely clear even in what terms one could describe the tail behavior for groups of rank greater than one.

4. It would be quite interesting to determine the class of Lie groups for which the law of large numbers is valid in the global formulation given above. It is proved for all nilpotent Lie groups and simply connected decomposable solvable Lie groups [19, 20]. However, as G. A. Margulis noted, for semisimple Lie groups with infinite center, this law is no longer valid. In fact, let $\widetilde{G}$ be a simply connected semisimple Lie group with infinite center, G be its quotient by the free component of the center C. We denote by $\widetilde{K}$ the universal covering of the group K. Fixing a fundamental domain $\widetilde{K}$ in K_0 with compact closure, we can represent any element $\widetilde{\kappa} \in \widetilde{K}$ uniquely in the form $\widetilde{\kappa} = (c, \kappa)$, where the "rotation number" c lies in C and $\widetilde{\kappa} \mapsto \kappa$. Using the Cartan decomposition $G = K(\exp \mathfrak{a}^+)K$, this representation can be extended to the entire group $\widetilde{G}$. It is clear from the results of Sec. 2 that for any element $g \in G$ such that $\lim \delta(g^n)/n > 0$ the factors κ_n and κ'_n in the decomposition $g^n = \kappa_n(\exp \alpha_n)\kappa'_n$ stabilize (up to factors from Z) and $\alpha_n / n \to \alpha$. Hence for $\widetilde{g}^n = (c_n, g_n)$ the limit $\lim c_n/n = c \in C$ exists. Obviously one can choose $\widetilde{g}$ so that $c \neq 0$. On the other hand, if $\widetilde{g}_n = \widetilde{h}_1 \ldots \widetilde{h}_n = (c'_n, g_n)$, where $\widetilde{h}_i$ are independent and have distribution $\beta\delta_{\widetilde{g}} + (1-\beta)\delta_{\widetilde{e}}$ then $\lim c'_n/n = \beta c$. It is easy to see that the function $\widetilde{\delta}(c, g) = |c| + \delta(g)$ is a principal gauge on $\widetilde{G}$ (cf. point 1.2) and in order that two elements of $\widetilde{G}$ be distant from one another by $o(n)$ it is necessary that their "rotation numbers" differ by $o(n)$. But as we just showed, the "mean number of rotations" for the sequence of powers of any element $\widetilde{g} \in \widetilde{G}$ necessarily lies in C, at the same time that for the sequence of partial products of random variables it can also not lie in C (for irrational β). Thus, for the group $\widetilde{G}$ the global law of large numbers does not hold.

4. Connection with Classical Lyapunov Exponents

Definition [1-3]. Let V be a finite-dimensional vector space over the field Φ ($= \mathbb{R}$ or $\mathbb{C}$), $\|\cdot\|$ be a norm in V. The sequence $\{A_t\}_{t=1}^{\infty} \subset GL(V)$ is called Lyapunov regular if the following conditions hold:

1) the limit

$$\lim \log |\det A_t| / t = \chi_{\det}$$

exists;

2) for any $v \in V \setminus \{0\}$ the limit

$$\lim \log \|A_t v\| / t = \chi(v)$$

exists;

3) if $\{0\} = V_0 \subset V_1 \subset \ldots \subset V_k = V$ is the filtration of χ corresponding to the function V, i.e., $\chi(v) = \chi_i$ for $v \in V_i \setminus V_{i-1}$ and $\chi_1 < \ldots < \chi_k$, then

$$\chi_{\det} = \sum_i (\dim V_i - \dim V_{i-1})\chi_i .$$

The numbers χ_i are called the Lyapunov characteristic exponents of the sequence $\{A_t\}$ the dimensions $\dim V_i - \dim V_{i-1}$, their multiplicities. (The definition obviously does not depend on the choice of norm.)

In V we fix a Euclidean (resp., Hermitian) form θ. By K we denote the subgroup of $SL(V)$ preserving the form θ. We identify the symmetric space $SL(V)/K$ with the set of positive self-adjoint operators (with respect to the form θ) from $SL(V)$ (the choice of the form θ is equivalent to fixing a point $x_0 = K$ of the symmetric space).

THEOREM 4.1. Let $\{A_t\}_{t=1}^{\infty} \subset GL(V)$ and the limit $\lim \log |\det A_t|/t = \chi_{\det}$, exist; then the following conditions are equivalent:

1) the sequence $\{A_t\}$ is Lyapunov regular;

2) there exists a positive self-adjoint operator Λ such that $\log \|A_t \Lambda^{-t}\| = o(t)$;

3) $\log \|A_{t+1} A_t^{-1}\| = o(t)$ and the limit $\Lambda = \lim (A_t^* A_t)^{1/2t}$ exists;

4) the sequence $x_t = (A_t^* A_t)^{1/2} / |\det A_t|^{1/n}$, where $n = \dim V$, is regular in the symmetric space $SL(V)/K$.

In particular, if $\{A_t\} \subset SL(V)$, then the regularity of the sequence $\{A_t\}$ in the sense of Lyapunov is equivalent to the regularity of the sequence $\{A_t^*\}$ of elements of the group $SL(V)$ in the sense of the definition of Sec. 2. The operator Λ from condition (2) is defined uniquely and coincides with the operator Λ from condition (3), its eigenvalues are $\exp(\chi_i - \chi_{\det})$ and the corresponding eigenspaces are $V_i \ominus V_{i-1}$ (orthogonal complement with respect to the form θ). The vector of exponents of the sequence $\{x_t\}$ is formed by the numbers $\chi_i - \chi_{\det}$, taken according to their multiplicities and positioned in increasing order.

Proof. Normalization of A_t reduces the general case to the case $|\det A_t| = 1$; then the equivalence of conditions (2), (3), and (4) follows from the results of Section 2. To prove the implication $(2) \Longrightarrow (1)$ we note that if $B_t = A_t \Lambda^{-t}$ then $\log\|B_t^{-1}\| = o(t)$, since $|\det B_t| = 1$ and the Lyapunov regularity of the sequence $A_t = B_t \Lambda^t$ follows from from the regularity of the sequence of powers $\{\Lambda^t\}$. To prove the opposite implication $(1) \Longrightarrow (2)$ we define Λ as in the formulation of the theorem and we consider the basis $\{e_i\}$ in V, formed of eigenvectors of Λ. Then obviously $\log \|A_t \Lambda^{-t} e_i\| = o(t)$ for all e_i so $\log \|A_t \Lambda^{-t}\| = o(t)$. The description of the vector of exponents of the sequence $\{x_t\}$ follows directly from the description of the operator Λ.

With the help of the theorem proved one can now carry the results of Secs. 2 and 3 over to sequences $\{A_t\} \subset GL(V)$. In particular, it follows from Theorem 3.1 that if the increments $A_{t+1} A_t^{-1}$ form a stationary stochastic process, and their one-dimensional distribution has finite first moment (i.e., $\log \|A_{t+1} A_t^{-1}\| = o(t)$ and $\log \|A_t A_{t+1}^{-1}\| = o(t)$), then the sequence $\{A_t\}$ is Lyapunov regular, which is the content of the multiplicative ergodic theorem of Oseledets [3].

Remark. In the general situation from the existence of the limit $\Lambda = \lim (A_t^* A_t)^{1/2t}$ (in geometric language this means convergence of the directing vectors of the geodesics $(x_0 x_t)$ and the existence of the limit $\lim d(x_0, x_t)/t$) the Lyapunov regularity of the sequence $\{A_t\}$ (or the regularity of the sequence of points $\{x_t\}$ of the symmetric space no longer follows. This is so only when $\log \| A_{t+1} A_t^{-1} \| = o(t)$ (i.e., $d(x_t, x_{t+1}) = o(t)$). If the increments $A_{t+1} A_t^{-1}$ form a stationary process and their one-dimensinal distribution has finite first moment, then this condition holds (cf. the proof of Theorem 3.1). Hence for such sequences $\{A_t\}$ Lyapunov regularity is equivalent to the a priori weaker condition of existence of the limit $\Lambda = \lim (A_t^* A_t)^{1/2t}$ (cf. [21]).

Theorem 4.1 combined with Theorems 2.3 and 2.5 lets us get the following criterion for Lyapunov regularity.

THEOREM 4.2. A sequence $\{A_t\} \subset GL(V)$ is Lyapunov regular if an only if $\log \| A_{t+1} A_t^{-1} \| = o(t)$ and for any $k \leqslant \dim V$ the limit

$$\xi_k = \lim \log \| A_t^{\wedge k} \| / t$$

exists (here $A^{\wedge k}$ is the k-th exterior power of A), here the exponents of the sequence $\{A_t\}$ are the numbers $\chi_k = \xi_k - \xi_{k-1}$.

THEOREM 4.3. Let the matrices $A_t \in GL(V)$ be upper quasitriangular in some basis of the space V; we denote by A_t^i their diagonal blocks. The sequence $\{A_t\}$ is Lyapunov regular if and only if $\log \| A_{t+1} A_t^{-1} \| = o(t)$ and all sequences of diagonal blocks $\{A_t^i\}$ are Lyapunov regular. The collection of exponents of the sequence $\{A_t\}$ is formed by combining the collections of exponents of the sequences $\{A_t^i\}$. In particular, the sequence $\{A_t\}$ of triangular matrices is Lyapunov regular if and only if $\log \| A_{t+1} A_t^{-1} \| = o(t)$ and the limits $\lambda^i = \lim \log |a_t^i| / t$ exist (here a_t^i are the diagonal elements of the matrix A_t); the numbers λ^i form the collection of exponents of the sequence $\{A_t\}$.

Theorems 4.1-4.3 carry over to the case when the parameter t assumes continuous values. Here the condition $\log \| A_{t+1} A_t^{-1} \| = o(t)$ is replaced by the condition (cf. Section 2)

$$\sup \{ \log \| A_{t+\tau} A_t^{-1} \| : 0 \leqslant \tau \leqslant 1 \} = o(t). \tag{*}$$

The linear differential equation $\dot{x} = B(t) x$ in the space V is called regular, if its fundamental matrix $A(t) \in GL(V)$ is Lyapunov regular. If $\log^+ \| B(t) \| = o(t)$ then (*) holds (the converse is generally false). Thus, Theorems 4.2 and 4.3 give necessary and sufficient conditions for the regularity of linear systems with subexponential growth of coefficients. For the case of bounded coefficients and triangular matrices, Theorem 4.3 is the Lyapunov criterion of [1], and Theorem 4.2 is also familiar for the case of bounded coefficients [2]†. We stress that (*) is a necessary condition for the system to be regular.

†In [2] these assertions are formulated without any assumptions about the boundedness of the coefficients of $B(t)$ but this condition is used there essentially. Some kind of restrictions on the nondiagonal coefficients are also missing in the formulatin of Lyapunov's criterion in the paper of V. M. Millionshchikov "Regular linear systems" in Vol. 4 of the Mathematical Encyclopedia (in the original paper of Lyapunov the condition of boundedness of the coefficients is explicitly formulated), although, as will be shown below, Lyapunov's criterion is invalid without such restrictions.

If it fails (here, naturally $\log^+\|B(t)\| \neq o(t)$), then even a triangular system may not be regular even if the exact means of the diagonal elements of B(t) exist. Simplest example:

$$B(t)=\begin{pmatrix} 0 & e^t \\ 0 & 0 \end{pmatrix},\ A(t)=\begin{pmatrix} 1 & e^t-1 \\ 0 & 1 \end{pmatrix}.$$

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