SOME REMARKS ABOUT THE ZARISKI TOPOLOGY OF THE CREMONA GROUP

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Abstract. For an algebraic variety X we study the behavior of algebraic functions from an algebraic variety to the group $\mathrm{Bir}(X)$ of birational maps of X and obtain, as application, some insight about the relationship between the so-called Zariski topology of $\mathrm{Bir}(X)$ and the algebraic structure of this group, where X is a rational variety.

1. Introduction

Let $\Bbbk$ be an algebraically closed field and denote by $\mathbb{P}^n$ the projective space of dimension n over $\Bbbk$. The set $\mathrm{Bir}(\mathbb{P}^n)$ of birational maps $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is the so-called Cremona group of $\mathbb{P}^n$. For an element $f \in \mathrm{Bir}(\mathbb{P}^n)$ there exist homogeneous polynomials of the same degree $f_0, \dots, f_n \in k[x_0, \dots, x_n]$, without nontrivial common factors, such that if $\mathbf{x} = (x_0 : \cdots : x_n)$ is not a common zero of the f_i's, then $f(\mathbf{x}) = \big(f_0(\mathbf{x}) : \cdots : f_n(\mathbf{x})\big)$. The (algebraic) degree of f is the common degree of the f_i's, and is denoted by $\deg(f)$.

A natural way to produce an "algebraic family" of birational maps is to consider a birational map $f = (f_0 : \cdots : f_n) \in \mathrm{Bir}(\mathbb{P}^n)$ and to allow the coefficients of the f_i's vary in an affine (irreducible) $\Bbbk$-variety T. That is, we consider polynomials $f_0, \dots, f_n \in \Bbbk[T] \otimes \Bbbk[x_0, \dots, x_n]$, homogeneous and of the same degree in $\mathbf{x}$ and we define $\varphi : T \to \mathrm{Bir}(\mathbb{P}^n)$ by

$$\varphi(t, \mathbf{x}) = \big(f_0(t, \mathbf{x}) : \cdots : f_n(t, \mathbf{x})\big);$$

in particular we assume that for all $t \in T$ the map $\varphi_t := \varphi(t, \cdot) : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is birational.

As pointed out by Serre in [Ser08, §1.6] there exists an unique topology on $\mathrm{Bir}(\mathbb{P}^n)$ which makes any such algebraic family a continuous function, designed in *loc. cit* as the *Zariski Topology* of $\mathrm{Bir}(\mathbb{P}^n)$. Moreover, we can replace $\mathbb{P}^n$ with an irreducible algebraic variety X of dimension n and the same holds for $\mathrm{Bir}(X)$.

The aim of this work is to study the behavior of this "morphisms" $T \to \mathrm{Bir}(X)$ and obtain, as application, some insight about the relationship between the topology and the algebraic structure of the group $\mathrm{Bir}(X)$, where X is a rational variety.

More precisely, in Section 2 we present some basic results about $\mathrm{Bir}(X)$ that show the relationship between the algebraic structure and the Zariski topology.

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In Section 3, the main one, we deal with the case $X = \mathbb{P}^n$, or more generally the case where X is a rational variety(see Lemma 2). We begin by stating two deep results about the connectedness and simplicity of $\mathrm{Bir}(\mathbb{P}^n)$ proved in [Bl2011] and [CaLa2011] (Proposition 16) and extract as an easy consequence that a nontrivial normal subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ has trivial centralizer. Next we prove that for a morphism $\varphi : T \to \mathrm{Bir}(\mathbb{P}^n)$, the function $t \mapsto \deg(\varphi_t)$ is lower semicontinuous (§3.2). This result has some nice consequences:

(a) every Cremona transformation of degree d is a specialization of Cremona transformations of degree $> d$;
(b) the degree map $\deg : \mathrm{Bir}(\mathbb{P}^n) \to \mathbb{Z}$ is lower semicontinuous and every morphism $T \to \mathrm{Bir}(\mathbb{P}^n)$ restricts to a dense open set as a morphism of algebraic varieties (§3.3);
(c) a morphism $T \to \mathrm{Bir}(\mathbb{P}^n)$ maps constructible sets into constructible sets (§3.4);
(d) the Zariski topology of $\mathrm{Bir}(\mathbb{P}^n)$ is not Noetherian (§3.5);
(e) there exist (explicit, non canonical) closed immersions of $\mathrm{Bir}(\mathbb{P}^{n-1}) \hookrightarrow \mathrm{Bir}(\mathbb{P}^n)$.
(f) the subgroup consisting of the elements $f \in \mathrm{Bir}(\mathbb{P}^n)$ which stabilize the set of lines passing through a fixed point is closed (§3.6).

Some weeks before this work was finished Blanc and Furter posted a preprint in arXiv (see arXiv:1210.6960v1) where, among other interesting things, they also obtain some of our results as for example item (a) above.

2. Generalities

Following [De1997, §2] a birational map $\varphi : T \times X \dashrightarrow T \times X$, where T and X are $\Bbbk$-varieties and X is irreducible, is said to be a *pseudo-automorphism of* $T \times X$, *over* T, if there exists a dense open subset $U \subset T \times X$ such that:

(a) φ is defined on U;
(b) $U_t := U \cap \big(\{t\} \times X\big)$ is dense in $\{t\} \times X$ for all $t \in T$, and
(c) there exists a morphism $f : U \to X$ such that $\varphi|_U(t, x) = \big(t, f(t, x)\big)$, and $\varphi|_{U_t} : U_t \to \{t\} \times X$ is a birational morphism.

In particular, a pseudo-automorphism φ as above induces a family $T \to \mathrm{Bir}(X)$ of birational maps $\varphi_t : X \dashrightarrow X$. Following [Bl2011] we call this family an *algebraic family* in $\mathrm{Bir}(X)$ or a *morphism* from T to $\mathrm{Bir}(X)$.

We will identify a morphism $\varphi : T \to \mathrm{Bir}(X)$ with its corresponding pseudo-automorphism and denote $\varphi_t = \varphi(t)$.

Note that if $\varphi : T \to \mathrm{Bir}(X)$ is a morphism, the map $\psi : T \to \mathrm{Bir}(X)$ defined by $\psi_t = \varphi_t^{-1}$ is also a morphism where φ_t^{-1} denotes the inverse map of φ_t.

We say $\mathcal{F} \subset \mathrm{Bir}(X)$ is *closed* if its pullback under every morphism $T \to \mathrm{Bir}(X)$ is closed in T, for all T. This defines the so-called *Zariski topology* on $\mathrm{Bir}(X)$ ([Mu1974], [Ser08, §1.6], [Bl2011]).

In order to define the Zariski topology, as above, it suffices to consider morphisms from an affine variety T. Indeed, notice that a subset $F \subset T$ is closed if and only if there exists a cover by open sets $T = \cup V_i$, with V_i affine, such that $F \cap V_i$ is closed in V_i, for all i. Then we may restrict a pseudo-automorphism $\varphi : T \times X \dashrightarrow T \times X$ to each $V_i \times X$ and obtain a pseudo-automorphism $\varphi_i : V_i \times X \dashrightarrow V_i \times X$, for every i. The assertion follows easily from the previous remark. Clearly, we may also suppose T is irreducible.

Unless otherwise explicitly stated, in the sequel we always suppose T is affine and irreducible.

Lemma 1. *Let X be an algebraic variety, and endow $\mathrm{Bir}(X)$ with a topology $\mathcal{T}$. Let $Z \subset \mathrm{Bir}(X)$ be a locally closed set with respect to $\mathcal{T}$ such that $\mathcal{T}$ induces a structure of algebraic variety on Z. Then the $\mathcal{T}$-topology of Z is finer than the induced Zariski topology, that is, if $\mathcal{F} \subset \mathrm{Bir}(X)$ is a Zariski closed subset, then $\mathcal{F} \cap Z$ is $\mathcal{T}$-closed in Z.*

Proof. Let $\varphi : Z \times X \to Z \times X$ be given by $\varphi(z, x) = \big(z, z(x)\big)$. Cleary, φ is a pseudo-automorphism such that $\varphi^{-1}(\mathcal{F}) = \mathcal{F} \cap Z$. Since $\mathcal{F}$ is closed in $\mathrm{Bir}(X)$ for the Zariski topology, it follows that $\mathcal{F} \cap Z$ is closed in Z. □

Lemma 2. *Let $F : X \dashrightarrow Y$ be a birational morphism between two algebraic varieties. Then the map $F^* : \mathrm{Bir}(Y) \to \mathrm{Bir}(X)$ defined by $F^*(f) = F^{-1} \circ f \circ F$ is a homeomorphism, with inverse $(F^{-1})^*$.*

Proof. The result follows once we observe that $\varphi : T \times Y \dashrightarrow T \times Y$ is a pseudo-automorphism if and only if $(\mathrm{id} \times F^{-1}) \circ \varphi \circ (\mathrm{id} \times F) : T \times X \dashrightarrow T \times X$ is a pseudo automorphism. □

We consider $\mathrm{Bir}(X) \times \mathrm{Bir}(Y) \subset \mathrm{Bir}(X \times Y)$ by taking $(f, g) \in \mathrm{Bir}(X) \times \mathrm{Bir}(Y)$ into the rational map $F : X \times Y \to X \times Y$ defined as $F(x, y) = \big(f(x), g(y)\big)$.

Lemma 3. *Let X, Y be algebraic varieties and $F \in \mathrm{Bir}(X \times Y)$ a birational map; write $F(x, y) = \big(F_1(x, y), F_2(x, y)\big)$ for $(x, y) \in X \times Y$ in the domain of F. Then $F \in \mathrm{Bir}(X) \times \mathrm{Bir}(Y) \subset \mathrm{Bir}(X \times Y)$ if and only if there exist dense open subsets $U \subset X$, $V \subset Y$ such that F is defined on $U \times V$ and $F_1(x, y) = F_1(x, y')$, $F_2(x, y) = F_1(x', y)$ for $x, x' \in U$, $y, y' \in V$,*

Proof. First suppose there exist $f \in \mathrm{Bir}(X)$ and $g \in \mathrm{Bir}(Y)$ such that $F(x, y) = \big(f(x), g(y)\big)$. Consider nonempty open sets $U \subset X$ and $V \subset Y$ such that f and g are defined on U and V respectively. Hence, F_1 and F_2 are defined on $U \times V$ and we have that $F_1(x, y) = f(x)$ and $F_2(x, y) = g(y)$, from which the "only if part" follows.

Conversely, suppose there exist nonempty open sets U and V as stated. Then F_1 and F_2 induce morphisms $f : U \to X$ and $g : V \to Y$ such that $F(x, y) = \big(f(x), g(y)\big)$ for $(x, y) \in U \times V$. Since $U \times V$ is dense in $X \times Y$, this completes the proof. $\square$

Proposition 4. *If X, Y are algebraic varieties, then $\mathrm{Bir}(X) \times \mathrm{Bir}(Y) \subset \mathrm{Bir}(X \times Y)$ is a closed subgroup.*

Proof. In view of Lemma 2, we can assume that $X \subset \mathbb{A}^n, Y \subset \mathbb{A}^m$ are affine varieties. Let $\varphi : T \times X \times Y \dashrightarrow T \times X \times Y$ be a pseudo-automorphism (over T). Then

$$\varphi(t, x, y) = \big(t, f_1(t, x, y), \ldots, f_n(t, x, y), g_1(t, x, y), \ldots, g_m(t, x, y)\big),$$

where $f_i, g_j \in \Bbbk(T \times X \times Y)$ are rational functions on $T \times X \times Y$ (of course, f_i, g_j verify additional conditions).

Let $A := \varphi^{-1}\big(\mathrm{Bir}(X) \times \mathrm{Bir}(Y)\big)$ and denote by $\overline{A}$ the closure of A in T. Following Lemma 3 it suffices to prove that the restrictions of the f_i's (resp. the g_j's) to $\overline{A} \times X \times Y$ do not depend on y (resp. on x), which implies $A = \overline{A}$.

Up to restrict φ to each irreducible component of $\overline{A}$ we may suppose that A is dense in T. By symmetry we only consider the case relative to the f_i's and write $f = f_i$ for such a rational function.

Since the poles of f are contained in a proper subvariety of $T \times X \times Y$, we deduce that there exists $y_0 \in Y$ such that the restriction of f to $T \times X \times \{y_0\}$ induces a rational function on this subvariety. If $p : T \times X \times Y \to T \times X \times \{y_0\}$ denotes the morphism $(t, x, y) \mapsto (t, x, y_0)$ we conclude $f \circ p$ is a rational function on $T \times X \times Y$.

Our assumption implies f coincides with $f \circ p$ along $A \times X \times Y$, which is dense in $T \times X \times Y$, so $f = f \circ p$ and the result follows. $\square$

Remark 5. Two pseudo-automorphisms $\varphi : T \times X \dashrightarrow T \times X$ and $\psi : T \times Y \dashrightarrow T \times Y$ induce a morphism $(\varphi, \psi) : T \to \mathrm{Bir}(X) \times \mathrm{Bir}(Y)$, that is, an algebraic family in $\mathrm{Bir}(X) \times \mathrm{Bir}(Y)$. As in the proof of Proposition 4, it follows from Lemma 3 that $\mathcal{F} \subset \mathrm{Bir}(X) \times \mathrm{Bir}(Y)$ is closed if and only if $(\varphi, \psi)^{-1}(\mathcal{F})$ is closed for every pair φ, ψ. Moreover, it is easy to prove that the topology on $\mathrm{Bir}(X) \times \mathrm{Bir}(Y)$ induced by the Zariski topology of $\mathrm{Bir}(X \times Y)$ is the *unique* topology for which all the morphisms (φ, ψ) are continuous.

Observe that the Zariski topology of $\mathrm{Bir}(X) \times \mathrm{Bir}(Y)$ is finner that the product topology of the Zariski topologies of its factors.

Proposition 6. *If $\varphi, \psi : T \to \mathrm{Bir}(X)$ are morphisms, then $t \mapsto \varphi_t \circ \psi_t$ defines an algebraic family in $\mathrm{Bir}(X)$. Moreover, the product homomorphism $\mathrm{Bir}(X) \times \mathrm{Bir}(X) \to \mathrm{Bir}(X)$ and the inversion map $\mathrm{Bir}(X) \to \mathrm{Bir}(X)$ are continuous.*

Proof. To prove the first assertion it suffices to note that the family $t \mapsto \varphi_t \circ \psi_t$ corresponds to the pseudo-automorphism $\varphi \circ \psi : T \times X \dashrightarrow T \times X$. Appling Remark 5, the first part of the second assertion follows. Indeed, if $\mathcal{F} \subset \mathrm{Bir}(X)$ is a closed subset, then

$(\varphi,\psi)^{-1}\big(m^{-1}(\mathcal{F})\big) = (\varphi\circ\psi)^{-1}(\mathcal{F})$. For the rest of the proof it suffices to note that for a family φ as above the map $t \mapsto \psi_t^{-1}$ defines an algebraic family. □

Lemma 7. *The Zariski topology on* $\mathrm{Bir}(X)$ *is T1. In particular, if* $\varphi,\psi : T \to \mathrm{Bir}(X)$ *are two morphisms, then the subset* $\big\{t\in T; \varphi(t)=\psi(t)\big\}$ *is closed.*

Proof. It suffices to show that $id\in\mathrm{Bir}(X)$ is a closed point. Without loss of generality we may suppose $X\subset\mathbb{P}^m$ is a projective variety. Then a morphism $\varphi : T\to\mathrm{Bir}(X)$ may be represented as

$$\varphi_t = \big(f_0(t,x):\cdots:f_m(t,x)\big), t\in T, x\in X$$

where $f_i\in\Bbbk[t,x_0,\ldots,x_m]$, $i=0,\ldots,m$, are homogeneous of same degree in the variables $x_0,\ldots,x_m$. Therefore

$$\begin{aligned}\big\{t\in T;\varphi(t)=id\big\} &= \bigcap_{i,j=0}^{m}\big\{t\in T : x_jf_i(t,x)=x_if_j(t,x),\ \forall x\in X\big\}\\ &= \bigcap_{i,j=0,x\in X}^{m}\big\{t\in T : x_jf_i(t,x)=x_if_j(t,x)\big\}.\end{aligned}$$

Since for all i,j the equations

$$x_jf_i(t,x)-x_if_j(t,x)=h_1(x)=\cdots=h_\ell(x)=0$$

define a closed set in $T\times X$, and X is projective we deduce $\big\{t\in T:\varphi(t)=id\big\}$ is closed in T. □

Corollary 8. *Let* $\psi : Y\to\mathrm{Bir}(X)$ *be a morphism, where* Y *is a projective variety. Then* $\psi(Y)$ *is closed.*

Proof. A morphism $\varphi : T\to\mathrm{Bir}(X)$ induces a morphism $\phi : T\times Y\to\mathrm{Bir}(X)$ defined by $(t,y)\mapsto\varphi(t)\circ\psi(y)^{-1}$. Then $\phi^{-1}\big(\{id\}\big)=\big\{(t,y);\varphi(t)=\psi(y)\big\}$ is closed in $T\times Y$. The projection of this set onto the first factor is exactly $\varphi^{-1}\big(\psi(Y)\big)$ which is closed. □

Corollary 9. *The centralizer of an element* $f\in\mathrm{Bir}(X)$ *is closed. In particular, the centralizer* $C_{\mathrm{Bir}(X)}(G)$ *of a subgroup* $G\subset\mathrm{Bir}(X)$ *is closed.* □

Proof. Since the commutator map $c_f : \mathrm{Bir}(X)\to\mathrm{Bir}(X)$, $c_f(h)=hfh^{-1}f^{-1}$, is continuous, $c_f^{-1}\big(\{id\}\big)$ is closed. □

Another consequence of Lemma 7 (and Remark 5) is that for an arbitrary topological subspace $A\subset\mathrm{Bir}(X)$ and a point $f\in\mathrm{Bir}(X)$, the natural identification map $\{f\}\times A\to A$ is an homeomorphism. As in [Sha, Chap.I, Thm. 3] we obtain:

Corollary 10. *If* $A,B\subset\mathrm{Bir}(X)$ *are irreducible subspaces, then* $A\times B$ *is an irreducible subspace of* $\mathrm{Bir}(X)\times\mathrm{Bir}(X)$.

Proposition 11. *The irreducible components of* $\mathrm{Bir}(X)$ *do not intersect. Moreover,* $\mathrm{Bir}(X)^0$*, the unique irreducible component of* $\mathrm{Bir}(X)$ *which contains id, is a normal (closed) subgroup.*

Proof. Let A, B be irreducible components containing id. Corollary 10 implies $A \cdot B$ is irreducible. Since $id \in A \cap B$ then $A \cup B \subset A \cdot B$ from which it follows $A = A \cdot B = B$. This proves the uniqueness of $\mathrm{Bir}(X)^0$.

The rest of the proof works as in [FSRi, Chapter 3, Thm. 3.8]. □

We have also the following easy result:

Proposition 12. *Let* $H \subset \mathrm{Bir}(X)$ *be a subgroup.*

(a) The closure $\overline{H}$ *of* H *is a subgroup. Moreover, if* H *is normal, then* $\overline{H}$ *is normal.*

(b) If H *contains a dense open set, then* $H = \overline{H}$*.*

Proof. The proof of this result follows the same arguments that the analogous case for algebraic groups (see [FSRi, Chapter 3, Section 3]). For example, in order to prove the second part of (a) it suffices to note that since Int_f is an homeomorphism, then $\mathrm{Int}_f(\overline{H}) = \overline{\mathrm{Int}_f(H)}$. □

3. The Cremona group

Now we consider the case $X = \mathbb{P}^n$; we fix homogeneous coordinates $x_0, \dots, x_n$ in $\mathbb{P}^n$. As in the introduction, if $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is a birational map, the *degree* of f is the minimal degree $\deg(f)$ of homogeneous polynomials in $\Bbbk[x_0, \dots, x_n]$ defining f.

3.1. Connectedness and simplicity.

In [Bl2011, Thms. 4.2 and 5.1] Jérémy Blanc proves the following two results:

Theorem 13 (J. Blanc)**.** $\mathrm{Bir}(\mathbb{P}^2)$ *does not admit nontrivial normal closed subgroups.*

Theorem 14 (J. Blanc)**.** *If* $f, g \in \mathrm{Bir}(\mathbb{P}^n)$*, then there exists a morphism* $\theta : U \to \mathrm{Bir}(\mathbb{P}^n)$*, where* U *is an open subset of* $\mathbb{A}^1$ *containing* $0, 1$*, such that* $\theta(0) = f, \theta(1) = g$*. In particular* $\mathrm{Bir}(\mathbb{P}^n)$ *is connected.*

In Theorem 14 the open set U is irreducible and the morphism θ is continuous. Hence we deduce that $\mathrm{Bir}(\mathbb{P}^n)$ is irreducible.

On the other hand, in [CaLa2011] Serge Cantat and Sthéphane Lamy prove the following result:

Theorem 15 (S. Cantat-S. Lamy)**.** $\mathrm{Bir}(\mathbb{P}^2)$ *is not a simple (abstract) group, i.e., it contains a non trivial normal subgroup.*

In fact they prove that for a "very general" birational map $f \in \mathrm{Bir}(\mathbb{P}^2)$ of degree d, with $d \gg 0$, the minimal normal subgroup containing f is nontrivial. From Theorems 13 and 15 it follows that all non trivial normal subgroup in $\mathrm{Bir}(\mathbb{P}^2)$ is dense.

Putting all together we obtain:

Proposition 16. *Let $G \subset \mathrm{Bir}(\mathbb{P}^2)$ be a nontrivial normal subgroup. Then $C_{\mathrm{Bir}(\mathbb{P}^2)}(G) = \{id\}$.*

Proof. Suppose $C_{\mathrm{Bir}(\mathbb{P}^2)}(G) \neq \{id\}$. The closure $\overline{G}$ of G is a normal subgroup, then it coincides with the entire Cremona group. If $f \in C_{\mathrm{Bir}(\mathbb{P}^2)}(G)$, then G is contained in the centralizer of f, which is closed. We deduce that f commute with all the elements of $\mathrm{Bir}(\mathbb{P}^2)$, that is $C_{\mathrm{Bir}(\mathbb{P}^2)}(G)$ coincides with the center $Z(\mathrm{Bir}(\mathbb{P}^2))$ of $\mathrm{Bir}(\mathbb{P}^2)$. Since $Z(\mathrm{Bir}(\mathbb{P}^2)) = \{id\}$, the result follows. For the convenience of the reader we give a proof of the well known fact that $Z(\mathrm{Bir}(\mathbb{P}^2)) = \{id\}$.

Recall that $\mathrm{Bir}(\mathbb{P}^2)$ is generated by quadratic transformations, i.e. maps of the form $g_1\sigma g_2$ where $g_1, g_2 \in \mathrm{PGL}(3, \Bbbk)$ and $\sigma = (x_1x_2 : x_0x_2 : x_0x_1)$ is the *standard* quadratic transformation. Take $f \in Z(\mathrm{Bir}(\mathbb{P}^2))$. If $L \subset \mathbb{P}^2$ is a general line, then we may construct a quadratic transformation σ_L which contracts L to a point and such that f is well defined in this point. Since $f\sigma_L = \sigma_L f$ and we may suppose f is well defined and injective on an open set of L we deduce f transforms L into a curve contracted by σ_L, that is, the strict transform of L under f is a line, and then $f \in \mathrm{PGL}(3, \Bbbk)$, so $f \in Z(\mathrm{PGL}(3, \Bbbk)) = \{id\}$. □

3.2. **Degree and semicontinuity.**

Let $\varphi : T \to \mathrm{Bir}(\mathbb{P}^n)$ be a morphism, where T is an affine irreducible variety. We may represent φ in the form

$$\varphi(t, \mathbf{x}) = \big(f_0(t, \mathbf{x}) : \cdots : f_n(t, \mathbf{x})\big)$$

where $f_i \in \Bbbk[T] \otimes \Bbbk[\mathbf{x}] = \Bbbk[T] \otimes \Bbbk[x_0, \ldots, x_n]$ are polynomials which are homogeneous of same degree in $x_0, \ldots, x_n$. We suppose this degree minimal among all possible such representations for φ and denote it by $\mathrm{Deg}(\varphi)$. For $t \in T$ we denote by $\deg(\varphi_t)$ the usual algebraic degree of the map $\varphi_t : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$; this is the minimal degree of the homogeneous polynomials defining φ_t. Clearly $\deg(\varphi_t) \leq \mathrm{Deg}(\varphi)$ for all $t \in T$.

Consider the ideal $I(\varphi) \subset \Bbbk[T] \otimes \Bbbk[\mathbf{x}]$ generated by $f_0, \ldots, f_n$. Then $I(\varphi)$ defines a subvariety $X^\varphi \subset T \times \mathbb{A}^{n+1}$. Notice that X^φ is stable under the action of $\Bbbk^*$ on $T \times \mathbb{A}^{n+1}$ defined by $\lambda \cdot (t, x) \mapsto (t, \lambda x)$. Moreover, the projection $\pi : X^\varphi \to T$ onto the first factor is equivariant and, by definition, surjective. The function $t \mapsto \dim \pi^{-1}(t)$ is upper-semicontinuous, from which we deduce $T_n := \{t; \dim \pi^{-1}(t) \geq n\}$ is closed in T. Since $\pi^{-1}(t) = X^\varphi \cap \big((\{t\} \times \mathbb{A}^{n+1}\big)$, it follows that $\dim \pi^{-1}(t) > n$ implies $\pi^{-1}(t) = \{t\} \times \mathbb{A}^{n+1}$ which contradicts that fact that φ_t is well defined. Hence:

Lemma 17. *Let $\varphi : T \to \mathrm{Bir}(\mathbb{P}^n)$ be a morphism. Then $t \in T_n$ if and only if one of the following assertions holds*

(a) there is a codimension 1 subvariety $X_t^\varphi \subset \mathbb{A}^{n+1}$ such that $\pi^{-1}(t) = \{t\} \times X_t^\varphi$.

(b) $\deg(\varphi_t) < \mathrm{Deg}(\varphi)$.

Proposition 18. *Let* $\varphi : T \to \mathrm{Bir}(\mathbb{P}^n)$ *be a morphism. Then the set* $U_\varphi := \{t \in T : \deg(\varphi_t) = \mathrm{Deg}(\varphi)\}$ *is dense and open.*

Proof. Let $T_n \subset T$ be the closed subset introduced above. We will prove that $U_\varphi = T\backslash T_n$.

Suppose for a moment $T_n = T$ and denote by $Z \subset T \times \mathbb{A}^{n+1}$ the zero set defined by $I(\varphi)$. Then Z has codimension 1; denote by Z_1 the union of codimension 1 irreducible components of Z. If $T' \subset T$ is an affine dense open set consisting of smooth points of T, then $Z_1' := Z_1 \cap (T' \times \mathbb{A}^{n+1})$ is dense in Z_1 and Z_1' is the zero set of a reduced polynomial $g \in \mathbb{k}[T'] \otimes \mathbb{k}[\mathbf{x}]$, homogeneous in $x_0, \ldots, x_n$ whose degree in $\mathbf{x}$ is lesser that $\mathrm{Deg}(\varphi)$. Since g extends to an element in $\mathbb{k}(T) \otimes \mathbb{k}[\mathbf{x}]$, multiplying it by an element in $\mathbb{k}[T]$ we may suppose $g \in \mathbb{k}[T] \otimes \mathbb{k}[\mathbf{x}]$.

By construction every polynomial f_i factors as $f_i = gh_i$ for a polynomial h_i whose coefficients are, *a priori*, in the algebraic closure of $\mathbb{k}(T)$. Since these coefficients satisfy (at least) a compatible linear system of equations with coefficients in $\mathbb{k}[T]$, *a posteriori* we deduce $h \in \mathbb{k}(T) \otimes \mathbb{k}[\mathbf{x}]$. Then for a suitable $a \in \mathbb{k}[T]$ we may suppose $af_i = gh_i$ with $h_i \in \mathbb{k}[T] \otimes \mathbb{k}[\mathbf{x}]$ for all $i = 0, \ldots, n$.

The morphism $\psi : T \to \mathrm{Bir}(\mathbb{P}^n)$ induced by the pseudo-automorphism

$$(t, x) \mapsto \big(t, (h_0(t, \mathbf{x}) : \cdots : h_n(t, \mathbf{x}))\big)$$

coincides with φ but $\mathrm{Deg}(\psi) < \mathrm{Deg}(\varphi)$: contradiction. Thus $T_n \neq T$.

We conclude that $T\backslash T_n$ is a dense open set consisting of points $t \in T$ such that $\deg(\varphi_t) = \mathrm{Deg}(\varphi)$. ☐

Clearly the function $t \mapsto \deg(\varphi_t)$ takes finitely many values, say $d_1 = \mathrm{Deg}(\varphi) > d_2 > \cdots > d_\ell \geq 1$. Consider the decomposition $T\backslash U_\varphi = X_1 \cup \cdots \cup X_r$ in irreducible components. We may restrict φ to each X_i and apply Proposition 18 to conclude $\deg(\varphi_t) = d_2$ for t in an open set (possibly empty for some i) $U_i \subset X_i$ and $\deg(\varphi_t) < d_2$ on $X_i\backslash U_i$, $i = 1, \ldots, r$. Repeating the argument with d_3, and so on we deduce:

Theorem 19. *Let* $\varphi : T \to \mathrm{Bir}(\mathbb{P}^n)$ *be a morphism. Then*

(a) There exists a stratification by locally closed sets $T\backslash U_\varphi = \cup_{j=1}^{\ell} V_j$ *such that* $\deg(\varphi_t)$ *is constant on* V_j*, for all* $j = 1, \ldots, \ell$.

(b) The function $\deg \circ \varphi : T \to \mathbb{N}$, $t \mapsto \deg(\varphi_t)$*, is lower-semicontinuous.*

Corollary 20. *Every Cremona transformation of degree* d *is specialization of Cremona transformations of degrees* $> d$.

Proof. Let f be a Cremona transformation of degree d. Consider a morphism $\theta : T \to \mathrm{Bir}(\mathbb{P}^n)$, where T is a dense open set in $\mathbb{A}^1$ containing $0, 1$ such that $\theta(0) = f$ and $\theta(1)$ is a transformation of degree $e > d$ (Theorem 14). Then $\mathrm{Deg}(\theta) \geq e$ and the proof follows from Proposition 18. ☐

Corollary 21. *The degree function* $\deg : \mathrm{Bir}(\mathbb{P}^n) \to \mathbb{N}$ *is lower-semicontinuous, i.e. for all* d *the subset* $\mathrm{Bir}_{\leq d}(\mathbb{P}^n)$ *of birational maps of degree* $\leq d$ *is closed. In particular, a subset* $\mathcal{F} \subset \mathrm{Bir}(\mathbb{P}^n)$ *is closed if and only if* $\mathcal{F} \cap \mathrm{Bir}(\mathbb{P}^n)_{\leq d}$ *is closed for all* $d > 0$.

Proof. The assertion relative to semicontinuity is a direct consequence of Theorem 19(b). For the last assertion we note that if $\varphi : T \to \mathrm{Bir}(\mathbb{P}^n)$ is a morphism and $e = \mathrm{Deg}(\varphi)$, then $\varphi^{-1}(\mathcal{F}) = \varphi^{-1}\big(\mathcal{F} \cap \mathrm{Bir}(\mathbb{P}^n)_{\leq e}\big)$. ☐

Remark 22. Note that $\mathrm{Bir}(\mathbb{P}^n) = \bigcup_{d\geq 1} \mathrm{Bir}(\mathbb{P}^n)_{\leq d}$, with $\mathrm{Bir}(\mathbb{P}^n)_{\leq d} \subsetneq \mathrm{Bir}(\mathbb{P}^n)_{\leq d+1}$ and $\mathrm{Bir}(\mathbb{P}^n)_1 = \mathrm{PGL}(n+1, \Bbbk)$.

3.3. **Algebraization of morphisms.**

In this paragraph we deal with the morphisms $\varphi : T \to \mathrm{Bir}(\mathbb{P}^n)$ and their relationship with the stratification described in Theorem 19. We consider the locally closed sets $\mathrm{Bir}(\mathbb{P}^n)_d := \mathrm{Bir}(\mathbb{P}^n)_{\leq d} \backslash \mathrm{Bir}(\mathbb{P}^n)_{\leq d-1}$, where $d \geq 2$. If $\mathrm{Deg}(\varphi) = d$, then $U_\varphi = \varphi^{-1}(\mathrm{Bir}(\mathbb{P}^n)_d)$.

For integers d, n, r, with $d, n > 0$ and $r \geq 0$, we consider the vector space $V = \Bbbk[x_0, \ldots, x_n]_d^{r+1}$ of $(r+1)$-uples of d-forms. Notice that the projective space $\mathbb{P}_{(d,n,r)} = \mathbb{P}(V)$ consisting of dimension 1 subspaces in V has dimension $N(d, n, r) = \binom{n+d}{d}(r+1) - 1$.

Theorem 23. *We have the following assertions:*

(a) $\mathrm{Bir}(\mathbb{P}^n)_d$ *is a quasi projective variety whose topology coincides with the topology induced by* $\mathrm{Bir}(\mathbb{P}^n)$*; in particular* $\mathrm{Bir}(\mathbb{P}^n)_{\leq d}$ *is a finite union of quasi projective varieties.*

(b) If $\varphi : T \to \mathrm{Bir}(\mathbb{P}^n)$ *is a morphism, then the induced map* $U_\varphi \to \mathrm{Bir}(\mathbb{P}^n)_d$ *is a morphism of algebraic varieties.*

Proof. Let $e < d$ be a non negative integer number. Consider the projective spaces $\mathbb{P}_{(d,n,n)}$, $\mathbb{P}_{(d-e,n,n)}$ and $\mathbb{P}_{(e,n,0)}$. Then there exists a "Segre type" morphism $s : \mathbb{P}_{(d-e,n,n)} \times \mathbb{P}_{(e,n,0)} \to \mathbb{P}_{(d,n,n)}$ which to a pair of elements $(g_0 : \cdots : g_n) \in \mathbb{P}_{(d-e,n,n)}$, $(f) \in \mathbb{P}_{(e,n,0)}$ it associates $(g_0 f : \cdots : g_n f)$. We denote by $\mathcal{W}_e \subset \mathbb{P}_{(d,n,n)}$ the image of s, which is a projective subvariety.

Now consider the open set $\mathcal{U} \subset \mathbb{P}_{(d,n,n)}$ consisting of points $(f_0 : f_1 : \cdots : f_n)$ where the Jacobian determinant $\partial(f_0, f_1, \ldots, f_n)/\partial(x_0, \ldots, x_n)$ is not identically zero. Clearly, points $(f_0 : f_1 : \cdots : f_n) \in \mathbb{P}_{(d,n,n)} \cap \mathcal{U}$ may be identified with dominant rational maps $\mathbb{P}^n \to \mathbb{P}^n$ defined by homogeneous polynomials (without common factors) of degree $\leq d$. Under this identification, points in $\mathcal{U}_d := \big[\mathbb{P}_{(d,n,n)} \backslash \big(\cup_{e=1}^{d-1} \mathcal{W}_e\big)\big] \cap \mathcal{U}$ correspond to dominant rational maps defined by polynomials of degree exactly d.

As it follows readily from [RPV2001, Annexe B, Pro. B], the open set $\mathrm{Bir}(\mathbb{P}^n)_d = \mathrm{Bir}(\mathbb{P}^n) \cap \mathcal{U}_d$ is closed in $\mathcal{U}_d$. Hence it is a quasi projective variety. Moreover, taking into account Lemma 1 the last assertion in (a) follows from part (b).

In order to prove (b) we represent φ as

$$\varphi(t) = \big(f_0(t, \mathbf{x}) : \cdots : f_n(t, \mathbf{x})\big),$$

for suitable polynomials $f_i \in \Bbbk[T][x_0,\dots,x_n]$, homogeneous in $\mathbf{x}$. If $t_0 \in U_\varphi$ is a (closed) point, then the polynomials $f_0(t_0,\mathbf{x}),\cdots,f_n(t_0,\mathbf{x}) \in \Bbbk[\mathbf{x}]$ do not admit a common factor. Hence the homogeneous coordinates of $\varphi(t_0)$ define a point in $\mathbb{P}_{(d,n,n)}$ which varies algebraically in t_0. This completes the proof. □

3.4. **Chevalley type Theorem.**

Theorem 24. *Let X be a rational variety. If $\varphi : T \to \mathrm{Bir}(X)$ is a morphism and $C \subset T$ is a constructible set, then $\varphi(C)$ is constructible and contains a open subset of $\overline{\varphi(C)}$.*

Proof. By Lemma 2 we may suppose $X = \mathbb{P}^n$ and φ with degree $d = \mathrm{Deg}(\varphi)$. Hence $\overline{\varphi(T)} \subset \mathrm{Bir}(\mathbb{P}^n)_{\leq d}$; we consider the morphism $\varphi_0 : U_0 = U_\varphi \to \mathrm{Bir}(\mathbb{P}^n)_d$ induced by φ.

On the other hand, Theorem 19 gives a stratification $T \setminus U_0 = \cup V_j^\ell$ by locally closed sets such that $d_j := \deg\bigl(\varphi(t)\bigr)$ is constant on each V_j; set $\varphi_j : V_j \to \mathrm{Bir}(\mathbb{P}^n)_{d_j}$ the morphism induced by φ on V_j.

We deduce the result by using Theorem 23 and applying the standard Chevalley Theorem to the morphisms $\varphi_0, \varphi_1, \dots, \varphi_\ell$. □

3.5. **Cyclic closed subgroups.**

Corollary 25. *Let $\{f_m\} \subset \mathrm{Bir}(\mathbb{P}^n)$ be a infinite sequence of birational maps. Then $\{f_m\}$ is closed if and only if $\lim_{m\to\infty} \deg(f_m) = \infty$. In particular, the Zariski topology on $\mathrm{Bir}(\mathbb{P}^n)$ is not Noetherian.*

Proof. Let $\varphi : T \to \mathrm{Bir}(\mathbb{P}^n)$ be a morphism, with $\mathrm{Deg}(\varphi) = d$. Then there exists m_0 such that $\deg(f_m) \geq d$ for all $m \geq m_0$, and thus $\varphi^{-1}\bigl(\{f_m\}\bigr)$ is finite. Hence, the only if follows from Corollary 21 and Theorem 23.

Conversely, suppose that $\liminf_{m\to\infty} \deg(f_m) = d < \infty$. Then there exist infinitely many f_i whose degree is d. Hence, $\{f_m\} \cap \mathrm{Bir}(\mathbb{P}^n)_d$ is an infinite countable subset of the algebraic variety and thus it is not closed.

□

Corollary 26. *Let $f \in \mathrm{Bir}(\mathbb{P}^n)$ be a birational map of degree d. The cyclic subgroup $\langle f \rangle$ generated by f is closed if and only if either f is of finite order or $\lim_{m\to\infty} \deg(f^m) = \infty$.*

Example 27. When $n = 2$ the behavior of the sequence $\deg(f^m)$ is well known. Indeed, following [DiFa2001], if $\langle f \rangle$ is infinite, then the sequence $\deg(f^m)$ satisfies exactly one of the following:

(a) $\deg(f^m) \leq b$ for some positive $b \in \mathbb{R}$.
(b) $am \leq \deg(f^m) \leq bm$ for some positive $a, b \in \mathbb{R}$.
(c) $am^2 \leq \deg(f^m) \leq bm^2$ for some positive $a, b \in \mathbb{R}$.
(d) $am^d \leq \deg(f^m) \leq bm^d$ for some positive $a, b \in \mathbb{R}$ where $d = \deg(f)$.

Hence the infinite cyclic group $\langle f\rangle$ is not closed only when the sequence $\deg(f^m)$ satisfies (a) above.

3.6. **Some big closed subgroups.**

Let $o\in\mathbb{P}^n$ be a point. Consider the subgroup $\mathrm{St}_o(\mathbb{P}^n)\subset\mathrm{Bir}(\mathbb{P}^n)$ of birational transformations which stabilize (birationality) the set of lines passing through o. If o' is another point $\mathrm{St}_o(\mathbb{P}^n)$ and $\mathrm{St}_{o'}(\mathbb{P}^n)$ may be conjugated by mean of a linear automorphism; in the sequel we fix $o=(1:0:\cdots:0)$. In [Do2011] the group $\mathrm{St}_o(\mathbb{P}^n)$ is introduced in a different form and is called the *de Jonquières subgroup of level* $n-1$ (see also [Pa2000]).

Let $\pi:\mathbb{P}^n\dashrightarrow\mathbb{P}^{n-1}$ be the projection of center o defined by

$$(x_0:x_1:\cdots:x_n)\mapsto(x_1:\cdots:x_n).$$

Then $\mathrm{St}_o(\mathbb{P}^n)=\{f\in\mathrm{Bir}(\mathbb{P}^n):\pi f=f\pi\}$. Moreover, note that $\mathrm{St}_o(\mathbb{P}^n)$ is the semidirect product

$$1\longrightarrow\mathrm{Jon}_o(\mathbb{P}^n)\longrightarrow\mathrm{St}_o(\mathbb{P}^n)\xrightarrow{\rho}\mathrm{Bir}(\mathbb{P}^{n-1})\longrightarrow 1$$

where $\mathrm{Jon}_o(\mathbb{P}^n)=\{f\in\mathrm{Bir}(\mathbb{P}^n):\pi f=\pi\}$ and ρ is the evident homomorphism. The morphism $\sigma:\mathrm{Bir}(\mathbb{P}^{n-1})\to\mathrm{Bir}(\mathbb{P}^n)$ given by

$$(h_1:\cdots:h_n)\mapsto(x_0h_1:x_1h_1:\cdots:x_1h_n)$$

is injective and such that $\sigma\big(\mathbb{P}^{n-1}\big)\subset\mathrm{St}_o(\mathbb{P}^n)$. Clearly, $\rho{\circ}\sigma=id$.

Moreover, we affirm that ρ is continuos, and σ is a continuos closed immersion. Indeed, if $\varphi:T\to\mathrm{Bir}(\mathbb{P}^n)$ is a morphism then the composition $\rho{\circ}\varphi$ defines a morphism $T\to\mathrm{Bir}(\mathbb{P}^{n-1})$; therefore ρ is a continuous function. Clearly, σ is continuos. In order to prove, among other things, that σ is a closed immersion we need the following:

Lemma 28. *Let $f\in\Bbbk[T]\otimes\Bbbk[x_0,\dots,x_n]$ be a polynomial, homogeneous in* $\mathbf{x}$*; denote by* $\deg_{x_0}(f)$ *its degree in x_0. Then for all integer $m\geq 0$ and $i=0,\dots,n$ the sets*

$$R=\big\{t\in T:x_i|f(t,\mathbf{x})\big\}\ ,\ S_m=\big\{t\in T;\deg_{x_0}(f)\leq m\big\}$$

are closed in T.

Proof. Let $a_1,\dots,a_N\in\Bbbk[T]$ be the coefficients of f as polynomial in $x_0,\dots,x_n$. It is clear that R and S_m are defined as common zeroes of a subset of the polynomials $\{a_1,\dots,a_N\}\subset\Bbbk[T]$. □

Theorem 29. *The subgroups* $\mathrm{Jon}_o(\mathbb{P}^n)$ *and* $\mathrm{St}_o(\mathbb{P}^n)$ *are closed and* $\sigma(\mathrm{Bir}(\mathbb{P}^{n-1})$ *is closed in* $\mathrm{Bir}(\mathbb{P}^n)$*. In particular, σ is a closed immersion.*

Proof. Let $\varphi:T\to\mathrm{Bir}(\mathbb{P}^n)$ be a morphism, say with $\mathrm{Deg}(\varphi)=d$. In order to prove that $\varphi^{-1}\big(\mathrm{Jon}_o(\mathbb{P}^n)\big)$ is closed it suffices to consider a net (t_ξ) in $\varphi^{-1}\big(\mathrm{Jon}_o(\mathbb{P}^n)\big)$, where ξ varies in a directed set, and show that every limit point $t_\infty\in T$ of that net satisfies $\varphi(t_\infty)\in\mathrm{Jon}_o(\mathbb{P}^n)$. Let t_∞ be such a limit point and $T=\cup_{j=0}^{l}V_j$ be the stratification given by Theorem 19(a), where $V_0 = U_\varphi$. Then there exists j such that the subnet $(t_\xi) \cap V_j$ has t_∞ as limit point. Thus, we can assume $t_\xi \in U_\varphi$ for all ξ, that is that $\deg(\varphi_{t_\xi}) = d$. Write

$$\varphi(t, \mathbf{x}) = \big(f_0(t, \mathbf{x}) : \cdots : f_n(t, \mathbf{x})\big),$$

where $f_i \in \Bbbk[T] \otimes \Bbbk[\mathbf{x}]$ are homogeneous in $\mathbf{x} = \{x_0, \ldots, x_n\}$ of degree d. From the description given in [Pa2000, §2] it follows that for all ξ there exists a homogeneous polynomial $q_\xi \in \Bbbk[\mathbf{x}]$ such that:

(a) $f_i(t_\xi, \mathbf{x}) = x_i q_\xi(\mathbf{x})$, for $i > 0$;
(b) $f_0(t_\xi, \mathbf{x})$ and $q_\xi(\mathbf{x})$ have degrees ≤ 1 in x_0;
(c) $f_0(t_\xi, \mathbf{x}) q_\xi(\mathbf{x})$ has degree ≥ 1 in x_0.

By Lemma 28, when t_ξ specializes to t_∞, then $\varphi_\xi = \varphi(t_\xi)$ specializes to the birational map $\varphi_{t_\infty} = (f : x_1 q : \cdots : x_n q) : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$, where $f(\mathbf{x})$ and $x_i q(\mathbf{x})$, $i > 0$, are polynomials in $\mathbf{x}$ of degree d and with degree ≤ 1 in x_0. Suppose that f and q admit a common factor $h \in \Bbbk[x_0, \ldots, x_n]$, of degree ≥ 0. Since the limit map φ_{t_∞} is birational (of degree $\leq d$) we deduce that $h \in \Bbbk[x_1, \ldots, x_n]$: otherwise h would have degree 1 in x_0 and the map φ_{t_∞} would be defined by polynomials in $x_1, \ldots, x_n$ contradicting birationality. Hence $f' := f/h$ and $q' := q/h$ satisfy the conditions (b) and (c) above. We conclude $\varphi_{t_\infty} = (f' : x_1 q' : \cdots : x_n q')$. Applying again the description of [Pa2000, §2], we deduce that $\pi \varphi_{t_\infty} = \pi$, that is $\varphi_{t_\infty} \in \mathrm{Jon}_o(\mathbb{P}^n)$, which proves $\mathrm{Jon}_o(\mathbb{P}^n)$ is closed.

In order to prove that $\sigma\big(\mathrm{Bir}(\mathbb{P}^{n-1})\big)$ is closed, consider a net $(t_\xi) \subset \varphi^{-1}\big(\sigma(\mathbb{P}^{n-1})\big)$, with limit point t_∞. As before, we can assume that $t_\xi \in U_\varphi$ for all ξ. With the notation introduced above we have that

(a) $f_i(t_\xi, \mathbf{x}) = x_1 h_{i,\xi}(\mathbf{x})$, for $i > 0$, and
(b) $f_0(t_\xi, \mathbf{x}) = x_0 h_{1,\xi}(\mathbf{x})$,

where $(h_{1,\xi} : \cdots : h_{n,\xi}) : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}$ is birational. From Lemma 28 we obtain that $h_{i,\xi}$ specializes to a polynomial $h_i \in \Bbbk[x_1, \ldots, x_n]$, $i > 0$ and that $\varphi_{t_\infty} = (x_0 h_1 : x_1 h_1 : \cdots : x_1 h_n)$. Since $\pi \varphi_{t_\infty} = \varphi_{t_\infty} \pi$ we conclude that $\varphi_{t_\infty} \in \mathrm{St}_o(\mathbb{P}^n)$ and thus $(h_1 : \cdots : h_n) \in \mathrm{Bir}(\mathbb{P}^{n-1}$ ([Pa2000, Prop. 2.2]). Since $\sigma\big((h_1 : \cdots : h_n)\big) = \varphi_{t_\infty}$, it follows that $\sigma\big(\mathrm{Bir}(\mathbb{P}^{n-1}\big)$ is closed.

Finally, since for elements $f \in \mathrm{Jon}_o(\mathbb{P}^n)$ and $h \in \mathrm{Bir}(\mathbb{P}^{n-1})$ the product $f \rtimes h$ is the composition $f \circ \sigma(h)$, then $\mathrm{St}_o(\mathbb{P}^n) = \mathrm{Jon}_o(\mathbb{P}^n) Im(\sigma)$ (product in $\mathrm{Bir}(\mathbb{P}^n)$). The fact that $\mathrm{St}_o(\mathbb{P}^n)$ is closed follows then from the two assertions we have just proved together with the continuity of the functions $\rho : \mathrm{St}_o(\mathbb{P}^n) \to \mathrm{Bir}\big(\mathbb{P}^{n-1}\big)$, the group product and the group inversion. Indeed, let $(f_\xi \rtimes h_\xi)$ be a net in $\mathrm{St}_o(\mathbb{P}^n)$ which specializes to $s \in \mathrm{Bir}(\mathbb{P}^n)$. Then $\rho(f_\xi \rtimes h_\xi) = \rho(1 \rtimes h_\xi) = h_\xi$ specializes to $\rho(s) = h \in \mathrm{Bir}(\mathbb{P}^{n-1})$. Since $(f_\xi \rtimes h_\xi) \cdot (1 \rtimes h_\xi^{-1}) = f_\xi \rtimes 1 \in \mathrm{Jon}_o(\mathbb{P}^n)$, the net $(f_\xi \rtimes 1)$ specializes to $s\sigma(h^{-1}) \in \mathrm{Jon}_o(\mathbb{P}^n)$. Thus $s \in \mathrm{St}_o(\mathbb{P}^n)$. □

Remark 30. More generally, for $\ell = 1, \ldots, n$, the map $\sigma_\ell : \mathrm{Bir}(\mathbb{P}^{n-1}) \to \mathrm{Bir}(\mathbb{P}^n)$ defined by

$$\sigma_\ell\big((h_1 : \cdots : h_n)\big) = (x_0 h_\ell : x_\ell h_1 : \cdots : x_\ell h_n)$$

is a continuous, closed, homomorphism whose image is contained in $\mathrm{St}_o(\mathbb{P}^n)$ and such that $\rho\sigma_\ell = id$. In this notation, the map σ of Theorem 29 is σ_1. Moreover, one has

$$\bigcap_{\ell=1}^{n} \sigma_\ell\big(\mathrm{Bir}(\mathbb{P}^{n-1})\big) = \{id\}.$$

If $\mathcal{U}_\ell$ is the dense open set $\mathrm{Bir}(\mathbb{P}^n)\backslash\sigma_\ell\big(\mathrm{Bir}(\mathbb{P}^{n-1})\big)$, then $\mathrm{Bir}(\mathbb{P}^n) - \{id\} = \bigcup_{\ell=1}^{n} \mathcal{U}_\ell$.

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