RANDOM AND MEAN LYAPUNOV EXPONENTS FOR $\mathrm{GL}_n(\mathbb{R})$

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Abstract. We consider orthogonally invariant probability measures on $\mathrm{GL}_n(\mathbb{R})$ and compare the mean of the logs of the moduli of eigenvalues of the matrices to the Lyapunov exponents of random matrix products independently drawn with respect to the measure. We give a lower bound for the former in terms of the latter. The results are motivated by Dedieu-Shub[DS03]. A novel feature of our treatment is the use of the theory of spherical polynomials in the proof of our main result.

1. Introduction and main result

This paper concerns a simple case of a family of linear cocycles. Our motivation for these considerations originates with the study of the entropy of diffeomorphisms of closed manifolds. Let $\pi : \mathcal{V} \to X$ be a finite-dimensional vector bundle. The basic object of interest is the iteration of fiberwise linear maps $\mathcal{A}$ of π which cover a map $f : X \to X$ of the base. The *cocycle* is described in the following diagram by the bundle map $\mathcal{A} : \mathcal{V} \to \mathcal{V}$ which satisfies $\pi \circ \mathcal{A} = f \circ \pi$.

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{\mathcal{A}} & \mathcal{V} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{f} & X \end{array} \tag{1.1}$$

See Ruelle [Rue79], Mañe [Mn87], and Viana [Via14] for extensions. We give four basic examples of this setup.

Example 1.1. The base X is one point. (This is the object of our paper.)

Example 1.2. X is a closed manifold M, $\mathcal{V}$ is the tangent bundle TM of M, f is a smooth (at least $C^{1+\alpha}$ endomorphism of M) and $\mathcal{A} = Tf$, the derivative of f. This is the *derivative cocycle*. Note that the kth-iterate of Tf is given by

$$(Tf)^k(x, v) = (f^k(x), Tf(f^{k-1}(x)) \cdots Tf(x)v), \qquad (x, v) \in TM.$$

Example 1.3. Let $\mathcal{V} \xrightarrow{\pi} X$ be a fixed vector bundle and $\mathscr{F}$ a family of bundle maps $(\mathcal{A}, f)$ as in (1.1), with $\mathcal{A} : \mathcal{V} \to \mathcal{V}$ fibrewise linear and $f : X \to X$ a base map. Assume given a finite measure μ on $\mathscr{F}$.

Then *random products* of independent elements of $\mathscr{F}$, drawn with respect to the measure μ, are described by the following cocycle. Let $\mathcal{G} = \mathscr{F}^{\mathbb{N}}$ with the product measure $\mu^{\mathbb{N}}$. Writing elements of $\mathcal{G}$ as

$$(\mathcal{A}_i, f_i)_i = (\cdots, (\mathcal{A}_n, f_n), \cdots, (\mathcal{A}_0, f_0))$$

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we define $\sigma : \mathcal{G} \to \mathcal{G}$ by $\sigma((\mathcal{A}_i, f_i)_i) = (\mathcal{A}_{i+1}, f_{i+1})_i$, that is, shift to the right and delete the first term. Then, the map $\mathcal{H} : \mathcal{G} \times \mathcal{V} \to \mathcal{G} \times \mathcal{V}$, given by

$$\mathcal{H}((\mathcal{A}_i, f_i)_i, v) = (\sigma((\mathcal{A}_i, f_i)_i), \mathcal{A}_0(v)), \qquad ((\mathcal{A}_i, f_i)_i, v) \in \mathcal{G} \times \mathcal{V}$$

defines the cocycle

$$\begin{tikzcd}
\mathcal{G} \times \mathcal{V} \arrow[r, "\mathcal{H}"] \arrow[d, "\mathrm{Id}_{\mathcal{G}} \times \pi"'] & \mathcal{G} \times \mathcal{V} \arrow[d, "\mathrm{Id}_{\mathcal{G}} \times \pi"] \\
\mathcal{G} \times X \arrow[r, "h"] & \mathcal{G} \times X
\end{tikzcd}$$

where the base map $h : \mathcal{G} \times X \to \mathcal{G} \times X$ is given by $h((\mathcal{A}_i, f_i)_i, x) = (\sigma((\mathcal{A}_i, f_i)_i), f_0(x))$, where $\pi(v) = x$.

The k^{th}-iterate of the cocycle $\mathcal{H}$, is given by

$$\mathcal{H}^k((\mathcal{A}_i, f_i)_i, v) = (\sigma^k((\mathcal{A}_i, f_i)_i), \mathcal{A}_{k-1} \cdots \mathcal{A}_0(v)), \qquad ((\mathcal{A}_i, f_i)_i, v) \in \mathcal{G} \times \mathcal{V},$$

which yields the products of random i.i.d. elements of the measure space $(\mathscr{F}, \mu)$.

Example 1.4. Let $f : X \to X$ and $\phi : X \to \mathrm{GL}_n(\mathbb{R})$. Let

$$\begin{tikzcd}
X \times \mathbb{R}^n \arrow[r, "\mathcal{A}"] \arrow[d, "\pi"'] & X \times \mathbb{R}^n \arrow[d, "\pi"] \\
X \arrow[r, "f"] & X
\end{tikzcd} \tag{1.2}$$

be defined by $\mathcal{A}(x, v) = (f(x), \phi(x)v)$. The functions f and ϕ are frequently called *linear cocyles* in the literature and $\mathcal{A}$ the associated linear extension. Here we use linear cocycle (or just cocycle) for both. In this case the k-th iterate of $\mathcal{A}$ is given by

$$\mathcal{A}^k(x, v) = (f^k(x), \phi(f^{k-1}(x)) \cdots \phi(f(x))\phi(x)v), \qquad (x, v) \in X \times \mathbb{R}^n.$$

We now return to the general setting of a finite dimensional vector bundle $\mathcal{V} \xrightarrow{\pi} X$ and cocycle as in (1.1). Assume that π has a *Finsler structure*, i.e. a norm on each fiber of $\mathcal{V}$. Consider the limit

$$\lim_n \frac{1}{n} \log \frac{\|\mathcal{A}^n(v)\|}{\|v\|}, \tag{1.3}$$

for a given nonzero vector $v \in \mathcal{V}$. If the limit (1.3) exists we call it a *Lyapunov exponent* of $\mathcal{A}$. We refer the reader to the expository article of Wilkinson [Wil17] for an introduction to Lyapunov exponents.

When X is a finite measure space, subject to various measurability and integrability conditions, the Oseledets Theorem [Ose68] says that for all $v \in \mathcal{V}$ the limit (1.3) exists almost surely and coincides with one of the real numbers

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n.$$

(See also Gol'dsheid-Margulis[GdM89], Guivarc'h-Raugi[GR89], Ruelle[Rue79], and Viana[Via14].)

For a real-valued function ψ, let $\psi^+(x) = \max(0, \psi(x))$. Then the theorem of Pesin [Pes77] and Ruelle [Rue78] implies that in the setting of Example 1.2, if

$f : M \to M$ preserves a measure μ, absolutely continuous with respect to Lebesgue, and $\mathcal{A}$ is the derivative cocycle, we have

$$\int_M \sum_i \lambda_i^+(x)\, dx = h_\mu(f), \tag{1.4}$$

where $h_\mu(f)$ is the entropy of f with respect to μ. From a dynamical systems perspective, knowing when $h_\mu(f)$ is positive and how large it may be is of great interest. But the Lyapunov exponents of the derivative cocycle are generally difficult to compute, even to show positivity of the integral (1.4). On the other hand the Lyapunov exponents of a *random product* are frequently easy to show positive.

One attempt to approach the problem is to consider diffeomorphism or more generally cocycles that belong to *rich* families $\mathscr{F}$, and to prove that $\int_M \sum_i \lambda^+(x,f)\, dx$ is positive for at least some elements of the family by comparing with Lyapunov exponents of random products. It is not clear what the notion of *rich* should be to carry out this program of bounding the average Lyapunov exponents by those of random products.

There is some success in Pujals-Robert-Shub[PRS06], Pujals-Shub[PS08], De la Llave-Shub-Simó[dlLSS08], and Dedieu-Shub[DS03], and an extensive discussion in Burns-Pugh-Shub-Wilkinson[BPSW01] for derivative cocycles. A notion of rich which comes close for the circle and two sphere is $\mathrm{O}_n(\mathbb{R})$ invariance. Indeed the paper [DS03] was important in this direction. Write the eigenvalues of a matrix $A \in \mathrm{GL}_n(\mathbb{C})$ according to their multiplicity as $\lambda_i(A)$ with

$$|\lambda_1(A)| \geq |\lambda_2(A)| \geq \cdots \geq |\lambda_n(A)|.$$

Let $r_1 \geq \ldots \geq r_n$ be the Lyapunov exponents of the random product. The main Theorem of [DS03] is:

Theorem (Theorem 1, [DS03])**.** *If μ is a unitarily invariant measure on $\mathrm{GL}_n(\mathbb{C})$, satisfying the integrability condition*

$$A \in \mathrm{GL}_n(\mathbb{C}) \mapsto \log^+(\|A\|) \text{ and } \log^+(\|A^{-1}\|) \quad \text{are} \quad \mu\text{-integrable},$$

then

$$\int_{A\in\mathrm{GL}_n(\mathbb{C})} \sum_{i=1}^k \log|\lambda_i(A)|\, d\mu(A) \geq \sum_{i=1}^k r_i.$$

In [DS03] and [BPSW01] it is asked if a similar theorem holds for $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{O}_n(\mathbb{R})$ perhaps with a constant c_n depending on n. Here we prove that it does. Our main theorem is the following.

Theorem 1. *There exists a universal constant $c_n > 0$ such that, if μ is an orthogonally invariant measure on $\mathrm{GL}_n(\mathbb{R})$ satisfying the integrability condition $A \in \mathrm{GL}_n(\mathbb{R}) \mapsto \log^+(\|A\|)$ and $\log^+(\|A^{-1}\|) \in L^1(\mathrm{GL}_n(\mathbb{R}), \mu)$, then*

$$\int_{A\in\mathrm{GL}_n(\mathbb{R})} \left(\sum_{i=1}^k \log|\lambda_i(A)|\right)^+ d\mu(A) \geq c_n \left(\sum_{i=1}^k r_i\right)^+$$

for any $k, 1 \leq k \leq n$.

Let $SL_n(\mathbb{R})$ be the *special linear group* of $n \times n$ matrices with determinant 1. Then we have the following result.

Corollary 1. *There exists a universal constant $c_n' > 0$ such that, if μ is an orthogonally invariant probability measure on $SL_n(\mathbb{R})$, then,*

$$\int_{A\in SL_n(\mathbb{R})} \sum_{i=1}^{k} \log|\lambda_i(A)|\, d\mu(A) \geq c_n' \left(\sum_{i=1}^{k} r_i\right)^{+}.$$

□

The proof of the corollary follows immediately since, for all $A \in SL(n,\mathbb{R})$, $\prod_{j=1}^{k} |\lambda_j(A)| \geq 1$, $(k = 1,\dots,n)$.

Some special cases of our main result Theorem 1 have been previously established. For $n = 2$ the result is proved in [DS03] and Avila-Bochi [AB02]. Rivin [Riv05] proves the case $n > 2, k = 1$. (Both [AB02] and [Riv05] prove more general results in these restricted settings, from which the stated results can be derived.)

We conclude this introduction with an outline of the remainder of the paper and a sketch of the ideas used in the proof of Theorem 1. The sums $\sum_{i\leq k} r_i$ of the random Lyapunov exponents appearing in Theorem 1 admit a geometric interpretation relating them to an integral over the Grassmannian manifold $\mathbb{G}_{n,k}$ of k dimensional subspaces of $\mathbb{R}^n$. We use this relation in Section 2 to reduce the proof of Theorem 1 to a comparison of an integral on the the orthogonal group to an integral on the Grassmannian. This comparison is effected by applying the coarea formula to the two projections Π_1, Π_2 of the manifold $\mathbb{V}_A$ of fixed k-dimensional subspaces

$$\mathbb{V}_A = \{(U,g) \in \mathrm{O}_n(\mathbb{R}) \times \mathbb{G}_{n,k} : (UA)_{\#} g = g\} \text{ for fixed } A \in \mathrm{GL}_n(\mathbb{R}).$$

This use of the coarea formula, presented in Sections 3 and 4 is similar to the approach of [DS03]. Our main point of departure from the earlier paper comes in Section 5 in our treatment of bounding an integral of the normal Jacobian of the projection Π_1. We use the theory of spherical polynomials for the symmetric space G/K for $G = \mathrm{GL}_n(\mathbb{R})$ and $K = \mathrm{O}_N(\mathbb{R})$. Our Theorem 4 is a consequence of a positivity results for Jack polynomials due to Knop-Sahi [KS97]. This approach highlights a difficulty in extending the results of [DS03] to our setting. In the case of $G = \mathrm{GL}_n(\mathbb{C}), K = \mathrm{U}_n(\mathbb{C})$, the associated spherical polynomials are simply Schur polynomials, thus permitting a more direct treatment in the earlier work using the Vandermonde determinant, see Section 4.5 of [DS03].

We hope that the results and techniques of this paper stimulate further interactions between the ergodic theory of cocycles and harmonic analyses on symmetric spaces. One appealing direction is the investigations of families of cocycles which have elements with $\int_{x\in X} \sum_i \lambda_i^+(x)\, dx$ positive. Especially interesting would be more rich families of dynamical systems which must have some elements of positive entropy. One approach for measure preserving families of dynamical systems would be to compare the Lyapunov exponents of the derivative cocycles of the family to the Lyapunov exponents of the random products of the cocycles of the family.

2. Proof of Theorem 1

Let $\mathbb{G}_{n,k}$ be the Grassmannian of k-dimensional subspaces of $\mathbb{R}^n$. Given $g \in \mathbb{G}_{n,k}$, let $\mathrm{O}(g)$ be the subgroup of $\mathrm{O}_n(\mathbb{R})$ that fixes g. For $A \in \mathrm{GL}_n(\mathbb{R})$ we denote by $A_{\#}$ the mapping corresponding to the natural induced action on $\mathbb{G}_{n,k}$ and by $A|_g$ the restriction of A to the subspace g.

Consider the Riemannian metric on $\mathrm{O}_n(\mathbb{R})$ coming from its embedding in the space of $n \times n$ matrices with the natural inner product $\langle A, B\rangle = \mathrm{tr}(A\,{}^tB)$. As a Lie group, this Riemannian structure on $\mathrm{O}_n(\mathbb{R})$ is left and right invariant and it induces a Riemannian structure on $\mathbb{G}_{n,k}$ as a homogeneous space of $\mathrm{O}_n(\mathbb{R})$. We denote by $\mathrm{vol}\,\mathrm{O}_n(\mathbb{R})$ and $\mathrm{vol}\,\mathbb{G}_{n,k}$ the Riemannian volumes of the orthogonal group and Grassmannian respectively, and note the relation

$$\mathrm{vol}\,\mathbb{G}_{n,k} = \frac{\mathrm{vol}\,\mathrm{O}_n(\mathbb{R})}{\mathrm{vol}\,\mathrm{O}_k(\mathbb{R}) \cdot \mathrm{vol}\,\mathrm{O}_{n-k}(\mathbb{R})}. \tag{2.1}$$

Define the constant

$$c_{n,k} = \frac{\mathrm{vol}\,\mathrm{O}_k(\mathbb{R}) \cdot \mathrm{vol}\,\mathrm{O}_{n-k}(\mathbb{R})}{\binom{n}{k}}. \tag{2.2}$$

Theorem 2. *For any $A \in \mathrm{GL}_n(\mathbb{R})$ we have*

$$\int_{U\in \mathrm{O}_n(\mathbb{R})} \left[\sup_{\substack{g\in\mathbb{G}_{n,k}:\\ (UA)_\# g = g}} \left(\log^+ |\det UA|_g|\right)\right] d\mathrm{O}_n(\mathbb{R}) \geq c_{n,k} \int_{g\in\mathbb{G}_{n,k}} \log^+ |\det A|_g|\, d\mathbb{G}_{n,k}.$$

If we integrate instead with respect to the Haar measure dU on $\mathrm{O}_n(\mathbb{R})$ and the invariant probability measure dg on $\mathbb{G}_{n,k}$ we get

$$\int_{U\in \mathrm{O}_n(\mathbb{R})} \left[\sup_{\substack{g\in\mathbb{G}_{n,k}:\\ (UA)_\# g = g}} \left(\log^+ |\det UA|_g|\right)\right] dU \geq \frac{1}{\binom{n}{k}} \int_{g\in\mathbb{G}_{n,k}} \log^+ |\det A|_g|\, dg. \tag{2.3}$$

This follows immediately from Theorem 2 and (2.1). The proof of Theorem 2 is given in Sections 2 and 5.

Note that Theorem 2 implies a slightly more general result.

Theorem 3. *There is a constant $c_n > 0$ such that, if μ is an orthgononally invariant probability measure on $\mathrm{GL}_n(\mathbb{R})$, then,*

$$\int_{A\in \mathrm{GL}_n(\mathbb{R})} \left[\sup_{\substack{g\in\mathbb{G}_{n,k}:\\ A_\# g = g}} \log^+ |\det A|_g| \right] d\mu \geq c_n \int_{A\in \mathrm{GL}_n(\mathbb{R})} \int_{g\in\mathbb{G}_{n,k}} \log^+ |\det A|_g|\, d\mathbb{G}_{n,k}\, d\mu.$$

Proof. Disintegrate the measure along orbits and apply the inequality of Theorem 2, orbit by orbit. Note that $|\det A|_g|$ is constant on an orbit. □

Proof of Theorem 1. Pointwise we have

$$\left(\sum_{i=1}^k \log|\lambda_i(A)|\right)^+ \geq \sup_{\substack{g\in\mathbb{G}_{n,k}:\\ A_\# g = g}} \log^+ |\det A|_g|, \quad (A \in \mathrm{GL}_n(\mathbb{R})),$$

where the supremum on the right hand side is defined to be 0 if the set of $g \in \mathbb{G}_{n,k}$ such that $A_\# g = g$ is empty.

Then, for finishing the proof of Theorem 1 it suffice to identify the right hand side of the expression in Theorem 3 in terms of $\left(\sum_{i=1}^k r_i\right)^+$.

As in the proof of Theorem 3 in [DS03], and surely in many other places in the reference,

$$\sum_{i=1}^{k} r_i = \int_{A\in \mathrm{GL}_n(\mathbb{R})} \int_{g\in \mathbb{G}_{n,k}} \log \Big| \det A|_g \Big| \, d\mathbb{G}_{n,k}\, d\mu$$

so

$$\begin{aligned} \left(\sum_{i=1}^{k} r_i\right)^{+} &= \left(\int_{A\in \mathrm{GL}_n(\mathbb{R})} \int_{g\in \mathbb{G}_{n,k}} \log | \det A|_g | \, d\mathbb{G}_{n,k}\, d\mu\right)^{+} \\ &\le \int_{A\in \mathrm{GL}_n(\mathbb{R})} \int_{g\in \mathbb{G}_{n,k}} \log^{+} | \det A|_g | \, d\mathbb{G}_{n,k}\, d\mu. \end{aligned}$$

□

We will give the proof of Theorem 2 in Sections 2 and 5 after some preparations in the next section.

3. Manifold of fixed subspaces

Let $A \in \mathrm{GL}_n(\mathbb{R})$, and define the manifold of fixed k-dimensional subspaces

$$\mathbb{V}_A := \{(U, g) \in \mathrm{O}_n(\mathbb{R}) \times \mathbb{G}_{n,k} : \ (UA)_{\#} g = g\}$$

Let $\Pi_1 : \mathbb{V}_A \to \mathrm{O}_n(\mathbb{R})$ and $\Pi_2 : \mathbb{V}_A \to \mathbb{G}_{n,k}$ be the associated projections.

$$\begin{tikzcd} & \mathbb{V}_A \arrow[dl, bend right, "\Pi_1"'] \arrow[dr, bend left, "\Pi_2"] & \\ \mathrm{O}_n(\mathbb{R}) & & \mathbb{G}_{n,k} \end{tikzcd}$$

Given $g \in \mathbb{G}_{n,k}$, one has

$$\Pi_2^{-1}(g) = \{(U, g) : U \in \mathrm{O}_n(\mathbb{R}),\ A_{\#} g = (U^{-1})_{\#} g\}.$$

By abusing notation we will write $\Pi_2^{-1}(g) = \Pi_1 \Pi_2^{-1}(g)$, which can be identified with the product space $\mathrm{O}(g) \times \mathrm{O}(g^{\perp})$, which we in turn identify with $\mathrm{O}(k) \times \mathrm{O}(n-k)$. Similarly, given $U \in \mathrm{O}_n(\mathbb{R})$, we identify $\Pi_1^{-1}(U)$ with

$$\{g \in \mathbb{G}_{n,k} : \ \text{fixed by } (UA)_{\#}\}.$$

Remark 2. Note that, on a set of full measure in $\mathrm{O}_n(\mathbb{R})$, the fiber $\Pi_1^{-1}(U)$ is finite and $\#\Pi_1^{-1}(U)$ is bounded above by $\binom{n}{k}$. This follows from the fact that the set of $U \in \mathrm{O}_n(\mathbb{R})$ such that UA has repeated eigenvalues, is a proper subvariety of $\mathrm{O}_n(\mathbb{R})$ defined by the discriminant of the characteristic polynomial of UA. Therefore a k-dimensional invariant subspace for UA, where U lies in the complement of the algebraic subvariety described above, corresponds to a choice of k-eigenvalues for UA, and corresponding eigenspaces,

The tangent space to the Grassmannian $\mathbb{G}_{n,k}$ at g, can be identified in a natural way with the set of linear maps $\mathrm{Hom}(g, g^{\perp})$, i.e., any subspace $g' \in \mathbb{G}_{n,k}$, in a neighborhood of g can be represented as the graph of a unique map in $\mathrm{Hom}(g, g^{\perp})$. More precisely, if we denote by π_g and $\pi_{g^{\perp}}$ the orthogonal projections of $\mathbb{R}^n = g \oplus g^{\perp}$

into g and $g^\perp$ respectively, then, $g' \in \mathbb{G}_{n,k}$ such that $g' \cap g^\perp = \{0\}$, is the graph of the linear map $\pi_{g^\perp} \circ ((\pi_g)|_{g'})^{-1}$.

Lemma 3. *Let $B \in \mathrm{GL}_n(\mathbb{R})$, and $g \in \mathbb{G}_{n,k}$ such that $B_\# g = g$. Then, the induced map $\mathcal{L}_B : \mathrm{Hom}(g, g^\perp) \to \mathrm{Hom}(g, g^\perp)$, on local charts, is given by*

$$\mathcal{L}_B(\varphi) = [\pi_{g^\perp}(B|_{g^\perp})] \circ \varphi \circ \big([\pi_g(B|_g)] + [\pi_g(B|_{g^\perp})] \circ \varphi)\big)^{-1}$$

Furthermore, its derivative at g, represented by $0 \in \mathrm{Hom}(g, g^\perp)$, is given by

$$D\mathcal{L}_B(0)\dot\varphi = [\pi_{g^\perp}(B|_{g^\perp})] \circ \dot\varphi \circ [\pi_g(B|_g)]^{-1}$$

Let us denote by NJ_{Π_1} and NJ_{Π_2} the normal Jacobians of the maps Π_1 and Π_2 respectively. (See [DS03, Section 3.1].)

Lemma 4 ([DS03, Section 3.2])**.** *Given $(U, g) \in \mathbb{V}_A$, one has*

- $\mathrm{NJ}_{\Pi_1}(U, g) = |\det \mathrm{Id} - D\mathcal{L}_{UA}(g)|$*;*
- $\mathrm{NJ}_{\Pi_2}(U, g) = 1$*.*

In Section 5 we will need the normal Jacobian written more explicitly. To this end, choose bases $v_1, \ldots, v_k$ for g and $v_{k+1}, \ldots, v_n$ for its orthogonal complement $g^\perp$. In terms of the basis $v_1, \ldots, v_n$ of $\mathbb{R}^n$, a linear map $B : \mathbb{R}^n \to \mathbb{R}^n$ which satisfies $Bg = g$ is represented by a matrix of the form

$$\left(\begin{array}{c|c} B_1 & * \\ \hline 0 & B_2 \end{array}\right).$$

By Lemma 3, if X is the matrix representing $\dot\varphi$ in this basis, then $D\mathcal{L}_B(0)\dot\varphi$ is represented by the matrix $B_2 X B_1^{-1}$.

Lemma 5. *Let $(U, g) \in \mathbb{V}_A$ and let*

$$\left(\begin{array}{c|c} B_1 & * \\ \hline 0 & B_2 \end{array}\right).$$

represent the map UA in the basis $v_1, \ldots, v_n$ defined above. Then

$$\det(\mathrm{Id} - D\mathcal{L}_{UA}(g)) = \det(\mathrm{Id} - B_2 \otimes {}^tB_1^{-1})$$

4. Proof of Theorem 2

Let $\phi : \mathbb{G}_{n,k} \to \mathbb{R}$ be an integrable function, and let $\hat\phi : \mathbb{V}_A \to \mathbb{R}$ be its lift to $\mathbb{V}_A$, i.e. $\hat\phi$ is given by $\hat\phi := \phi \circ \Pi_2$. (Note that given $g \in \mathbb{G}_{n,k}$, $\hat\phi$ is constant in the fiber $\Pi_2^{-1}(g)$, and its value coincides with the value of ϕ at g.)

For a set of full measure of $U \in \mathrm{O}_n(\mathbb{R})$ (cf. Remark 2) we have

$$\binom{n}{k} \sup_{g \in \Pi_1^{-1}(U)} (\phi(g)) \geq \#\Pi_1^{-1}(U) \sup_{g \in \Pi_1^{-1}(U)} (\phi(g)) \geq \sum_{g \in \Pi_1^{-1}(U)} \phi(g). \tag{4.1}$$

By the coarea formula applied to Π_1 we get

$$\int_{U \in \mathrm{O}_n(\mathbb{R})} \left(\sum_{g \in \Pi_1^{-1}(U)} \phi(g) \right) d\mathrm{O}_n(\mathbb{R}) = \int_{\mathbb{V}_A} \hat\phi(U, g)\, \mathrm{NJ}_{\Pi_1}(U, g)\, d\mathbb{V}_A \tag{4.2}$$

On the other hand, applying the coarea formula to the projection Π_2,

$$\int_{\mathbb{V}_A} \hat{\phi}(U,g)\,\mathrm{NJ}_{\Pi_1}(U,g)\,d\mathbb{V}_A = \int_{g\in\mathbb{G}_{n,k}} \left(\int_{U\in\Pi_2^{-1}(g)} \phi(g)\,\mathrm{NJ}_{\Pi_1}(U,g)\,d\Pi_2^{-1}(g) \right) d\mathbb{G}_{n,k}, \tag{4.3}$$

where we have used the fact that $\mathrm{NJ}_{\Pi_2} = 1$.

Then from (4.1), (4.2), (4.3) and Lemma 4 we have

$$\begin{aligned} \int_{U\in \mathrm{O}_n(\mathbb{R})} &\left(\sup_{g\in\Pi_1^{-1}(U)} (\phi(g)) \right) d\mathrm{O}_n(\mathbb{R}) \\ &\geq \binom{n}{k}^{-1} \int_{g\in\mathbb{G}_{n,k}} \phi(g) \left[\int_{U\in\Pi_2^{-1}(g)} \mathrm{NJ}_{\Pi_1}(U,g)\,d\Pi_2^{-1}(g) \right] d\mathbb{G}_{n,k} \\ &= \binom{n}{k}^{-1} \int_{g\in\mathbb{G}_{n,k}} \phi(g) \left[\int_{U\in\Pi_2^{-1}(g)} |\det \mathrm{Id} - D\mathcal{L}_{UA}(g)|\,d\Pi_2^{-1}(g) \right] d\mathbb{G}_{n,k} \end{aligned} \tag{4.4}$$

Specialize now to $\phi : \mathbb{G}_{n,k} \to \mathbb{R}$ given by

$$\phi(g) := \log^+ \det A|g, \quad g \in \mathbb{G}_{n,k}.$$

In particular

$$\sup_{g\in\Pi_1^{-1}(U)} \phi(g) = \sup_{\substack{g\in\mathbb{G}_{n,k}: \\ (UA)_\# g = g}} (\log^+ \det(UA)|g).$$

Now, the proof of Theorem 2, follows from Theorem 4 below which is used to bound the bracketed inner integral in (4.4); this together with the nonnegativity of ϕ proves Theorem 2.

Theorem 4. *Given $g \in \mathbb{G}_{n,k}$, one has*

$$\int_{U\in\Pi_2^{-1}(g)} (\det \mathrm{Id} - D\mathcal{L}_{UA}(g))\; d\Pi_2^{-1}(g)(U) \geq \mathrm{vol}\,\mathrm{O}_k(\mathbb{R}) \cdot \mathrm{vol}\,\mathrm{O}_{n-k}(\mathbb{R}).$$

The proof is given in the following section.

5. Proof of Theorem 4

For fixed $g \in \mathbb{G}_{n,k}$ choose $U_0 \in \mathrm{O}_n(\mathbb{R})$ such that $U_0 A g = g$. Then

$$\Pi_2^{-1}(g) = \{VU_0 : V \in \mathrm{O}_k(\mathbb{R}) \times \mathrm{O}_{n-k}(\mathbb{R})\},$$

where we continue to identify $\mathrm{O}_k(\mathbb{R}) \times \mathrm{O}_{n-k}(\mathbb{R})$ with $\mathrm{O}(g) \times \mathrm{O}(g^\perp)$. We have

$$\begin{aligned} \int_{U\in\Pi_2^{-1}(g)} &\det\left(\mathrm{Id} - D\mathcal{L}_{UA}(g)\right)\, d\Pi_2^{-1}(g)(U) \\ &= \int_{V\in \mathrm{O}_k(\mathbb{R})\times \mathrm{O}_{n-k}(\mathbb{R})} \det(\mathrm{Id} - D\mathcal{L}_{VU_0A}(g))\, d\Pi_2^{-1}(g)(VU_0) \\ &= \mathrm{vol}\,\mathrm{O}_k(\mathbb{R}) \cdot \mathrm{vol}\,\mathrm{O}_{n-k}(\mathbb{R}) \\ &\qquad \times \int_{\psi_1\in \mathrm{O}_k(\mathbb{R})} \int_{\psi_2\in \mathrm{O}_{n-k}(\mathbb{R})} \det(\mathrm{Id} - (\psi_2 B_2) \otimes {}^t(\psi_1 B_1)^{-1})\, d\psi_2\, d\psi_1 \end{aligned}$$

where $d\psi_1, d\psi_2$ are the Haar measures on $\mathrm{O}_k(\mathbb{R})$ and $\mathrm{O}_{n-k}(\mathbb{R})$ The last equality follows from Lemma 5, with

$$B_1 = \pi_g((U_0 A)|_g) \text{ and } B_2 = \pi_{g^\perp}((U_0 A)|_{g^\perp})$$

More generally, for $B_1 \in \mathrm{GL}_k(\mathbb{R}), B_2 \in \mathrm{GL}_{n-k}(\mathbb{R})$ we consider the integral of the characteristic polynomial

(5.1)
$$\mathcal{J}(B_1, B_2; u) = \int_{\psi_1 \in \mathrm{O}_k(\mathbb{R})} \int_{\psi_2 \in \mathrm{O}_{n-k}(\mathbb{R})} \det\left(\mathrm{Id} - u(\psi_2 B_2) \otimes {}^t(\psi_1 B_1)^{-1}\right) d\psi_2 \, d\psi_1.$$

Therefore Theorem 4 is equivalent to

$$\mathcal{J}(B_1, B_2; 1) \geq 1. \tag{5.2}$$

In fact we will prove an explicit formula for the integral, expressing the coefficients of the characteristic polynomial $\mathcal{J}(B_1, B_2; u)$ as polynomials in the squares of the singular values of B_1 and B_2^{-1} with positive integer coefficients.

We complete the proof of the Theorem 4 and inequality (5.2) in several steps. First we use the representation theory of the general linear group to factor the double integral into a linear combination of a product of two integrals over $\mathrm{O}_n(\mathbb{R})$ and $\mathrm{O}_{n-k}(\mathbb{R})$ respectively. Next each orthogonal group integral is identified with a *spherical polynomial*. Finally, the theorem follows from an identity between spherical polynomials and Jack polynomials, and a positivity result for the latter due to Knop and Sahi [KS97].

5.1. **Orthogonal group integrals.** We begin by expanding the characteristic polynomial in the integrand as a sum of traces:

$$\det\left(\mathrm{Id} - u(\psi_2 B_2) \otimes {}^t(\psi_1 B_1)^{-1}\right) = \sum_{j=0}^{k(n-k)} (-u)^j \operatorname{tr} \bigwedge^j (\psi_2 B_2 \otimes {}^t(\psi_1 B_1)^{-1})$$

Next, decompose the exterior powers of the tensor product as

$$\bigwedge^j (\psi_2 B_2 \otimes {}^t(\psi_1 B_1)^{-1}) = \sum_{\lambda : |\lambda| = j} \rho_{\lambda'}(\psi_2 B_2) \otimes \rho_\lambda({}^t(\psi_1 B_1)^{-1}), \tag{5.3}$$

where

- the sum is over all partitions λ of j with at most k rows and $n-k$ columns,
- λ' is the partition conjugate to λ and
- $\rho_\lambda, \rho_{\lambda'}$ are the irreducible representations of $\mathrm{GL}_k(\mathbb{R})$ and $\mathrm{GL}_{n-k}(\mathbb{R})$ associated to the partitions λ, λ', respectively.

See, for example, Exercise 6.11 of Fulton-Harris [FH91]. Since the trace of a tensor product of two matrices is the product of the two traces, we may write

$$\det\left(\mathrm{Id} - u(\psi_2 B_2) \otimes {}^t(\psi_1 B_1)^{-1}\right) = \sum_{j=0}^{k(n-k)} (-u)^j \sum_{|\lambda| = j} \operatorname{tr} \rho_{\lambda'}(\psi_2 B_2) \cdot \operatorname{tr} \rho_\lambda(\psi_1 B_1).$$

Integrating over $\mathrm{O}_k(\mathbb{R}) \times \mathrm{O}_{n-k}(\mathbb{R})$ we find that $\mathcal{J}(A,B;u)$ is equal to

(5.4)
$$\sum_{j=0}^{k(n-k)} (-u)^j \sum_{|\lambda|=j} \left(\int_{\psi_2 \in \mathrm{O}_{n-k}(\mathbb{R})} \operatorname{tr} \rho_{\lambda'}(\psi_2 B_2)\, d\psi_2 \right) \cdot \left(\int_{\psi_1 \in \mathrm{O}_k(\mathbb{R})} \operatorname{tr} \rho_{\lambda}(\psi_1 B_1)\, d\psi_1 \right).$$
$$= 1 + \sum_{\substack{1 \le j \le k(n-k) \\ |\lambda|=j}} (-u)^j F_{\lambda'}(B_2) F_{\lambda}(B_1),$$

where for $M \in \mathrm{GL}_N(\mathbb{R})$ and μ a partition of j with at most N parts, we define

$$F_\mu(M) = \int_{\psi \in \mathrm{O}_N(\mathbb{R})} \operatorname{tr} \rho_\mu(\psi M)\, d\psi \tag{5.5}$$

Theorem 4 follows from the following more explicit result.

Theorem 5. *Let $M \in \mathrm{GL}_N(\mathbb{R})$ and $\mu = (\mu_1, \ldots, \mu_r)$ with $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_r > 0$ be a partition of k of at most N parts.*

1. *If any of the parts μ_i is odd, then $F_\mu(M) = 0$.*
2. *If all the parts μ_i are even, then $F_\mu(M)$ is an even polynomial in the singular values of M with positive coefficients.*

5.2. **Spherical polynomials.** The proof of Theorem 5 involves the theory of spherical polynomials for the symmetric space G/K where $G = \mathrm{GL}_N(\mathbb{R})$ and $K = \mathrm{O}_N(\mathbb{R})$, and Jack polynomials. We recall these briefly.

Let $\mathcal{P}_N$ be the set of partitions with at most N parts, thus

$$\mathcal{P}_N = \{\mu \in \mathbb{Z}^N \mid \mu_1 \ge \mu_2 \ge \cdots \ \ge \mu_N \ge 0\}.$$

For $\mu \in \mathcal{P}_N$, let (ρ_μ, V_μ) be the corresponding representation of G, and let V_μ^* be the contragredient representation. A matrix coefficient of V_μ is a function on G of the form

$$\phi_{u,v}(M) = \langle u, \rho_\mu(M) v \rangle,$$

where $u \in V_\mu^*$ and $v \in V_\mu$. We write $\mathcal{F}_\mu$ for the span of matrix coefficients of V_μ. Then $\mathcal{F}_\mu$ is stable under left and right multiplication by G, and one has a $G \times G$-module isomorphism

$$V_\mu^* \otimes V_\mu \approx \mathcal{F}_\mu, \quad u \otimes v \mapsto \phi_{u,v}.$$

Theorem 6. *Let μ be a partition in $\mathcal{P}_N$. Then the following are equivalent*

1. *μ is even, that is, $\mu_i \in 2\mathbb{Z}$ for all i.*
2. *V_μ has a spherical vector, that is, a vector fixed by K.*
3. *V_μ^* has a spherical vector.*
4. *$\mathcal{F}_\mu$ contains a spherical polynomial ϕ_μ, that is, a function satisfying*
$$\phi_\mu(kgk') = \phi_\mu(g), \ g \in G, \ k, k' \in K.$$

The spherical vector v_μ and spherical polynomial ϕ_μ are unique up to scalar multiple, and the latter is usually normalized by the requirement $\phi_\mu(e) = 1$, which fixes it uniquely.

Proof. This follows from the Cartan-Helgason theory of spherical representations [Hel84, Theorem V.4.1]. ☐

We now connect the polynomial F_μ to ϕ_μ.

Theorem 7. *Let $F_\mu(M)$ be as in (5.5). If μ is even then $F_\mu = \phi_\mu$, otherwise $F_\mu = 0$.*

Proof. If $\{v_i\}, \{u_i\}$ are dual bases for V_λ, V_λ^* then $\operatorname{tr}\rho_\mu(M) = \sum_i \phi_{u_i,v_i}(M)$, thus the character $\chi_\mu(M) = \operatorname{tr}\rho_\mu(M)$ is an element of $\mathcal{F}_\mu$. Since $\mathcal{F}_\mu$ is stable under the left action of K, it follows that $F_\mu(M) = \int_K \chi_\mu(kM)\,dk$ is in $\mathcal{F}_\mu$ as well.

We next argue that F_μ is $K\times K$ invariant. For this we compute as follows:

$$F_\mu(k_1Mk_2) = \int_K \chi_\mu(kk_1Mk_2)\,dk = \int_K \chi_\mu(k_2kk_1M)\,dk = F_\mu(M)$$

Here the first equality holds by definition, the second is a consequence the invariance of the trace character – $\chi_\mu(AB) = \chi_\mu(BA)$, and the final equality follows from the $K\times K$ invariance of the Haar measure dk.

By Theorem 6 this proves that F_μ is a multiple of ϕ_μ if μ is even, and $F_\mu = 0$ otherwise. To determine the precise multiple we need to compute the following integral for even μ :

$$F_\mu(e) = \int_K \chi_\mu(k)\,dk.$$

By Schur orthogonality, this integral is the multiplicity of the trivial representation in the restriction of V_μ to K, which is 1 if μ is even. Thus we get $F_\mu = \phi_\mu$, as desired. ☐

5.3. **Jack polynomials.** Jack polynomials $J_\lambda^{(\alpha)}(x_1,\dots,x_N)$ are a family of symmetric polynomials in N variables whose coefficients depend on a parameter α. The main result of [KS97] is that these coefficients are themselves positive integral polynomials in the parameter α.

Spherical functions correspond to Jack polynomials with $\alpha = 2$. More precisely, we have

$$\phi_\mu(g) = \frac{J_\lambda^{(2)}(a_1,\dots,a_N)}{J_\lambda^{(2)}(1,\dots,1)}, \quad \mu = 2\lambda \tag{5.6}$$

where $a_1,\dots,a_N$ are the eigenvalues of the symmetric matrix g^Tg; in other words, the a_i are the squares of the singular values of g,

We can now finish the proof of Theorem 5.

Proof of Theorem 5. Part (1) follows from Theorem 7. Part (2) follows from formula (5.6) and the positivity of Jack polynomials as proved in [KS97] ☐

5.4. **Examples.** We conclude this section with two low rank examples of the characteristic polynomials $\mathcal{J}(A,B;u)$ for $A\in\mathrm{GL}_k(\mathbb{R}), B\in\mathrm{GL}_{n-k}(\mathbb{R})$. As we may assume A and B are diagonal, let us write

$$A = \operatorname{diag}(a_1,\dots,a_k) \text{ and } B = \operatorname{diag}(b_1,\dots,b_k).$$

The case $n=4, k=2$. Here we consider the integral

$$\mathcal{J}(A,B;u) = \int_{\psi_1\in\mathrm{O}_2(\mathbb{R})}\int_{\psi_2\in\mathrm{O}_2(\mathbb{R})} \det\left(\mathrm{Id} - u(\psi_2B)\otimes {}^t(\psi_1A)^{-1}\right)\,d\psi_2\,d\psi_1. \tag{5.7}$$

As we are essentially integrating over the circle, it is easy to compute this directly and see that

$$\mathcal{J}(A,B;u) = 1 + \frac{\det(B)^2}{\det(A)^2}u^4 = 1 + \frac{b_1^2b_2^2}{a_1^2a_2^2}u^4. \tag{5.8}$$

The case $n = 6, k = 2$. In this case we use (5.4) to compute

$$\mathcal{J}(A, B; u) = \int_{\psi_1 \in \mathrm{O}_2(\mathbb{R})} \int_{\psi_2 \in \mathrm{O}_4(\mathbb{R})} \det \left(\mathrm{Id} - u(\psi_2 B) \otimes {}^t(\psi_1 A)^{-1} \right) \, d\psi_2 \, d\psi_1$$

for $A \in \mathrm{GL}_2(\mathbb{R}), B \in \mathrm{GL}_4(\mathbb{R})$. Write

$$\mathcal{J}(A, B; u) = 1 + c_2 u^2 + c_4 u^4 + c_6 u^6 + c_8 u^8.$$

By part 1 of Theorem 5 we immediately see that $c_2 = c_6 = 0$ because there are no partitions λ of 2 or 6 for which both λ and its conjugate λ' have only even parts. The only even partition of $k = 8$ with at most 2 parts and with even conjugate is $\lambda = (4, 4)$. For V the standard two-dimensional representation of $\mathrm{GL}_2(\mathbb{R})$, we have that $\rho_\lambda(V) = \mathrm{sym}^4(\Lambda^2 V)$ is the fourth power of the determinant representation. Hence

$$F_\lambda(A^{-1}) = \det A^{-4}.$$

Similarly for W the standard four-dimensional representation of $\mathrm{GL}_4(\mathbb{R})$, the conjugate $\lambda' = (2, 2, 2, 2)$ and $\rho_{\lambda'}(W) = \mathrm{sym}^2(\Lambda^4(W))$ is the square of the determinant. Hence $F_{\lambda'}(B) = \det B^2$ and

$$c_8 = \frac{\det(B)^2}{\det(A)^4}.$$

The only even partition of $k = 4$ with even conjugate is $\lambda = \lambda' = (2, 2)$. In this case $\rho_\lambda(V)$ is the square of the determinant representation. The dimension 20 representation $\rho_\lambda(W)$ is a quotient of $\mathrm{sym}^2(\Lambda^2(W))$ with a unique $\mathrm{O}_4(\mathbb{R})$-fixed vector, namely, the image of

$$v = (e_1 \wedge e_2)^2 + (e_1 \wedge e_3)^2 + (e_1 \wedge e_4)^2 + (e_2 \wedge e_3)^2 + (e_2 \wedge e_4)^2 + (e_3 \wedge e_4)^2.$$

It is readily seen that the trace $\rho_\lambda(B)$ restricted to the span of v is

$$\sum_{1 \le i < j \le 4} b_i^2 b_j^2.$$

Then, including the normalizing factor of $1/J^{(2)}_{(1,1)}(1, 1, 1, 1) = 1/6$ we conclude that

$$c_4 = \frac{\frac{1}{6}(b_1^2 b_2^2 + b_1^2 b_3^2 + b_1^2 b_4^2 + b_2^2 b_3^2 + b_2^2 b_4^2 + b_3^2 b_4^2)}{a_1^2 a_2^2}.$$

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