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ON CREMONA TRANSFORMATIONS OF $\mathbb{P}^3$ WITH ALL POSSIBLE BIDEGREES

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1. Introduction

The aim of this note is to correct a mistake in the proof of Theorem [Pa2000-2, Théorème. 2.2] which was pointed me out by Igor Dolgachev. More precisely, the example [Pa2000-2, Exemple 2.1] is wrong, and it was used to prove [Pa2000-2, Lemme 2.1] which is needed, in turn, to obtain that Theorem. Apparently that example can not be rewritten in order to use it as before, then we propose here a more precise and explicit construction of Cremona transformations of $\mathbb{P}^3$ (see § 2, specially Lemma 2) which, together with their inverses, provide all possible bidegrees (Theorem 3 and Corollary 4).

Acknowledge I would like thank Igor Dolgachev for point me out a mistake in [Pa2000-2, Exemple 2.1].

2. Main construction and results

Let $\mathbb{P}^3$ be the projective space over the complex number field $\mathbb{C}$; we fix homogeneous coordinates w, x, y, z on $\mathbb{P}^3$.

We recall that a Cremona transformation of $\mathbb{P}^3$ is a birational map $F : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$. We say F has *bidegree* (d, e) when F and its inverse F^{-1} are defined by homogeneous polynomials, without non trivial common factors, of degrees d and e respectively; notice that in this case F^{-1} has bidegree (e, d). If $V \subset \mathbb{P}^3$ is a dense open set where F^{-1} is well defined and injective, and $L \subset \mathbb{P}^3$ is a line with $L \cap V \neq \emptyset$, then e is the degree of the closure of $F^{-1}(L \cap V)$ (see for example [Pa2000-2, Proposition 1.1]).

If $X \subset \mathbb{P}^2$ is a curve and $p \in \mathbb{P}^2$ we denote by $\mathrm{mult}_p(X)$ the multiplicity of X at p. If $S, S' \subset \mathbb{P}^3$ are surfaces and $C \subset S \cap S'$ is an irreducible component, we denote by $\mathrm{mult}_C(S, S')$ the intersection multiplicity of S and S' along C.

Consider a rational map $T : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ defined by

$$T = (g : qt_1 : qt_2 : qt_3)$$

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where $t_1, t_2, t_3 \in \mathbb{C}[x,y,z]$ are homogeneous of degree r, without non trivial common factors, and $g, q \in \mathbb{C}[w,x,y,z]$ are homogeneous of degrees $d, d-1$, with $d \geq r \geq 1$ and g irreducible. We know that T is birational when $\tau := (t_1 : t_2 : t_3) : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is birational and g, q vanish at $o = (1:0:0:0)$ with orders $d-1$ and $\geq d-r-1$ respectively (see [Pa2000-1, Proposition 2.2]).

On the other hand, consider $2r-1$ points $p_0, p_1, \ldots, p_{2r-2}$ in $\mathbb{P}^2$, $r \geq 2$, satisfying the following condition:

> There exist curves $X_r, Y_{r-1} \subset \mathbb{P}^2$ of degrees $r, r-1$, respectively, with X_r irreducible, such that $\mathrm{mult}_{p_0}(X_r) = r-1$, $\mathrm{mult}_{p_0}(Y_r) \geq r-2$ and $p_i \in X_r \cap Y_{r-1}$ for $i = 1, \ldots, 2r-2$.

Hence *loc. cit* also implies there exists a plane Cremona transformation defined by polynomials of degree r which vanish at p_0 with order $r-1$ and with order 1 at the other points: indeed, if we suppose $p_0 = (1:0:0)$ and take polynomials t_1 and f, of degrees r and $r-1$, defining X_r and Y_{r-1} respectively, then $(t_1 : yf : zf) : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is a Cremona transformation as required; such a transformation is said to be *associated* to the points $p_0, p_1, \ldots, p_{2r-2}$.

Remark 1. The transformations satisfying the condition quoted above are general cases of the so-called *de Jonquières transformations* (see [dJo1864] or [Alb2000, Def. 2.6.10]). We note that the Enriques criterion [Alb2000, Thm. 5.1.1] may be also used to prove that a set of $2r-2$ points $p_0, p_1, \ldots, p_{2r-2}$ with assigned multiplicities $r-1, 1, \ldots, 1$, and satisfying the condition above, defines a de Jonquières transformation.

Fix $r = d$ and take a homogeneous irreducible polynomial $g = wA(x,y,z) + B(x,y,z)$ of degree d; that is, in this case the polynomial q is a nonzero complex number. Denote by $T_{g,\tau}$ the Cremona transformation defined by

$$T_{g,\tau} = (g : t_1 : t_2 : t_3)$$

where $\tau = (t_1 : t_2 : t_3)$ is associated to $2d-2$ points satisfying the condition quoted above.

We have

Lemma 2. *Let $d \geq 2$ be an integer number. There exist g and τ such that:*

(a) $T_{g,\tau}$ has bidegree $(d, 2d-1-m)$ for $0 \leq m \leq d-1$.

(b) $T_{g,\tau}$ has bidegree $(d, d^2 - \ell^2 - m)$ for $0 \leq \ell < d-1$ and $0 \leq m \leq 2d-2$.

Proof. We identify $\mathbb{P}^2$ with the plane $\{w = 0\} \subset \mathbb{P}^3$. Fix a point $p_0 \in \mathbb{P}^2$.

To prove (a) we first choose g vanishing along a line op_0 with order $d-1$ and being general with respect to this condition: for example, if $p_0 = (0:1:0:0)$ we take

$g = wA + B$ with
$$A = A_{d-1}(y,z), B = xB_{d-1}(y,z) + B_d(y,z),$$
where A_i, B_i are general homogeneous polynomials of degree i. Hence $A = 0$ defines the union of $d-1$ distinct lines in $\mathbb{P}^2$ passing through p_0 and $B = 0$ defines an irreducible degree d curve with an ordinary singular point of multiplicity $d-1$ at p_0.

Notice that, by construction, in the open set $\mathbb{P}^2 - \{p_0\}$ the curves $A = 0$ and $B = 0$ intersect at $d(d-1)-(d-1)^2 = d-1$ points; in particular, if $m \leq d-1$, there exist m points $p_1, \ldots, p_m$ satisfying $A(p_i) = B(p_i) = 0$ for $1 \leq i \leq m$. Therefore, we consider m such points and choose $p_{m+1}, \ldots, p_{2d-2}$ with $A(p_j) \neq 0, B(p_j) = 0$, for all $j = m+1, \ldots, 2d-2$, and in order that $p_0, p_1, \ldots, p_{2d-2}$ satisfy the condition quoted above. Let τ be the plane Cremona transformation associated to these points.

Now consider a general member in the linear system defining $T_{g,\tau}$, that is a degree d irreducible surface, S say, with equation of the form
$$ag + a_1t_1 + a_2t_2 + a_3t_3 = 0,$$
where $a, a_1, a_2, a_3 \in \mathbb{C}$ are general. Therefore S has an ordinary singularity of multiplicity $d-1$ at the generic point of the line op_0 and is smooth at the generic point of the line op_i for $1 \leq i \leq m$. If S' is another general member of that linear system, then there exists a rational irreducible curve Γ of degree $e = \deg(T_{g,\tau}^{-1})$ such that the intersection scheme $S \cap S'$ is supported on
$$\Gamma \cup (\cup_{i=0}^{m} op_i)$$
where
$$\mathrm{mult}_\Gamma(S,S') = 1; \mathrm{mult}_{op_0}(S,S') = (d-1)^2; \mathrm{mult}_{op_i}(S,S') = 1, i = 1, \ldots, m.$$

We conclude $e = d^2 - (d-1)^2 - m = 2d - 1 - m$, which proves the assertion (a).

To prove (b) we work analogously but by choosing $g = wA + B$ with
$$A = \sum_{i=\ell}^{d-1} x^{d-1-i} A_i(y,z), B = \sum_{j=\ell}^{d} x^{d-j} B_j(y,z)$$
where A_i, B_i are general homogeneous polynomials of degree i. Since $\ell \leq d-2$ there exist points $p_1, \ldots, p_{2d-2}$ such that $A(p_i) = B(p_i) = 0$ for $1 \leq i \leq m$ and $A(p_j) \neq 0$ $B(p_j) = 0$ for $j = m+1, \ldots, 2d-2$: indeed, in the open set $\mathbb{P}^2 - \{p_0\}$, the curves $A = 0$ and $B = 0$ intersect at $d(d-1) - \ell^2 \geq d(d-1) - (d-2)^2 = 3d - 4$ points.

□

Theorem 3. *There exist Cremona transformations of bidegree (d,e) for $d \leq e \leq d^2$.*

Proof. By using part (a) of Lemma 2 we obtain Cremona transformations of bidegrees (d,e) for $d \leq e \leq 2d-1$.

Now we suppose $\ell < d-1$ and think of $e = d^2 - \ell^2 - m$ as a function $e(\ell, m)$ depending on ℓ and m; to complete the proof it suffices to prove that the image of this function contains $\{2d, 2d+1, \ldots, d^2\}$.

We note that $e(d-2, 2d-2) = 2d-2$ and $e(0,0) = d^2$; in other words, the part (b) of Lemma 2 implies there are Cremona transformations of bidegrees $(d, 2d-2)$ and (d, d^2). On the other hand $e(\ell, 0) - e(\ell-1, 2d-2) = 2(d-\ell) - 1 > 0$. Since $e(\ell, m)$ decreases with m, we easily obtain the result. $\square$

For $d = 2$ the Theorem gives transformations of bidegrees $(2,2), (2,3), (2,4)$, for $d = 3$ it gives transformations of bidegrees $(3,3), (3,4), \ldots, (3,9)$, and son on. By symmetry we deduce

Corollary 4. *There exist Cremona transformations of bidegree (d, e) for $\sqrt{d} \leq e \leq d^2$.*

Remark 5. The inequalities $\sqrt{d} \leq e \leq d^2$ are the unique obstructions to the degree of the inverse of a Cremona transformation of degree d in $\mathbb{P}^3$.

References

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