A note about the average number of real roots of a Bernstein polynomial system

Diego Armentano, Jean-Pierre Dedieu *

DRAFT, March 10, 2009

Abstract

We prove that the real roots of normal random homogeneous polynomial systems with $n+1$ variables and given degrees are, in some sense, equidistributed in the projective space $\mathbb{P}\left(\mathbb{R}^{n+1}\right)$. From this fact we compute the average number of real roots of normal random polynomial systems given in the Bernstein basis.

Keywords: Random polynomial system, real root, Bernstein basis

1 Introduction and main results

Due to a constant interest in CAGD on Bézier curves and Bernstein polynomials the question arises to describe theirs properties in terms of their coefficients when they are given in the Bernstein basis:

$$b_{d,k}(x) = \binom{d}{k} x^k (1-x)^{d-k},\ 0 \le k \le d,$$

in the case of univariate polynomials, and

$$b_{d,\alpha}(x_1, \ldots, x_n) = \binom{d}{\alpha} x_1^{\alpha_1} \ldots x_n^{\alpha_n} (1 - x_1 - \ldots - x_n)^{d-|\alpha|},\ |\alpha| \le d,$$

for polynomials in n variables. Here $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-integer, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, and

$$\binom{d}{\alpha} = \frac{d!}{\alpha_1! \ldots \alpha_n! (d - |\alpha|)!}$$

is the multinomial coefficient.

In this note we are interested in the average number of real roots of such equations or systems of equations when the coefficients are taken at random.

*Institut de Mathématiques, Université Paul Sabatier, 31062 Toulouse cedex 09, France (diego.armentano@math.univ-toulouse.fr, jean-pierre.dedieu@math.univ-toulouse.fr).
J.-P. Dedieu was supported by the ANR Gecko.

Let us denote by $\mathcal{P}_{(d)}$ the set of real polynomial systems in n variables, $F = (F_i)$, $1 \le i \le n$, where

$$F_i(x_1, \ldots, x_n) = \sum_{|\alpha| \le d_i} a_\alpha^{(i)} x_1^{\alpha_1} \ldots x_n^{\alpha_n} \left(1 - x_1 - \ldots - x_n\right)^{d_i - |\alpha|}.$$

Here $(d) = (d_1, \ldots, d_n)$ denotes the vector of degrees, $d_i \ge 1$, and $\deg f_i \le d_i$ for every i. The space $\mathcal{P}_{(d)}$ is equipped with the Euclidean structure defined by the norm

$$\|F\|^2 = \sum_{i=1}^{n} \sum_{|\alpha| \le d_i} \binom{d_i}{\alpha}^{-1} \left| a_\alpha^{(i)} \right|^2,$$

and the corresponding probability measure dF. In other words, the coefficients $a_\alpha^{(i)}$ of a polynomial system $F \in \mathcal{P}_{(d)}$ are independent normal random variables with mean equal to 0 and variances $\binom{d_i}{\alpha}$.

Define

$$\tau : \mathbb{R}^n \to \mathbb{P}\left(\mathbb{R}^{n+1}\right)$$

by

$$\tau(x_1, \ldots, x_n) = [x_1, \ldots, x_n, 1 - x_1 - \ldots - x_n].$$

Here $\mathbb{P}\left(\mathbb{R}^{n+1}\right)$ is the projective space associated with $\mathbb{R}^{n+1}$, $[y]$ is the class of the vector $y \in \mathbb{R}^{n+1}$, $y \neq 0$, for the equivalence relation defining this projective space. The (unique) orthogonally invariant probability measure in $\mathbb{P}\left(\mathbb{R}^{n+1}\right)$ is denoted by λ_n.

For any measurable set B in $\mathbb{R}^n$ we let $N_B(F)$ the number of roots of F lying in B, and by $\mathbb{E}\left(N_B(F)\right)$ the average number of $N_B(F)$ for $F \in \mathcal{P}_{(d)}$.

Theorem 1. *1. For any measurable set B in $\mathbb{R}^n$ we have*

$$\mathbb{E}\left(N_B(F)\right) = \lambda_n(\tau(B))\sqrt{d_1 \ldots d_n}.$$

In particular

2. $\mathbb{E}\left(N_{\mathbb{R}^n}(F)\right) = \sqrt{d_1 \ldots d_n}$,

3. $\mathbb{E}\left(N_{S_n}(F)\right) = \sqrt{d_1 \ldots d_n}/2^n$, *where*

$$S_n = \left\{x \in \mathbb{R}^n \ : \ x_i \ge 0 \text{ and } x_1 + \ldots + x_n \le 1\right\},$$

4. When $n = 1$, for any interval $I = [\alpha, \beta] \subset \mathbb{R}$, one has

$$\mathbb{E}\left(N_I(F)\right) = \frac{\sqrt{d}}{\pi}\left(\arctan(2\beta - 1) - \arctan(2\alpha - 1)\right).$$

This theorem is easily deduced from the next one which has its own interest and which is a consequence of Shub-Smale [10]. The fourth assertion in theorem 1 is deduced from the first assertion but it also can be derived from Crofton's formula like in Edelman-Kostlan [5].

Let us denote by $\mathcal{H}_{(d)}$ the space of real homogeneous polynomial systems in $n+1$ variables, $\mathcal{F} = (\mathcal{F}_i)$, $1 \le i \le n$, where

$$\mathcal{F}_i(x_1, \ldots, x_n, x_{n+1}) = \sum_{|\alpha| \le d_i} a_\alpha^{(i)} x_1^{\alpha_1} \ldots x_n^{\alpha_n} x_{n+1}^{d_i - |\alpha|}.$$

$(d) = (d_1, \ldots, d_n)$ denotes the vector of degrees, $d_i \ge 1$, and $\deg \mathcal{F}_i = d_i$ for every i. The space $\mathcal{H}_{(d)}$ is equipped with the Euclidean structure defined by the norm

$$\|\mathcal{F}\|^2 = \sum_{i=1}^{n} \sum_{|\alpha| \le d_i} \binom{d_i}{\alpha}^{-1} \left| a_\alpha^{(i)} \right|^2,$$

and the corresponding probability measure $d\mathcal{F}$.

The real roots of such a system consist in lines through the origin in $\mathbb{R}^{n+1}$ which are identified to points in $\mathbb{P}\left(\mathbb{R}^{n+1}\right)$. For any measurable set $\mathcal{B} \subset \mathbb{P}\left(\mathbb{R}^{n+1}\right)$ we denote by $N_{\mathcal{B}}(\mathcal{F})$ the number of roots of $\mathcal{F}$ lying in $\mathcal{B}$, and by $\mathbb{E}\left(N_{\mathcal{B}}(\mathcal{F})\right)$ the average number of $N_{\mathcal{B}}(\mathcal{F})$ for $\mathcal{F} \in \mathcal{H}_{(d)}$.

Theorem 2. *For any measurable set $\mathcal{B} \subset \mathbb{P}\left(\mathbb{R}^{n+1}\right)$ we have*

$$\mathbb{E}\left(N_{\mathcal{B}}(\mathcal{F})\right) = \lambda_n(\mathcal{B}) \sqrt{d_1 \ldots d_n}.$$

The first result about the average number of real roots of polynomial equations is due to Kac [6], [7], who gives the asymptotic value

$$\mathbb{E}\left(N_{\mathbb{R}}(F)\right) = \frac{2}{\pi} \log d$$

as d tends to infinity when the coefficients of the degree d univariate polynomial F in the basis of monomials are Gaussian centered independent random variables $N(0,1)$. But, when the variance of the $k-$th coefficient in the basis of monomials is equal to $\binom{d}{k}$ (Weyl's distribution), the average number is equal to

$$\mathbb{E}\left(N_{\mathbb{R}}(F)\right) = \sqrt{d}$$

like in the case of Bernstein polynomials (see Bogomolny-Bohias-Leboeuf [4] and also Edelman-Kostlan [5]).

Other results of the same vein have been obtained by Shub-Smale [10] who considered the case of homogeneous polynomial systems under Weyl's distribution and Rojas [9] for sparse systems. A general formula for $\mathbb{E}\left(N_B(F)\right)$ when the random functions F_i, $i = 1, \ldots, n$, are stochastically independent and their law is centered and invariant under the orthogonal group can be found in Azaïs-Wschebor [2], which includes the Shub-Smale result as a special case. The non-centered case is considered in Armentano-Wschebor [1].

2 Proof of theorem 2

For any measurable set $\mathcal{B} \subset \mathbb{P}\left(\mathbb{R}^{n+1}\right)$ let us define

$$\mu_n(\mathcal{B}) = \mathbb{E}\left(N_{\mathcal{B}}(\mathcal{F})\right).$$

We see that μ_n is an orthogonally invariant measure in $\mathbb{P}\left(\mathbb{R}^{n+1}\right)$. Thus it is equal to λ_n up to a multiplicative factor. This factor is equal to $\sqrt{d_1 \dots d_n}$ as it is easily seen from Shub-Smale [10] (see also [3] section 13.3). Therefore

$$\mathbb{E}\left(N_{\mathcal{B}}(\mathcal{F})\right) = \lambda_n(\mathcal{B})\sqrt{d_1 \dots d_n}.$$

3 Proof of theorem 1

Let us prove the first item. For any measurable set $B \subset \mathbb{R}^n$ we have by theorem 2 applied to $\mathcal{B} = \tau(B)$

$$\lambda_n(\tau(B))\sqrt{d_1 \dots d_n} = \mathbb{E}\left(N_{\tau(B)}(\mathcal{F})\right) = \int_{\mathcal{H}_{(d)}} N_{\tau(B)}(\mathcal{F}) d\mathcal{F}.$$

The map h which associates to $F \in \mathcal{P}_{(d)}$ the homogeneous system $\mathcal{F} \in \mathcal{H}_{(d)}$ obtained in substituing x_{n+1} to the affine form $(1 - x_1 - \dots - x_n)$ is an isometry between these two spaces so that

$$\int_{\mathcal{H}_{(d)}} N_{\tau(B)}(\mathcal{F}) d\mathcal{F} = \int_{\mathcal{P}_{(d)}} N_{\tau(B)}(h(F)) dF.$$

Since $N_{\tau(B)}(h(F)) = N_B(F)$ this last integral is equal to $\int_{\mathcal{P}_{(d)}} N_B(F) dF$.

To complete the proof of this theorem we notice that $\lambda_n(\tau(\mathbb{R}^n)) = 1$, $\lambda_n(\tau(S_n)) = 1/2^n$, and,

$$\lambda_1(\tau([\alpha, \beta])) = \frac{1}{\pi} \int_\alpha^\beta \frac{1}{t^2 + (1-t)^2}\, dt = \frac{\arctan(2\beta - 1) - \arctan(2\alpha - 1)}{\pi},$$

which follows from the computation of the length of the path $\{\tau(t)\}_{t \in [\alpha, \beta]} \subset \mathbb{P}(\mathbb{R})$.

References

[1] Armentano D., and M. Wschebor, *Random systems of polynomial equations. The expected number of roots under smooth analysis.* Bernoulli 15, No. 1, (2009), 249-266.

[2] Azaïs J.-M., and M. Wschebor, *On the roots of a random system of equations. The theorem of Shub and Smale and some extensions.* Fondations of Computational Mathematics (2005) 125-144.

[3] Blum, L., F. Cucker, M. Shub, and S. Smale, *Complexity and Real Computation*, Springer, 1998.

[4] Bogomolny E., O. Bohias, and P. Leboeuf, *Distribution of roots of random polynomials.* Phys. Rev. Letters, 68 (1992) 2726-2729.

[5] Edelman A., and E. Kostlan, *How many zeros of a random polynomial are real?* Bulletin of the AMS, 32 (1995) 1-37 and 33 (1996) 325.

[6] Kac M., *On the average number of real roots of a random algebraic equation.* Bull. Am. Math. Soc. 49 (1943) 314-320 and 938.

[7] Kac M., *On the average number of real roots of a random algebraic equation (II).* Proc. London Math. Soc. 50 (1949) 390-408.

[8] Kostlan E., *On the expected number of real roots of a system of radom polynomial equations.* In: Foundations of Computational Mathematics, Hong Kong 2002, 149-188. World Sci. Pub., 2002.

[9] Rojas M., *On the Average Number of Real Roots of Certain Random Sparse Polynomial Systems.* In: The Mathematics of Numerical Analysis. Edited by: James Renegar, Michael Shub, and Steve Smale. Lectures in Applied Mathematics. 32 (1996).

[10] Shub, M., and S. Smale, *Complexity of Bézout's Theorem II: Volumes and Probabilities* in: *Computational Algebraic Geometry,* F. Eyssette and A. Galligo eds., em Progress in Mathematics, vol. 109, Birkhäuser, 1993, 267-285.