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Partial actions of discrete groups and related structures

Fernando Abadie[1]

ABSTRACT

We give constructions of the groupoid of a partial action of a discrete group other than the ones given in [1]. We also show that Paterson's universal groupoid of the inverse semigroup of a discrete group agrees with the groupoid associated to the "universal" partial action of the group.

RESUMEN

Se dan construcciones del grupoide asociado a una acción parcial de un grupo discreto diferentes de las presentadas en [1]. También se muestra que el grupoide universal de Paterson del semigrupo inverso de un grupo discreto coincide con el grupoide asociado a la acción parcial "universal" del grupo.

To Alfredo Jones

Introduction

We have shown in [1] how to associate a locally compact groupoid $\mathcal{G}_\theta$ with every partial action θ on a locally compact Hausdorff space. We also observed that in the case of a topologically free partial action of a discrete group, the corresponding groupoid is nothing but a sheaf groupoid of germs. The purpose of the present paper is to compare this groupoid with some other groupoids which are obtained, in the special case of discrete groups, by combining results of Exel, Nica, Paterson and Sieben. The key step consists in passing from a partial action of G to an action of an associated $\tilde{F}$–inverse semigroup, and then to construct a groupoid via Nica's theory ([4]). We show below that the groupoids thus obtained coincide with ours. We also connect the work of Nica with that of Paterson on localizations. On the other hand, Paterson shows in [5] how to associate a "universal groupoid" with any inverse semigroup. We will see that the universal groupoid of $S(G)$ is the groupoid of a suitable partial action of G.

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Suppose that G is a discrete group and $\theta = (\{X_t\}_{t\in G}, \{\theta_t\}_{t\in G})$ is a partial action of G on a set X. This means that each X_t is a subset of X, with $X_e = X$ (where e is the unity element of G), $\theta_t : X_{t^{-1}} \to X_t$ is bijective, $\forall t \in G$, and θ_{st} is an extension of $\theta_s\theta_t$, $\forall s,t \in G$. Thus $\theta_e = id_X$, and $\theta_t^{-1} = \theta_{t^{-1}}$, $\forall t \in G$. If X is a topological space it is also required that each X_t is open and each θ_t is a homeomorphism.

The partial action θ on the set X can be thought of as the set of morphisms of a groupoid $\mathcal{G}_\theta$ in the following way. The set of objects of $\mathcal{G}_\theta$ is the space X, and, given $x, y \in X$, the set of morphisms $\mathcal{G}_\theta(x,y)$ from x into y is defined as $\mathcal{G}_\theta(x,y) := \{(y,t,x) \in X \times G \times X : x \in X_{t^{-1}} \text{ and } \theta_t(x) = y\}$. Composition of morphisms is given by $(z,s,y)(y,t,x) = (z,st,x)$. The fact that θ is a partial action ensures that $\mathcal{G}_\theta$ is a groupoid (see [1] for details). Note that the identity morphism at x is (x,e,x), and $(y,t,x)^{-1} = (x,t^{-1},y)$. If X is a locally compact Hausdorff space, we endow $\mathcal{G}_\theta$ with the topology inherited by the product topology on $X \times G \times X$. It turns out that $\mathcal{G}_\theta$ is a locally compact Hausdorff groupoid with this topology.

1. $\mathcal{G}_\theta$ in two steps

We briefly review some basic facts of the theory developed in [2]. The inverse semigroup $S(G)$ of a group G is defined by a set of generators $\{[t] : t \in G\}$ and the relations:

(i) $[s^{-1}][s][t] = [s^{-1}][st], \forall s,t \in G.$

(ii) $[s][t][t^{-1}] = [st][t^{-1}], \forall s,t \in G.$

(iii) $[s][e] = [s], \forall s \in G$

(iv) $[e][s] = [s], \forall s \in G.$

The semigroup $S(G)$ is characterized by the following universal property: if S is a semigroup and $f : G \to S$ is a map satisfying

$$f(s^{-1})f(s)f(t) = f(s^{-1})f(st), \forall s,t \in G \tag{1}$$

$$f(s)f(t)f(t^{-1}) = f(st)f(t^{-1}), \forall s,t \in G \tag{2}$$

$$f(s)f(e) = f(s), \forall s \in G \tag{3}$$

then there is a unique homomorphism $\tilde{f} : S(G) \to S$ such that $\tilde{f}([t]) = f(t)$, $\forall t \in G$. In particular, there is a homomorphism $\partial : S(G) \to G$ such that $\partial([t]) = t$, $\forall t \in G$. A consequence of the universal property of $S(G)$ is that there exists an involution $* : S(G) \to S(G)$ such that $[t]^* = [t^{-1}]$, $\forall t \in G$, and $\big(S(G), *\big)$ is an inverse semigroup. If $t \in G$, the element $\varepsilon_t := [t][t^{-1}]$ belongs to the semilattice $\mathcal{E}_{S(G)}$ of the idempotents of $S(G)$. Moreover, every $\sigma \in S(G)$ has a *standard form*: there exist unique disjoint subsets $\{t\}$ and $\{t_1, \ldots, t_k\}$ of G such that $\sigma = \varepsilon_{t_1} \cdots \varepsilon_{t_k}[t]$.

Every inverse semigroup S has a partial order defined by: $\sigma \leq \tau \iff \sigma\sigma^* = \tau\sigma^*$. A unital inverse semigroup S is called an $\tilde{F}$–inverse semigroup if every $\sigma \in S$, $\sigma \neq 0$, is majorized by a unique maximal element of S. If such condition holds for each $\sigma \in S$, it is said that S is an F–inverse semigroup.

PROPOSITION 1.1 *Let $S(G)$ be the inverse semigroup of the group G. We have:*

1. *If $\sigma \in S(G)$ and $t \in G$, then $\partial(\sigma) = t \iff \sigma \leq [t]$*
2. *The set of maximal elements of $S(G)$ is $\mathcal{M} = \{[t] : t \in G\}$*
3. *$S(G)$ is an F–inverse semigroup.*
4. *The maximal element that majorizes $[s][t]$ is $[st]$.*

Proof. To prove 1. note that if $\partial(\sigma) = t$ and $\sigma = \varepsilon_{s_1} \cdots \varepsilon_{s_k}[s]$, then $s = t$. Therefore $\sigma\sigma^* = \varepsilon_{s_1} \cdots \varepsilon_{s_k}\varepsilon_t = [t]\sigma^*$, so $\sigma \leq [t]$. Conversely, since homomorphisms of inverse semigroups are order preserving, we have $\partial(\sigma) \leq t$ whenever $\sigma \leq [t]$, and therefore $\partial(\sigma) = t$. Now 2., 3. and 4. follow directly from 1. $\square$

With every pair (S, α), where S is an $\tilde{F}$–inverse semigroup S and α a left action of S on a space X, Nica associates a groupoid by using a procedure comparable with the construction of the sheaf groupoid of germs (see [1, Remark 2.1]). Let $\mathcal{M}_S$ be the set of maximal elements of S. There is a partially defined product on $\mathcal{M}_S$: if $\mu, \mu' \in \mathcal{M}_S$ are such that their product $\mu\mu'$ in S is not zero, then $\mu \cdot \mu'$ is defined to be the unique element in $\mathcal{M}_S$ that majorizes $\mu\mu'$. The groupoid of (S, α) is $\mathcal{N}_\alpha := \{(\mu, x) : \mu \in \mathcal{M}_S, x \in \mathrm{dom}(\alpha_\mu)\}$ with product $(\mu, x)(\mu', x') = (\mu \cdot \mu', x')$ whenever $\alpha_\mu(x') = x$ and $\mu\mu' \neq 0$, inversion $(\mu, x)^{-1} = (\mu^*, \alpha_\mu(x))$, and with the product topology. The reader is referred to [4] for details.

Consider now a partial action of a discrete group G on a locally compact Hausdorff space X. By [2, Theorem 4.2] θ induces a unique action (also called θ) of $S(G)$ on X, such that $\theta_t = \theta_{[t]}$, $\forall t \in G$. Following [4] we may associate with $\big(S(G), \theta\big)$ a groupoid $\mathcal{N}_\theta$. In view of parts 2. and 4. of Proposition 1.1 and the comments above we have $\mathcal{N}_\theta = \{([t], x) : t \in G, x \in X_{t^{-1}}\}$, with the product $([s], y)([t], x) = ([st], x)$, for $([s], y), ([t], x) \in \mathcal{N}_\theta$ such that $\theta_{[t]}(x) = y$.

PROPOSITION 1.2 *The groupoids $\mathcal{G}_\theta$ and $\mathcal{N}_\theta$ are naturally isomorphic.*

Proof. It is clear that the map $\psi : \mathcal{G}_\theta \to \mathcal{N}_\theta$ given by $\phi(y, t, x) = ([t], x)$ is a homeomorphism with inverse $([t], x) \mapsto (\theta_t(x), t, x)$. Moreover:

$$\psi\big((z, s, y)(y, t, x)\big) = \psi(z, st, x) = ([st], x) = ([s], y)([t], x) = \psi(z, s, y)\psi(y, t, x).$$

Therefore ψ is an isomorphism of locally compact groupoids. $\square$

A partial action of the discrete group G on the C^*-algebra A is a partial action $\sigma = (\{D_t\}, \{\sigma_t\})$ on the set A with the requirement that every D_t is an ideal in A and every σ_t is an isomorphism of C^*-algebras. If A is commutative, then every partial action σ on A is the dual of a unique partial action θ on the spectrum X of A, that is: $D_t = C_0(X_t)$ and $\sigma_t(a) = a \circ \theta_{t^{-1}}$, $\forall t \in G$ and $a \in D_{t^{-1}}$, where X_t is an open subset of X (see [1] for details).

A covariant representation of the system (A, G, σ) on the Hilbert space H is a pair (π, u), where $\pi : A \to B(H)$ is a non-degenerate representation of A and $u : G \to B(H)$ is such that every u_t is a partial isometry on H with initial subspace $\pi(D_{t^{-1}})H$ and final space $\pi(D_t)H$, such that $u_t\pi(a)u_{t^{-1}} = \pi(\sigma_t(a))$, $\forall a \in D_{t^{-1}}$, $t \in G$, and $u_{st}h = u_s u_t h$, $\forall h \in \pi(D_{t^{-1}} \cap D_{t^{-1}s^{-1}})H$. Covariant representations of (A, G, σ) are in a bijective correspondence with non-degenerate representations of the crossed product $A \ltimes_\sigma G$. In one direction, the covariant representation (π, u) on H gives rise to a representation $\pi \times u : A \ltimes_\sigma G \to B(H)$ which is determined by $(\pi \times u)(a_t\delta_t) = \pi(a_t)u_t$, where $a_t\delta_t$ is such that $a_t \in D_t$, and $a_t\delta_t(r) = a_t$ if $r = t$, 0 otherwise (recall that the linear span of such elements is dense in $A \ltimes_\sigma G$).

In [6] Sieben associates an inverse semigroup $S_{\pi,u}$ with every covariant representation (π, u) of the system $\big(A, G, \sigma\big)$. Such inverse semigroup is given by $S_{\pi,u} = \{(\sigma_{t_1} \dots \sigma_{t_n}, u_{t_1} \dots u_{t_n}) : n \in \mathbb{N}, t_j \in G, \forall j = 1, \dots, n\}$ with the obvious operations.

Theorem 1.3 *Let σ be a partial action of the discrete group G on the C^*-algebra A, and suppose that (π, u) is a covariant representation of the partial dynamical system $\big(A, G, \sigma\big)$ on the Hilbert space H. Then*

1. *The map $G \to S_{\pi,u}$ given by $t \mapsto (\sigma_t, u_t)$ extends uniquely to a homomorphism $S(G) \to S_{\pi,u}$, which moreover is surjective.*

2. *If the representation $\pi \times u : A \ltimes_\sigma G \to B(H)$ is faithful, then $S_{(\pi,u)}$ is an $\tilde{F}$–inverse semigroup, whose maximal set is $\mathcal{M}_{\pi,u} = \{(\sigma_t, u_t) : t \in G, \sigma_t \neq 0\}$.*

Proof. Since σ is a partial action and u is a partial representation of G (because (π, u) is a covariant representation), identities (1), (2) and (3) are satisfied by the map $t \mapsto (\sigma_t, u_t)$, so the existence and uniqueness of the claimed homomorphism is guaranteed by the universal property of $S(G)$. The surjectivity of such a homomorphism follows from the fact that $\{(\sigma_t, u_t) : t \in G\}$ generates $S_{\pi,u}$.

To prove the second assertion note first that $\forall t_1, \ldots, t_n \in G$ we have that $(\sigma_{t_1} \ldots \sigma_{t_n}, u_{t_1} \ldots u_{t_n}) \leq (\sigma_{t_1 \cdots t_n}, u_{t_1 \cdots t_n})$. We will show that if $0 \neq (\sigma_s, u_s) \leq (\sigma_t, u_t)$, then $s = t$. Now, $(\sigma_s, u_s) \leq (\sigma_t, u_t) \iff (id_{D_s}, u_s u_s^*) = (\sigma_t \sigma_{s^{-1}}, u_t u_s^*)$. The equality of the first coordinates implies that $D_{s^{-1}} \subseteq D_{t^{-1}}$, $D_s \subseteq D_t$ and $\sigma_t(a) = \sigma_s(a)$, $\forall a \in D_{s^{-1}}$. The equality of the second coordinates implies that the partial isometry u_t agrees with u_s on the initial space of the latter. Thus u_t^* extends the partial isometry u_s^*, so that $u_t^* u_s = u_s^* u_s$, and therefore $u_s^* u_t = u_s^* u_s$. Identify A with the image of its universal representation, and consider the normal extension of π (which will be still denoted by π) to the enveloping von Neumann algebra A'' of A. As shown in [6], if p_r is the unit element of the strong closure of D_r, then $\pi(p_r) = u_r u_r^*$, $\forall r \in G$. Since $D_s \subseteq D_t$, we have that $a\delta_t \in A \ltimes_\sigma G$, for every $a \in D_s$. Therefore for $a \in D_s$ we have:

$$(\pi \times u)(a\delta_t) = \pi(a)u_t = \pi(ap_s)u_t = \pi(a)u_s u_s^* u_t = \pi(a)u_s = (\pi \times u)(a\delta_s).$$

Since $\pi \times u$ is faithful and $D_s \neq 0$, we must have that $s = t$. Therefore $S_{\pi,u}$ is an $\tilde{F}$–inverse semigroup. □

COROLLARY 1.4 *Let X be a locally compact Hausdorff space, $A = C_0(X)$, and σ the dual of the partial action θ on X. Suppose that (π, u) is a covariant representation of (A, G, σ) such that $\pi \times u$ is faithful. Then the map $(\sigma_{t_1} \ldots \sigma_{t_n}, u_{t_1} \ldots u_{t_n}) \mapsto \theta_{t_1} \ldots \theta_{t_n}$ defines a left action of $S_{\pi,u}$ on X, and the corresponding Nica groupoid is isomorphic to $\mathcal{G}_\theta$.*

Proof. By 1.3 $S_{\pi,u}$ is an $\tilde{F}$–inverse semigroup, whose maximal set is $\mathcal{M}_{\pi,u} = \{(\sigma_t, u_t) : t \in G, \sigma_t \neq 0\}$. Since θ is a partial action, it is clear that the given map defines a left action of $S_{\pi,u}$ on X. By definition the corresponding Nica groupoid is $\mathcal{N}_{\pi,u} = \{(m, x) : m \in \mathcal{M}_{\pi,u}, x \in \mathrm{dom}(m)\}$. Therefore $\mathcal{N}_{\pi,u} = \{(\sigma_t, u_t, x) : \sigma_t \neq 0 \text{ and } x \in X_{t^{-1}}\}$. Moreover, it is routine to check that the map $\mathcal{G}_\theta \to \mathcal{N}_{\pi,u}$ given by $(y, t, x) \mapsto (\sigma_t, u_t, x)$ is an isomorphism of locally compact groupoids. □

2. The universal groupoid of $S(G)$

In [5] Paterson associates a "universal groupoid" $\Gamma_{\mathbf{u}}$ with every inverse semigroup S. The crucial property of $\Gamma_{\mathbf{u}}$ is that it determines all S–groupoids. We refer the reader to [5] for complete information. For our purposes it will be enough to recall a concrete form of $\Gamma_{\mathbf{u}}$. Suppose that S is an inverse semigroup, and let $\mathcal{E}_S$ be its semilattice of idempotents. A semicharacter of $\mathcal{E}_S$ is a non–zero homomorphism $x : \mathcal{E} \to \{0,1\}$. Denote by X_S the space of non–zero semicharacters of $\mathcal{E}_S$ endowed with the product topology. For $\sigma \in S$ consider $D_\sigma := \{x \in X_S : x(\sigma\sigma^*) = 1\}$ and $R_\sigma := \{x \in X_S : x(\sigma^*\sigma) = 1\}$. There is a right action of S on $\mathcal{E}_S$ given by: $x \mapsto x \cdot \sigma$, $\forall x \in D_\sigma$, where $x \cdot \sigma(\varepsilon) := x(\sigma\varepsilon\sigma^*)$, $\forall \varepsilon \in \mathcal{E}_S$. This map is a partial homeomorphism in X_S, with domain D_σ and range R_σ.

If $\Sigma_S := \{(x,\sigma) : \sigma \in S, x \in D_\sigma\}$, there is an equivalence relation on Σ_S: $(x,\sigma) \sim (y,\tau) \iff x = y$ and there exists $\varepsilon \in \mathcal{E}$ such that $x(\varepsilon) = 1$ and $\varepsilon\sigma = \varepsilon\tau$. Denote by $\overline{(x,\sigma)}$ the class of (x,σ).

The underlying set of the universal groupoid of S is $\Gamma_{\mathbf{u}} = \{\overline{(x,\sigma)} : \sigma \in S, x \in D_\sigma\}$. The product of two elements $\overline{(x,\sigma)}, \overline{(y,\tau)} \in \Gamma_{\mathbf{u}}$ is defined only if $y = x \cdot \sigma$, and in this case we have $\overline{(x,\sigma)}\,\overline{(y,\tau)} = \overline{(x,\sigma\tau)}$. A basis for the topology of $\Gamma_{\mathbf{u}}$ is $\mathcal{U} := \{D(U,\sigma) : U \subseteq X_S \text{ is open}, \sigma \in S\}$, where $D(U,\sigma) := \{\overline{(x,\sigma)} : x \in U\}$. The inverse of $\overline{(x,\sigma)}$ is $\overline{(x \cdot \sigma, \sigma^*)}$. See [5, Theorem 4.3.1] for details. Let us give a description of $\Gamma_{\mathbf{u}}$ for the inverse semigroup $S(G)$ in terms of a partial action of G.

LEMMA 2.1 *If $(x,\sigma),(y,\tau) \in \Sigma_{S(G)}$, then $(x,\sigma) \sim (y,\tau) \iff x = y$ and $\partial(\sigma) = \partial(\tau)$.*

Proof. It is clear that $(x,\sigma) \sim (y,\tau) \Rightarrow x = y$ and $\partial(\sigma) = \partial(\tau)$. Suppose now that $x = y$ and $\partial(\sigma) = \partial(\tau)$. Note that every $\sigma \in S(G)$ satisfies $\sigma = \sigma\sigma^*[\partial(\sigma)]$. Therefore, if we let $\varepsilon := \sigma\sigma^*\tau\tau^*$, then $x(\varepsilon) = 1$, and

$$\varepsilon\sigma = \tau\tau^*\sigma\sigma^*[\partial(\sigma)] = \tau\tau^*\sigma\sigma^*[\partial(\tau)] = \sigma\sigma^*\tau\tau^*[\partial(\tau)] = \varepsilon\tau.$$

□

In [2] and [3] several C^*-algebras are successfully described and studied in terms of invariant sets of a certain partial action. This partial action is defined as follows. Let $\Omega := \{\omega : G \to \{0,1\} / \omega(e) = 1\}$ with the product topology, where G is a discrete group. Thus Ω is a compact Hausdorff space, which we will identify with the set $\{\omega \subseteq G : e \in \omega\}$ in the obvious way. For $t \in G$ consider the compact open subset $\Omega_t := \{\omega : t \in \omega\}$ of Ω. We have a partial action $\rho = (\{\rho_t\},\{\Omega_t\})$ of G on Ω given by $\rho_t : \Omega_{t^{-1}} \to \Omega_t$ such that $\rho_t(\omega) = t\omega$. We will show that the groupoid $\mathcal{G}_\rho$ is naturally isomorphic to the universal groupoid $\Gamma_{\mathbf{u}}$ of $S(G)$.

Suppose that $\omega \in \Omega$ and let $x^\omega : \mathcal{E}_{S(G)} \to \{0,1\}$ be given by $x^\omega(\varepsilon_{t_1} \cdots \varepsilon_{t_k}) = \chi_\omega(t_1) \cdots \chi_\omega(t_k)$, $\forall \varepsilon = \varepsilon_{t_1} \cdots \varepsilon_{t_k} \in \mathcal{E}_{S(G)}$, where χ_ω is the characteristic function of ω. Then x^ω is a semicharacter of $\mathcal{E}_{S(G)}$, and x^ω is non–zero because $x^\omega([e]) = 1$.

PROPOSITION 2.2 *The map: $h : \Omega \to X_{S(G)}$ given by $\omega \mapsto x^\omega$ is a homeomorphism, and $h(\Omega_t) = D_{[t]}$, $\forall t \in G$. Moreover, $h(\rho_{t^{-1}}\omega) = h(\omega) \cdot [t]$, $\forall t \in G, \omega \in \Omega_t$.*

Proof. For $x \in X_{S(G)}$ consider $\omega_x \subseteq G$ such that $\chi_{\omega_x}(t) = x(\varepsilon_t)$, $\forall t \in G$. Since $x([e]) = 1$, it follows that $\omega_x \in \Omega$. It is clear that $x \mapsto \omega_x$ is the inverse map of h. Note that $x \in D_{[t]} \iff x(\varepsilon_t) = 1 \iff t \in \omega_x \iff \omega_x \in \Omega_t$. Therefore $h(\Omega_t) = D_{[t]}$, $\forall t \in G$. Since both of Ω and $X_{S(G)}$ are considered with the product topologies, it is clear that h is a homeomorphism. Consider now $x \in D_{[t]}$, and let $s \in G$. By using successively the relations defining $S(G)$ we have $[t][s][s^{-1}][t^{-1}] = [ts][s^{-1}][t^{-1}][t][t^{-1}] = [ts][s^{-1}t^{-1}][t][t^{-1}] = \varepsilon_{ts}\varepsilon_t$. Thus

$$x \cdot [t](\varepsilon_s) = x([t][s][s^{-1}][t^{-1}]) = x(\varepsilon_{ts}\varepsilon_t) = x(\varepsilon_{ts})x(\varepsilon_t) = x(\varepsilon_{ts}).$$

Therefore, if $x \in D_{[t]}$, $s \in G$ we have $s \in \omega_{x\cdot[t]} \iff x(\varepsilon_{ts}) = 1 \iff \chi_{\omega^x}(ts) = 1 \iff ts \in \omega_x \iff s \in t^{-1}\omega_x$. This shows that $h(\rho_{t^{-1}}(\omega_x)) = h(\omega_x) \cdot [t]$. □

THEOREM 2.3 *Let $\Gamma_{\boldsymbol{u}}$ be the universal groupoid of the inverse semigroup $S(G)$. Then the map $\Phi : \mathcal{G}_\rho \to \Gamma_{\boldsymbol{u}}$ such that $\Phi(t\omega, t, \omega) = \overline{\big(h(t\omega), [t]\big)}$ is an isomorphism of locally compact groupoids, whose inverse is Ψ, given by $\Psi\big(\overline{(x,\sigma)}\big) = \big(h^{-1}(x), \partial(\sigma), \partial(\sigma)^{-1}h^{-1}(x)\big)$*

Proof. Note that Ψ is well defined by Lemma 2.1. Let $(t\omega, t, \omega) \in \mathcal{G}_\rho$. Then:

$$\Psi\Phi(t\omega, t, \omega) = \Psi\big(\overline{(h(t\omega), [t])}\big) = \big(h^{-1}\big(h(t\omega)\big), t, t^{-1}h^{-1}\big(h(t\omega)\big)\big) = (t\omega, t, \omega)$$

Now, if $\overline{(x,\sigma)} \in \Gamma_{\mathbf{u}}$ by Lemma 2.1 we have that $\overline{(x,\sigma)} = \overline{(x, [\partial(\sigma)])}$. Therefore:

$$\Phi\Psi\big(\overline{(x,\sigma)}\big) = \Phi\big(h^{-1}(x), \partial(\sigma), \partial(\sigma)^{-1}h^{-1}(x)\big) = \overline{\big(h\big(h^{-1}(x)\big), [\partial(\sigma)]\big)} = \overline{(x,\sigma)}$$

Hence Φ is a bijection with inverse Ψ. Now, consider elements $(st\omega, s, t\omega)$ and $(t\omega, t, \omega) \in \mathcal{G}_\rho$. Since $(st\omega, s, t\omega)(t\omega, t, \omega) = (st\omega, st, \omega)$, we have that $\Phi\big((st\omega, s, t\omega)(t\omega, t, \omega)\big) = \Phi(st\omega, st, \omega) = \overline{\big(h(st\omega), [st]\big)}$. On the other hand:

$$\begin{aligned}\Phi(st\omega, s, t\omega)\Phi(t\omega, t, \omega) &= \overline{\big(h(st\omega), [s]\big)}\ \overline{\big(h(t\omega), [t]\big)} \\ &= \overline{\big(h(st\omega), [s][t]\big)} \\ &= \overline{\big(h(st\omega), [st]\big)}.\end{aligned}$$

It follows that Φ is an isomorphism of groupoids. We show next that it is also a homeomorphism. Consider, for $t, s \in G$, $a \in \{0,1\}$ the sets:

$$V_{t,s,a} := \{(t\omega, t, \omega) \in \mathcal{G}_\rho : \chi_\omega(s) = 1\} \subseteq \mathcal{G}_\rho, \qquad U_{t,s,a} := D(U_{s,a}, [t]) \subseteq \Gamma_{\mathbf{u}},$$

where $U_{s,a} := \{x \in X_{S(G)} : x(\varepsilon_s) = 1\}$. The family $\mathcal{V} := \{V_{t,s,a} : t, s \in G,\, a \in \{0,1\}\}$ is a subbasis for the topology of $\mathcal{G}_\rho$, and the family $\mathcal{U} := \{U_{t,s,a} : t, s \in G,\, a \in \{0,1\}\}$ is a subbasis for the topology of $\Gamma_{\mathbf{u}}$. Thus to see that Φ is a homeomorphism it suffices to show that $\mathcal{U} = \{\Phi(V) : V \in \mathcal{V}\}$. Now: $\Phi(V_{t,s,a}) = \{\overline{(h(t\omega), [t])} : h(\omega)(\varepsilon_s) = a)\} = \{\overline{(h(\omega)\cdot[t]^*, [t])} : \big(h(\omega)\cdot[t]^*\big)\cdot[t](\varepsilon_s) = a)\} = \{\overline{(x, [t])} : x(\varepsilon_{ts}) = a\} = \{\overline{(x,[t])} : x \in U_{ts,a}\} = U_{t,ts,a}$. Therefore $\Phi(V) \in \mathcal{U}$, $\forall V \in \mathcal{V}$. Since $U_{t,s,a} = \Phi(V_{t,t^{-1}s,a})$, we conclude that $\mathcal{U} = \{\Phi(V) : V \in \mathcal{V}\}$. □

3. $\tilde{F}$–inverse semigroups and localizations

We end the paper by showing sketchily how Nica's theory connects with the groupoids of localizations defined by Paterson ([5, page 127]). We use the notation of Section 1. Recall that a localization is a right action β of an inverse semigroup S on a space X such that $\{\mathrm{dom}(\beta_\sigma) : \sigma \in S\}$ is a basis for the topology of X. Given a localization β, Paterson considers the set $\Theta := \{(x,\sigma) \in X \times S : x \in \mathrm{dom}(\beta_\sigma)\}$ with the equivalence relation $(x,\sigma) \sim (y,\tau) \iff x = y$ and there exists $\varepsilon \in \mathcal{E}_S$ such that $x \in \mathrm{dom}(\beta_\varepsilon)$ and $\varepsilon\sigma = \varepsilon\tau$. Then he defines the locally compact groupoid $\Gamma(X,S) := \{\overline{(x,\sigma)} : \sigma \in S, x \in \mathrm{dom}(\beta_\sigma)\}$, where the product is given by $\overline{(x,\sigma)}\,\overline{(x\cdot\sigma,\tau)} := \overline{(x,\sigma\tau)}$ ($x\cdot\sigma$ denotes $\beta_\sigma(x)$), the inversion by $\overline{(x,\sigma)}^{-1} := \overline{(x\cdot\sigma,\sigma^*)}$, and a basis for the topology is given by $\{D(U,\sigma) : \sigma \in S, U$ open subset of $\mathrm{dom}(\beta_\sigma)\}$, where $D(U,\sigma) := \{\overline{(x,\sigma)} : x \in U\}$.

Suppose now that α is a (left) action of an $\tilde{F}$–inverse semigroup S on X, and consider the inverse semigroup

$$L_\alpha := \{u \in \mathrm{PHom}(X) : u \text{ is extended by some } \alpha_\sigma\},$$

where $\mathrm{PHom}(X)$ stands for the inverse semigroup of the partial homeomorphisms of X.

Proposition 3.1 *Let S be an $\tilde{F}$–inverse semigroup and $S_\alpha := \{(\sigma,u) \in S \times L_\alpha : u \leq \alpha_\sigma\}$. Then S_α is an $\tilde{F}$–inverse semigroup, and $\mathcal{M}_{S_\alpha} = \{(\mu,\alpha_\mu) : \mu \in \mathcal{M}_S\}$.*

Proof. It is clear that S_α is an inverse semigroup. Note that $(\sigma,u) \leq (\tau,v) \iff \sigma \leq \tau$ and $u \leq v$. Thus if σ is non–zero and $\sigma \leq \mu$, with $\mu \in \mathcal{M}_S$, and if $(\sigma,u) \leq (\tau,v)$, then it must be $\tau \leq \mu$, and hence $(\sigma,u) \leq (\tau,v) \leq (\mu,\alpha_\mu)$, which is obviously a maximal element. This ends the proof. □ The left action

α induces a right action β of S_α on X via $\beta_{(\sigma,u)} := u^*$ ($\beta_{(\sigma,u)}$ is the restriction of α_{σ^*} to the range of u). Then β is a localization of S on X in the sense of Paterson , and therefore it has associated a groupoid Γ_α. By Proposition 3.1 we can identify a class $\overline{(x,(\sigma,u))}$ with a pair (x,μ), where $\sigma \leq \mu$, $\mu \in \mathcal{M}_S$, and $x \in \mathrm{dom}(\alpha_\mu)$ and then we have $\Gamma_\alpha = \{(x,\mu) : \mu \in \mathcal{M}_S, x \in \mathrm{dom}(\beta_\mu)\}$, $(x,\mu)(x\cdot\mu,\mu') = (x,\mu\cdot\mu')$, $(x,\mu)^{-1} = (x\cdot\mu,\mu^*)$, and Γ_α has the product topology. It is easily checked that the map $\mathcal{N}_\alpha \to \Gamma_\alpha$ given by $(\mu,x) \mapsto (x\cdot\mu^*,\mu)$ is an isomorphism of locally compact groupoids. Thus we have:

PROPOSITION 3.2 *The Nica groupoid of the action α is naturally isomorphic to the Paterson groupoid associated with the localization induced by α.*

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FERNANDO ABADIE
fabadie@cmat.edu.uy
Centro de Matemática. Facultad de Ciencias
Universidad de la República
Iguá 4225, 11400. Montevideo, Uruguay