MINIMIZING THE DISCRETE LOGARITHMIC ENERGY ON THE SPHERE: THE ROLE OF RANDOM POLYNOMIALS

DIEGO ARMENTANO, CARLOS BELTRÁN, AND MICHAEL SHUB

Abstract. We prove that points in the sphere associated with roots of random polynomials via the stereographic projection, are surprisignly well-suited with respect to the minimal logarithmic energy on the sphere. That is, roots of random polynomials provide a fairly good approximation to Elliptic Fekete points.

This paper deals with the problem of distributing points in the 2-dimensional sphere, in a way that the logarithmic energy is minimized. More precisely, let $x_1, \ldots, x_N \in \mathbb{R}^3$, and let

$$V(x_1, \ldots, x_N) = \ln \prod_{1 \le i < j \le N} \frac{1}{\|x_i - x_j\|} = - \sum_{1 \le i < j \le N} \ln \|x_i - x_j\| \tag{0.1}$$

be the logarithmic energy of the N-tuple $x_1, \ldots, x_N$. Here, $\| \cdot \|$ is the Euclidean norm in $\mathbb{R}^3$. Let

$$V_N = \min_{x_1, \ldots, x_N \in \mathbb{S}^2} V(x_1, \ldots, x_N)$$

denote the minimum of this function when the x_k are allowed to move in the unit sphere $\mathbb{S}^2 = \{x \in \mathbb{R}^3 : \|x\| = 1\}$. We are interested in N-tuples minimizing the quantity (0.1). These optimal N-tuples are usually called Elliptic Fekete Points. This problem has attracted much attention during the last years. The reader may find background in [6, 7] and references therein. It is considered an example of highly non-trivial optimization problem. In the list of Smale's problems for the XXI Century [12], problem number 7 reads

Problem 1. Can one find $x_1, \ldots, x_N \in \mathbb{S}^2$ such that

$$V(x_1, \ldots, x_N) - V_N \le c \ln N, \tag{0.2}$$

c a universal constant?

More precisely, Smale demands a real number algorithm in the sense of [5] that with input N returns a N-tuple $x_1, \ldots, x_N$ satisfying equation (0.2), and such that the running time is polynomial on N.

One of the main difficulties when dealing with problem 1 is that the value of V_N is not completely known. To our knowledge, the most precise result is the following, proved in [7, Th. 3.1 and Th. 3.2].

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Theorem 0.1. *Defining C_N by*

$$V_N = -\frac{N^2}{4}\ln\left(\frac{4}{e}\right) - \frac{N\ln N}{4} + C_N N,$$

we have

$$-0.112768770... \leq \liminf_{N\to\infty} C_N \leq \limsup_{N\to\infty} C_N \leq -0.0234973...$$

Thus, the value of V_N is not even known up to logarithmic precission, as required by equation (0.2).

The lower bound of Theorem 0.1 is obtained by algebraic manipulation of the formula for $V(x_1,\ldots,x_N)$, and the upper bound is obtained by the explicit (and difficult) construction of N-tuples $x_1,\ldots,x_N$ at which V attains small values.

In this paper we choose a completely different approach to this problem. First, assume that $y_1,\ldots,y_N$ are chosen randomly and independently on the sphere, with the uniform distribution. One can easily show that the expected value of the function $V(y_1,\ldots,y_N)$ in this case is,

$$\mathbb{E}(V(y_1,\ldots,y_N)) = -\frac{N^2}{4}\ln\left(\frac{4}{e}\right) + \frac{N}{4}\ln\left(\frac{4}{e}\right). \tag{0.3}$$

Thus, a random choice of points in the sphere with the uniform distribution already provides a reasonable approach to the minimal value V_N, accurate to the order of $O(N\ln N)$. It is a natural question whether other handy probability distributions, i.e. different from the uniform distribution in $(\mathbb{S}^2)^N$, may yield better expected values. We will give a partial answer to this question in the framework of random polynomials.

Part of the motivation of problem 1 is the search for a polynomial all of whose roots are well conditioned, in the context of [11]. On the other hand, roots of random polynomials are known to be well conditioned, for a sensible choice of the random distribution of the polynomial (see [10]). We make this connection more precise in the historical note at the end of the Introduction. This idea motivates the following approach:

Let f be a degree N polynomial. Let $z_1,\ldots,z_N\in\mathbb{C}$ be its complex roots. Let $z_k = u_k + iv_k$ and let

$$\hat{z}_k = \frac{(u_k, v_k, 1)}{1+u_k^2+v_k^2} \in \{x\in\mathbb{R}^3 : \|x-(0,0,1/2)\| = 1/2\}, \quad 1\leq k\leq N, \tag{0.4}$$

be the associated points in the Riemann Sphere, i.e. the sphere of diameter 1 centered at $(0,0,1/2)$. Note that the $\hat{z}_k$'s are the inverse image under the stereographic projection of the z_k's, seen as points in the 2-dimensional plane $\{(u,v,1) : u,v\in\mathbb{R}\}$. Finally, let

$$x_k = 2\hat{z}_k - (0,0,1) \in \mathbb{S}^2, \quad 1\leq k\leq N, \tag{0.5}$$

be the associated points in the unit sphere. Note that the $\hat{z}_k, x_k$ depend only on f, so we can consider the two following mappings

$$f\mapsto V(\hat{z}_1,\ldots,\hat{z}_N), \qquad f\mapsto V(x_1,\ldots,x_N).$$

These two mappings are well defined in the sense that they do not depend on the way we choose to order the roots of f. Our main claim is that the points $x_1,\ldots,x_N$ are well-distributed for the function of equation (0.1), if the polynomial f is chosen with a particular distribution. That is, we will prove the following theorem in Section 1.

Theorem 0.2 (Main)**.** *Let $f(X) = \sum_{k=0}^{N} a_k X^k \in \mathcal{P}_N$ be a random polynomial, such that the coefficients a_k are independent complex random variables, such that the real and imaginary parts of a_k are independent (real) Gaussian random variables centered at* 0 *with variance $\binom{N}{k}$. Then, with the notations above,*

$$\mathbb{E}\left(V(\hat{z}_1,\ldots,\hat{z}_N)\right) = \frac{N^2}{4} - \frac{N\ln N}{4} - \frac{N}{4}.$$

$$\mathbb{E}\left(V(x_1,\ldots,x_N)\right) = -\frac{N^2}{4}\ln\left(\frac{4}{e}\right) - \frac{N\ln N}{4} + \frac{N}{4}\ln\frac{4}{e}.$$

By comparison of theorems 0.1 and 0.2 and equation (0.3), we see that the value of $V(x_1,\ldots,x_N)$ is surpringsingly small at points coming from the solution set of random polynomials! In figure 1 below we have plotted (using Matlab) the roots $z_1,\ldots,z_{70}$ and associated points $x_1,\ldots,x_{70}$ of a polynomial of degree 70 chosen randomly.

Equivalently, one can take random homogeneous polynomials (as in the historical note at the end of this introduction) and consider its complex projective solutions, under the identification of $\mathbb{P}(\mathbb{C}^2)$ with the Riemann sphere.

There exist different approaches to the problem of actually producing N-tuples satisfying Inequality (0.2) above (see [13, 7, 4] and references therein), although none of them has been proved to solve problem 1 yet. In [3, 4] numerical experiments were done, designed to find local minima of the function V and involving massive computational effort. The method used there is a descent method which follows a gradient-like vector field. For the initial guess, N points are chosen at random in the unit sphere, with the uniform distribution.

Our Theorem 0.2 above suggests that better-suited initial guesses are those coming from the solution set of random polynomials. More especifically, consider the following numerical procedure:

(1) Guess $a_k \in \mathbb{C}$, $k = 0\ldots N$, complex random variables as in Theorem 0.2
(2) Construct the polynomial $f(X) = \sum_{k=0}^{N} a_k X^k$ and find its N complex solutions $z_1,\ldots,z_N \in \mathbb{C}$.
(3) Construct the associated points in the unit sphere $x_1,\ldots,x_N$ following equations (0.4,0.5).

In view of Theorem 0.2, it seems reasonable for a flow-based search optimization procedure that attempts to compute optimal $x_1,\ldots,x_N$, to start by executing the procedure described above and then following the desired flow. As it is well-known, item (2) of this procedure can only be done approximately. We may perform this task using some homotopy algorithm as the ones suggested in [9, 8, 1] which guarantee average polynomial running time, and produce arbitrarily close approximations to the z_k. In practice, it may be prefereable to construct the companion matrix of f and to compute its eigenvalues with some standard Linear Algebra method.

The choice of the probability distribution for the coefficients of $f(X)$ in Theorem 0.2 is not casual. That probability distribution corresponds to the classical unitarily invariant Hermitian structure in the space of homogeneous polynomials, recalled at the beginning of Section 1 below. This Hermitian structure is called by some authors

Bombieri-Weyl structure, or Kostlan structure, and it is a classical construction with many interesting properties. The reader may see [5] for background.

0.1. **Historical Note.** According to [12], part of the original motivation for problem 1 was the search for well conditioned homogeneous polynomials as in [11]. Given $g = g(X, Y)$ a degree N homogeneous polynomials with unknowns X, Y and complex coefficients, the condition number of g at a projective root $\zeta = (x, y) \in \mathbb{P}(\mathbb{C}^2)$ is defined by

$$\mu(g,\zeta) = N^{1/2}\frac{\|g\|\|\zeta\|^{N-1}}{|Dg(\zeta)\mid_{\zeta^\perp}|},$$

where $\|g\|$ is the Bombieri-Weyl norm of g and $Dg(\zeta)\mid_{\zeta^\perp}$ is the differential mapping of g at ζ, restricted to the complex orthogonal complement of ζ.

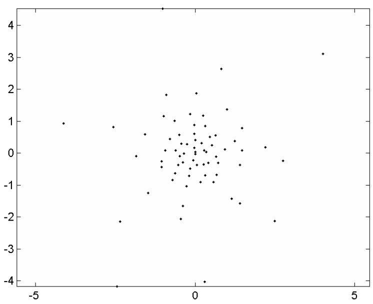
[10] proved that well-conditioned polynomials are highly probable. In [11] the problem was raised as to how to write a deterministic algorithm which produces a polynomial g all of whose roots are well-conditioned. It was also realised that a polynomial whose projective roots (seen as points in the Riemann sphere) have logarithmic energy close to the minimum as in Smale's problem after scaling to $\mathbb{S}^2$, are well conditioned.

From the point of view of [11], the ability to choose points at random already solves the problem. Here, instead of trying to use the logarithmic energy function $V(\cdot)$ to produce well-conditioned polynomials, we use the fact that random polynomials are well-conditioned, to try to produce low-energy N-tuples.

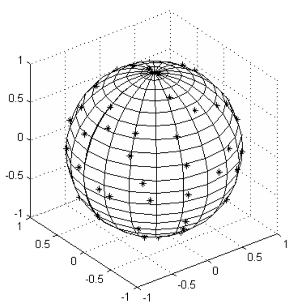
The relation between the condition number and the logarithmic energy is

$$V(\hat{z}_1,\ldots,\hat{z}_N) = \frac{1}{2}\sum_{i=1}^N \ln\mu(f,z_i) + \frac{N}{2}\sum_{i=1}^N \ln\sqrt{1+|z_i|^2} - \frac{N}{2}\ln\|f\| - \frac{N}{4}\ln N,$$

where the roots in $\mathbb{P}(\mathbb{C}^2)$ are $(z_i, 1)$, therefore f is monic.



(a) Complex roots z_k of f



(b) Associated points x_k in $\mathbb{S}^2$

FIGURE 1. The points z_k and x_k for a degree 70 polynomial f chosen at random (using Matlab). The reader may see that the points in the sphere are pretty well distributed.

1. Technical tools and proof of Theorem 0.2

As in the introduction, $f = f(X)$ denotes a polynomial of degree N with complex coefficients, $z_1, \ldots, z_N \in \mathbb{C}$ are the complex roots of f, and $\hat{z}_1, \ldots, \hat{z}_N$ and $x_1, \ldots, x_N$ are the associated points in the Riemann Sphere and $\mathbb{S}^2$ respectively defined by equations (0.4,0.5). Let $\mathcal{P}_N$ be the vector space of degree N polynomials with complex coefficients. As in [5, 2], we consider $\mathcal{P}_N$ endowed with the Bombieri-Weyl inner product, given by

$$\langle \sum_{k=0}^{N} a_k X^k, \sum_{k=0}^{N} b_k X^k \rangle = \sum_{k=0}^{N} \binom{N}{k}^{-1} a_k \overline{b_k}.$$

We denote the associated norm in $\mathcal{P}_N$ simply by $\| \cdot \|$. Let $f(X) = \sum_{k=0}^{N} a_k X^k$ be a random polynomial, where the a_k's complex random variables as in Theorem 0.2. Then, note that the expected value of some measurable function $\phi : \mathcal{P}_N \to \mathbb{R}$ satisfies

$$\mathbb{E}(\phi(f)) = \frac{1}{(2\pi)^{N+1}} \int_{f \in \mathcal{P}_N} \phi(f) e^{-\|f\|^2/2} \, d\mathcal{P}_N. \tag{1.1}$$

Let $W = \{(f, z) \in \mathcal{P}_N \times \mathbb{C} : f(z) = 0\}$ be the so-called solution variety, which is a complex smooth submanifold of $\subseteq \mathcal{P}_N \times \mathbb{C}$ of dimension $N + 1$. For $z \in \mathbb{C}$, let $W_z = \{f \in \mathcal{P}_N : f(z) = 0\}$ be the set of polynomials which have z as a root. We consider W_z endowed with the inner product inherited from $\mathcal{P}_N$.

Proposition 1.

$$V(\hat{z}_1, \ldots, \hat{z}_N) = (N-1) \sum_{i=1}^{N} \ln \sqrt{1 + |z_i|^2} - \frac{1}{2} \sum_{i=1}^{N} \ln |f'(z_i)| + \frac{N}{2} \ln |a_N|,$$

Proof. A simple algebraic manipulation yields

$$V(\hat{z}_1, \ldots, \hat{z}_N) = - \sum_{1 \le i < j \le N} \ln \|\hat{z}_i - \hat{z}_j\| = - \sum_{1 \le i < j \le N} \ln \frac{|z_i - z_j|}{\sqrt{1 + |z_i|^2} \sqrt{1 + |z_j|^2}} =$$

$$(N-1) \sum_{i=1}^{N} \ln \sqrt{1 + |z_i|^2} - \sum_{1 \le i < j \le N} \ln |z_i - z_j|.$$

Note that

$$f(X) = a_N \prod_{i=1}^{N} (X - z_i).$$

Thus,

$$f'(z_i) = a_N \prod_{i \neq j} (z_i - z_j),$$

and

$$|a_N|^N \prod_{i=1}^{N} \frac{1}{|f'(z_i)|} = \prod_{i=1}^{N} \prod_{j \neq i} \frac{1}{|z_i - z_j|} = \prod_{1 \le i < j \le N} \frac{1}{|z_i - z_j|^2}.$$

Thus,

$$- \sum_{1 \le i < j \le N} \ln |z_i - z_j| = \frac{1}{2} \left(- \sum_{i=1}^{N} \ln |f'(z_i)| + N \ln |a_N| \right),$$

and the proposition follows. □

The rest of the proof of Theorem 0.2 will consist on the computation of the expected values of the quantities in Proposition 1. The following lemma will be useful

Lemma 1.1. *For any $t \in \mathbb{R}$,*

$$\sum_{k=0}^{N} \binom{N}{k} t^{2k} = (1+t^2)^N,$$

$$\sum_{k=0}^{N} \binom{N}{k} k t^{2k-1} = Nt(1+t^2)^{N-1},$$

$$\sum_{k=0}^{N} \binom{N}{k} k^2 t^{2k-2} = N(1+t^2)^{N-2}(1+Nt^2).$$

Proof. The first equality is the classical binomial expansion. Differentiate it to get

$$2\sum_{k=0}^{N} \binom{N}{k} k t^{2k-1} = 2Nt(1+t^2)^{N-1},$$

and the second equality follows. Differentiate again to get

$$\sum_{k=0}^{N} \binom{N}{k} (2k^2-k) t^{2k-2} = N(1+t^2)^{N-1} + 2N(N-1)t^2(1+t^2)^{N-2}.$$

Hence,

$$2\sum_{k=0}^{N} \binom{N}{k} k^2 t^{2k-2} = \frac{1}{t}\sum_{k=0}^{N} \binom{N}{k} k t^{2k-1} + N(1+t^2)^{N-1} + 2N(N-1)t^2(1+t^2)^{N-2} =$$

$$N(1+t^2)^{N-1} + N(1+t^2)^{N-1} + 2N(N-1)t^2(1+t^2)^{N-2} = 2N(1+t^2)^{N-2}(1+Nt^2).$$

The last equality of the lemma follows. □

Proposition 2. *Let $\phi : W \to \mathbb{R}$ be a measurable function. Then,*

$$\int_{f\in\mathcal{P}_N} \sum_{z:f(z)=0} \phi(f,z)\, d\mathcal{P}_N = \int_{z\in\mathbb{C}} \frac{1}{(1+|z|^2)^N} \int_{f\in W_z} |f'(z)|^2 \phi(f,z)\, dW_z\, d\mathbb{C} \tag{1.2}$$

Proof. As in [5], we apply the smooth coarea formula to the double fibration

$$\begin{array}{ccc} & W & \\ \swarrow & & \searrow \\ \mathcal{P}_N & & \mathbb{C} \end{array}$$

to get the formula

$$\int_{f\in\mathcal{P}_N} \sum_{z:f(z)=0} \phi(f,z)\, d\mathcal{P}_N = \int_{z\in\mathbb{C}} \int_{f\in W_z} (DG_z(f)DG_z(f)^*)^{-1} \phi(f,z)\, dW_z\, d\mathbb{C},$$

where $G_z : U_f \to U_z$ is the implicit function defined in a neighborhood of f satisfies $g(G_z(g)) = 0$, and $DG_z(f)$ is the Jacobian matrix of G_z at f, writen in some orthonormal basis. By implicit differentiation, $DG_z(f)\dot{f} = -f'(z)^{-1}\dot{f}(z)$. Thus, in the orthonormal basis given by the monomials $\binom{N}{k}^{1/2} X^k$, $k = 0 \dots N$, the jacobian matrix is

$$DG_z(f) = -\frac{1}{f'(z)} \left(\binom{N}{0}^{1/2} z^0, \dots, \binom{N}{N}^{1/2} z^N \right).$$

We conclude that $DG_z(f)DG_z(f)^* = |f'(z)|^{-2} \sum_{k=0}^{N} \binom{N}{k} |z|^{2k} = |f'(z)|^{-2}(1 + |z|^2)^N$. The proposition follows. $\square$

Proposition 3. *Let $z \in \mathbb{C}$ and let $\phi : \mathbb{R} \to \mathbb{R}$ be a measurable function. Then,*

$$\int_{f \in W_z} \phi(|f'(z)|^2) e^{-\|f\|^2/2} \, dW_z = (2\pi)^N \int_0^\infty t\phi \left(t^2 N(1 + |z|^2)^{N-2} \right) e^{-t^2/2} \, dt.$$

Proof. Consider the mapping $\varphi : W_z \to \mathbb{C}$, $f(X) = \sum_{k=0}^{N} a_k X^k \mapsto w = f'(z) = \sum_{k=0}^{N} k a_k z^{k-1}$. Denote by $NJ\varphi(f)$ the Normal Jacobian of φ at f, that is $NJ\varphi(f) = \max_{\dot f \in W_z} \|D\varphi(f)\dot f\|^2$ (see [5, pag. 241] for references and background). Let $g_1, g_2 \in \mathcal{P}_N$ be the following polynomials,

$$g_1(X) = \sum_{k=0}^{N} \binom{N}{k} \overline{z}^k X^k, \quad g_2(X) = \sum_{k=0}^{N} k \binom{N}{k} \overline{z}^{k-1} X^k$$

Note that for any $f \in \mathcal{P}_N$ and $z \in \mathbb{C}$, we have

$$f(z) = \langle f, g_1 \rangle, \quad f'(z) = \langle f, g_2 \rangle.$$

Thus,

$$W_z = \{f \in \mathcal{P}_N : f(z) = 0\} = \{f \in \mathcal{P}_N : \langle f, g_1 \rangle = 0\},$$
$$D\varphi(f)\dot f = \dot f'(z) = \langle \dot f, g_2 \rangle.$$

Thus, if π is the orthogonal projection onto W_z, we have

$$NJ\varphi(f) = \|\pi(g_2)\|^2 = \|g_2\|^2 - \frac{|\langle g_1, g_2 \rangle|^2}{\|g_1\|^2} =$$

$$\sum_{k=0}^{N} \binom{N}{k} k^2 |z|^{2k-2} - \frac{\left(\sum_{k=0}^{N} \binom{N}{k} k |z|^{2k-1} \right)^2}{\sum_{k=0}^{N} \binom{N}{k} |z|^{2k}}.$$

From Lemma 1.1, we conclude

$$NJ\varphi(f) = N(1 + |z|^2)^{N-2}(1 + N|z|^2) - \frac{N^2 |z|^2 (1 + |z|^2)^{2N-2}}{(1 + |z|^2)^N} =$$

$$N(1 + |z|^2)^{N-2}(1 + N|z|^2) - N^2|z|^2(1 + |z|^2)^{N-2} = N(1 + |z|^2)^{N-2}$$

The coarea formula [5] then yields

$$\int_{f \in W_z} \phi(|f'(z)|^2) e^{-\|f\|^2/2} \, dW_z = \tag{1.3}$$

$$\frac{1}{N(1 + |z|^2)^{N-2}} \int_{w \in \mathbb{C}} \phi(|w|^2) \int_{\{f \in W_z : f'(z) = w\}} e^{-\|f\|^2/2} \, df \, d\mathbb{C}.$$

The set $\{f \in W_z : f'(z) = w\}$ is an affine subspace of $\mathcal{P}_N$ of dimension $N - 1$, defined by the equations $\langle f, g_1 \rangle = 0$, $\langle f, g_2 \rangle = w$, which are linear independent equations on the coefficients of f. One can compute the norm of the minimal norm element of this affine subspace using standard tools from Linear Algebra. This minimal norm turns to be equal to $|w|\nu$ where

$$\nu = \frac{1}{\sqrt{\|g_2\|^2 - \frac{|\langle g_1, g_2\rangle|^2}{\|g_1\|^2}}} = \frac{1}{\sqrt{NJ\varphi(f)}} = \frac{1}{\sqrt{N}(1+|z|^2)^{\frac{N-2}{2}}}.$$

Thus,

$$\int_{\{f\in W_z : f'(z)=w\}} e^{-\|f\|^2/2}\, df = (2\pi)^{N-1} \exp\left(-\nu^2|w|^2/2\right),$$

and

$$\int_{w\in\mathbb{C}} \phi(|w|^2) \int_{f\in W_z : f'(z)=w} e^{-\|f\|^2/2}\, df\, d\mathbb{C} = (2\pi)^N \int_0^\infty \rho\phi(\rho^2) e^{-\nu^2\rho^2/2}\, d\rho =$$

$$\frac{(2\pi)^N}{\nu^2} \int_0^\infty t\phi\left(\frac{t^2}{\nu^2}\right) e^{-t^2/2}\, dt = (2\pi)^N N(1+|z|^2)^{N-2} \int_0^\infty t\phi\left(\frac{t^2}{\nu^2}\right) e^{-t^2/2}\, dt.$$

From this and equation (1.3) we conclude,

$$\int_{f\in W_z} \phi(|f'(z)|^2) e^{-\|f\|^2/2}\, dW_z = (2\pi)^N \int_0^\infty t\phi\left(\frac{t^2}{\nu^2}\right) e^{-t^2/2}\, d\rho,$$

as wanted. □

Proposition 4. *Let $f(X) = \sum_{k=0}^N a_k X^k$ where the a_k are as in Theorem 0.2. Then,*

$$\mathbb{E}\left(\sum_{i=1}^N \ln\sqrt{1+|z_i|^2}\right) = \frac{N}{2}. \tag{1.4}$$

$$\mathbb{E}\left(\ln|a_N|\right) = \frac{\ln(2)-\gamma}{2}. \tag{1.5}$$

$$\mathbb{E}\left(\sum_{i=1}^N \ln|f'(z_i)|\right) = \frac{(\ln(2)-1-\gamma+\ln(N)+N)N}{2}. \tag{1.6}$$

Here, $\gamma \sim 0.5772156649$ is Euler's constant.

Proof. From equalities (1.1,1.2),

$$\mathbb{E}\left(\sum_{i=1}^N \ln\sqrt{1+|z_i|^2}\right) = \frac{1}{(2\pi)^{N+1}} \int_{f\in\mathcal{P}_N} \sum_{i=1}^N \ln\sqrt{1+|z_i|^2} e^{-\|f\|^2/2}\, d\mathcal{P}_N =$$

$$\frac{1}{(2\pi)^{N+1}} \int_{z\in\mathbb{C}} \frac{\ln\sqrt{1+|z|^2}}{(1+|z|^2)^N} \int_{f\in W_z} |f'(z)|^2 e^{-\|f\|^2/2}\, dW_z\, d\mathbb{C}.$$

From Proposition 3,

$$\int_{f\in W_z} |f'(z)|^2 e^{-\|f\|^2/2}\, dW_z = (2\pi)^N \int_0^\infty t^3 N(1+|z|^2)^{N-2} e^{-t^2/2}\, dt =$$

$$(2\pi)^N 2N(1+|z|^2)^{N-2}.$$

Thus,

$$\mathbb{E}\left(\sum_{i=1}^N \ln\sqrt{1+|z_i|^2}\right) = \frac{N}{\pi} \int_{z\in\mathbb{C}} \frac{\ln\sqrt{1+|z|^2}}{(1+|z|^2)^2}\, d\mathbb{C} =$$

$$= 2N \int_0^\infty \frac{\rho \ln \sqrt{1+\rho^2}}{(1+\rho^2)^2}\, d\rho = \frac{N}{2},$$

and equation (1.4) follows. Equation (1.5) is trivial, as

$$\mathbb{E}\left(\ln |a_N|\right) = \frac{1}{2\pi}\int_{a\in\mathbb{C}} \ln|a| e^{-|a|^2/2}\, d\mathbb{C} = \int_0^\infty \rho\ln(\rho) e^{-\rho^2/2}\, d\rho = \frac{\ln(2)-\gamma}{2}.$$

Now let us prove equation (1.6). Note that from the equalities (1.1,1.2),

$$\mathbb{E}\left(\sum_{i=1}^N \ln|f'(z_i)|\right) = \frac{1}{(2\pi)^{N+1}}\int_{f\in\mathcal{P}_N} e^{-\|f\|^2/2} \sum_{z\in\mathbb{C}:f(z)=0} \ln|f'(z_i)|\, d\mathcal{P}_N =$$

$$\frac{1}{(2\pi)^{N+1}}\int_{z\in\mathbb{C}} \frac{1}{(1+|z|^2)^N}\int_{f\in W_z} e^{-\|f\|^2/2}|f'(z)|^2 \ln|f'(z)|\, dW_z\, d\mathbb{C} =$$

From Proposition 3, we know that

$$\int_{f\in W_z} |f'(z)|^2 \ln|f'(z)| e^{-\|f\|^2/2}\, dW_z =$$

$$(2\pi)^N \int_0^\infty t\left(t^2 N(1+|z|^2)^{N-2}\right) \ln\sqrt{t^2 N(1+|z|^2)^{N-2}} e^{-t^2/2}\, dt =$$

$$(2\pi)^N N(1+|z|^2)^{N-2} \int_0^\infty t^3\left(\ln t + \ln\sqrt{N(1+|z|^2)^{N-2}}\right) e^{-t^2/2}\, dt =$$

$$(2\pi)^N N(1+|z|^2)^{N-2}\left(1-\gamma+\ln 2 + 2\ln\sqrt{N(1+|z|^2)^{N-2}}\right).$$

Thus,

$$\mathbb{E}\left(\sum_{i=1}^N \ln|f'(z_i)|\right) = \frac{N}{2\pi}\int_{z\in\mathbb{C}} \frac{1-\gamma+\ln 2+\ln(N(1+|z|^2)^{N-2})}{(1+|z|^2)^2}\, d\mathbb{C} =$$

$$N\left(1-\gamma+\ln 2+\ln N\right)\int_0^\infty \frac{\rho}{(1+\rho^2)^2}\, d\rho + N(N-2)\int_0^\infty \frac{\rho\ln(1+\rho^2)}{(1+\rho^2)^2}\, d\rho =$$

$$\frac{N}{2}\left(1-\gamma+\ln 2+\ln N\right) + N\frac{N-2}{2},$$

and equation (1.6) follows.

□

1.1. **Proof of Theorem 0.2.** From Proposition 1,

$$\mathbb{E}\left(V(\hat{z}_1,\ldots,\hat{z}_N)\right) = (N-1)\mathbb{E}\left(\sum_{i=1}^N \ln\sqrt{1+|z_i|^2}\right) - \frac{1}{2}\mathbb{E}\left(\sum_{i=1}^N \ln|f'(z_i)|\right) + \frac{N}{2}\mathbb{E}\left(\ln|a_N|\right),$$

which from Proposition 4 is equal to

$$\frac{N(N-1)}{2} - \frac{(\ln(2)-1-\gamma+\ln(N)+N)N}{4} + \frac{N(\ln(2)-\gamma)}{4},$$

and the first assertion of Theorem 0.2 follows. The second equality of Theorem 0.2 is the trivial, as the affine transformation in $\mathbb{R}^3$ that takes the $\hat{z}_k$'s into the x_k's is a traslation followed by a homotetia of dilation factor 2. Hence,

$$\|x_i - x_j\| = 2\|\hat{z}_i - \hat{z}_j\|, \quad 1\le i<j\le N,$$

and for any choice of $x_1, \dots, x_N$ we have

$$V(x_1, \dots, x_N) = V(\hat{z}_1, \dots, \hat{z}_N) - \frac{N(N-1)}{2} \ln 2.$$

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Centro de Matemática, Universidad de la República. Montevideo, Uruguay
E-mail address: `diego@cmat.edu.uy`

Departmento de Matemáticas, Universidad de Cantabria. Santander, Spain
E-mail address: `beltranc@gmail.com`

Department of Mathematics, University of Toronto. Toronto, Canada
E-mail address: `michael.shub@utoronto.ca`