Central Limit Theorem for the volume of the zero set of Kostlan-Shub-Smale random polynomial systems.

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Abstract

We state the Central Limit Theorem, as the degree goes to infinity, for the normalized volume of the zero set of a rectangular Kostlan-Shub-Smale random polynomial system. This paper is a continuation of *Central Limit Theorem for the number of real roots of Kostlan Shub Smale random polynomial systems* by the same authors in which the square case was considered. Our main tools are the Kac-Rice formula for the second moment of the volume of the zero set and an expansion of this random variable into the Itô-Wiener Chaos.
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1 Main result

The present paper extends the results in [2] to the case of rectangular systems. The proof follows the same lines.

Let $r < m$ and consider a rectangular system $\mathbf{Y}_d$ of r homogeneous polynomial equations in $m+1$ variables with common degree $d > 1$. More precisely, let $\mathbf{Y}_d = (Y_1, \dots, Y_r)$ with

$$Y_\ell(t) = \sum_{|\boldsymbol{j}|=d} a_{\boldsymbol{j}}^{(\ell)} t^{\boldsymbol{j}}; \quad \ell = 1, \dots, r,$$

where

1. $\boldsymbol{j} = (j_0, \dots, j_m) \in \mathbb{N}^{m+1}$ and $|\boldsymbol{j}| = \sum_{k=0}^m j_k$;
2. $a_{\boldsymbol{j}}^{(\ell)} = a_{j_0 \cdots j_m}^{(\ell)} \in \mathbb{R}$, $\ell = 1, \dots, r$, $|\boldsymbol{j}| = d$;
3. $t = (t_0, \dots, t_m) \in \mathbb{R}^{m+1}$ and $t^{\boldsymbol{j}} = \prod_{k=0}^m t_k^{j_k}$.

We say that $\mathbf{Y}_d$ has the Kostlan-Shub-Smale (KSS for short) distribution if the coefficients $a_{\boldsymbol{j}}^{(\ell)}$ are independent centered normally distributed random variables with variances

$$\operatorname{Var}\left(a_{\boldsymbol{j}}^{(\ell)}\right) = \binom{d}{\boldsymbol{j}} = \frac{d!}{j_0! j_1! \dots j_m!}.$$

We are interested in the zero level set of $\mathbf{Y}_d$. Since $\mathbf{Y}_d$ is homogeneous (for $\lambda \in \mathbb{R}$ it verifies $\mathbf{Y}_d(\lambda t) = \lambda^d \mathbf{Y}_d(t)$) its roots consist of lines through 0 in $\mathbb{R}^{m+1}$. Then, each root ray of $\mathbf{Y}_d$ in $\mathbb{R}^{m+1}$ corresponds exactly to two (opposite) roots of $\mathbf{Y}$ on the unit sphere S^m of $\mathbb{R}^{m+1}$. Hence, the natural place where to consider the zero set is S^m (or the projective space $\mathbb{P}(\mathbb{R}^m)$).

By Sard-type argument the zero level set of $\mathbf{Y}_d$ on S^m is, almost surely, a smooth submanifold of dimension $m-r$ (see for example Azaïs & Wschebor [4]). We denote by $\mathcal{V}_{\mathbf{Y}_d}(\mathbf{0})$ the $(m-r)$-volume of the zero level set (on the sphere).

Shub and Smale [11] and Kostlan [6] proved that $\mathbb{E}\left(\mathcal{V}_{\mathbf{Y}_d}(\mathbf{0})\right) = 2d^{r/2} c_{m,r}$, $r \le m$, where $c_{m,r}$ is the geometric measure of the sphere S^{m-r} as a submanifold of S^m. Letendre [7] proved that there exists $0 < V_\infty^r < \infty$ such that

$$\lim_{d \to \infty} \frac{\mathcal{V}_{\mathbf{Y}_d}(\mathbf{0})}{d^{r-m/2}} = V_\infty^r. \tag{1.1}$$

We now establish the CLT for the rectangular case.

Theorem 1. *Let $\mathbf{Y}_d$ be an $r \times (m+1)$ KSS homogeneous system. Then, if $r < m$, the standardized $(m-r)$-volume of the zero level set*

$$\bar{\mathcal{V}}_d = \frac{\mathcal{V}_{\mathbf{Y}_d}(\mathbf{0}) - \mathbb{E}\left[\mathcal{V}_{\mathbf{Y}_d}(\mathbf{0})\right]}{d^{\frac{r}{2} - \frac{m}{4}}}$$

converges in distribution as $d \to \infty$ towards a normal random variable with positive variance.

2 Outline of the proof

The proof follows the same lines as that of the square case in [2], but some technical points must be adapted. The main steps of the proof of Theorem 1 are the following.

- We expand the standardized $(m-r)$-volume of the zero level set of $\mathbf{Y}_d$ on S^m in the L^2 sense (chaotic expansion).
- Taking advantage of the structure of chaotic random variables, the CLT is easily obtained for each term in the expansion as well as for any finite sum of them.
- We use a convenient partition of the sphere and the existence of a local limit process in order to prove the negligibility (of the variance) of the tail of the expansion.

3 Preliminaries

Notation: We denote by S^m the unit sphere in $\mathbb{R}^{m+1}$ and its volume by κ_m. The variables s and t denote points on S^m and ds and dt denote the corresponding geometric measure. The variables u and v are in $\mathbb{R}^m$ and du and dv are the associated Lebesgue measure. The variables z and θ are reals and dz and $d\theta$ are the associated differentials.

As usual we use the Landau's big O and small o notation. The set $\mathbb{N}$ of natural numbers contains 0. Also, Const will denote a universal constant that might change from a line to another.

Hyperspherical coordinates: For $\theta = (\theta_1, \dots, \theta_{m-1}, \theta_m) \in [0,\pi)^{m-1} \times [0, 2\pi)$ we write $x^{(m)}(\theta) = (x_1^{(m)}(\theta), \dots, x_{m+1}^{(m)}(\theta)) \in S^m$ in the following way

$$x_k^{(m)}(\theta) = \prod_{j=1}^{k-1} \sin(\theta_j) \cdot \cos(\theta_k),\ k \le m \text{ and } x_{m+1}^{(m)}(\theta) = \prod_{j=1}^{m} \sin(\theta_j); \tag{3.1}$$

with the convention that $\prod_1^0 = 1$.

We use repeatedly in the sequel that for $h : [-1,1] \to \mathbb{R}$ it holds that

$$\int_{S^m \times S^m} h(\langle s,t \rangle) ds dt = \kappa_m \kappa_{m-1} \int_0^\pi \sin^{m-1}(\theta) h(\cos(\theta)) d\theta, \tag{3.2}$$

being $\langle \cdot, \cdot \rangle$ the usual inner product in $\mathbb{R}^{m+1}$.

Covariances: Direct computation yields

$$r_d(s,t) := \mathbb{E}\left(Y_\ell(s) Y_\ell(t)\right) = \langle s,t \rangle^d; \quad s,t \in \mathbb{R}^{m+1}.$$

As a consequence, the distribution of the system $\mathbf{Y}_d$ is invariant under the action of the orthogonal group in $\mathbb{R}^{m+1}$.

For $\ell = 1, \dots, r$, we denote by $Y'_\ell(t)$ the derivative (along the sphere) of $Y_\ell(t)$ at the point $t \in S^m$ and by $Y'_{\ell k}$ its k-th component on a given basis of the tangent space of S^m at the point t. We define the standardized derivative as

$$\overline{Y}'_\ell(t) := \frac{Y'_\ell(t)}{\sqrt{d}}, \quad \text{and} \quad \overline{\mathbf{Y}}'_d(t) := (\overline{Y}'_1(t), \dots, \overline{Y}'_r(t)), \tag{3.3}$$

where $\overline{Y}'_j(t)$ is a row vector. For $t \in S^m$, set also

$$\mathbf{Z}_d(t) = (Z_1(t), \dots, Z_{r(1+m)}(t)) = (\mathbf{Y}_d(t), \overline{\mathbf{Y}}'_d(t)). \tag{3.4}$$

The covariances

$$\rho_{k\ell}(s,t) = \mathbb{E}\left(Z_k(s) Z_\ell(t)\right), \quad k,\ell = 1,\dots,r(1+m), \tag{3.5}$$

are obtained via routine computations, see [2]. These computations are simplified using the invariance under isometries. For instance, if $k = \ell \le r$

$$\rho_{k\ell}(s,t) = \langle s,t\rangle^d = \cos^d(\theta), \quad \theta \in [0,\pi),$$

where θ is the angle between s and t.

When the indexes k or ℓ are larger than r the covariances involve derivatives of r_d. In fact, in [2] is shown that $\mathbf{Z}_d$ is a vector of $r(1+m)$ standard normal random variables whose covariances depend upon the quantities

$$\begin{aligned}
\mathcal{A}(\theta) &= -\sqrt{d}\cos^{d-1}(\theta)\sin(\theta), \\
\mathcal{B}(\theta) &= \cos^d(\theta) - (d-1)\cos^{d-2}(\theta)\sin^2(\theta), \\
\mathcal{C}(\theta) &= \cos^d(\theta), \\
\mathcal{D}(\theta) &= \cos^{d-1}(\theta).
\end{aligned} \tag{3.6}$$

Actually, we can write the variance-covariance matrix of the vector

$$(Y_\ell(s), Y_\ell(t), \overline{Y}'_\ell(s), \overline{Y}'_\ell(t))$$

in the following form

$$\left[\begin{array}{c|c|c} A_{11} & A_{12} & A_{13} \\ \hline A_{12}^T & I_m & A_{23} \\ \hline A_{13}^T & A_{23}^T & I_m \end{array}\right], \tag{3.7}$$

where I_m is the $m \times m$ identity matrix,

$$A_{11} = \begin{bmatrix} 1 & \mathcal{C} \\ \mathcal{C} & 1 \end{bmatrix}, \; A_{12} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ -\mathcal{A} & 0 & \cdots & 0 \end{bmatrix}, \; A_{13} = \begin{bmatrix} \mathcal{A} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

and $A_{23} = \operatorname{diag}([\mathcal{B},\mathcal{D},\dots,\mathcal{D}])_{m\times m}$.

Furthermore, when dealing with the conditional distribution of $(\overline{\mathbf{Y}}'_d(s), \overline{\mathbf{Y}}'_d(t))$ given that $\mathbf{Y}_d(s) = \mathbf{Y}_d(t) = 0$ the following expressions appear for the common variance and the correlation

$$\sigma^2(\theta) = 1 - \frac{\mathcal{A}(\theta)^2}{1-\mathcal{C}(\theta)^2}; \quad \rho(\theta) = \frac{\mathcal{B}(\theta)(1-\mathcal{C}(\theta)^2) - \mathcal{A}(\theta)^2\mathcal{C}(\theta)}{1-\mathcal{C}(\theta)^2-\mathcal{A}(\theta)^2}.$$

After scaling $\theta = z/\sqrt{d}$, we have the following bounds.

Lemma 1 ([2])**.** *There exists $0 < \alpha < \frac{1}{2}$ such that for $\frac{z}{\sqrt{d}} < \frac{\pi}{2}$ it holds that:*

$$|\mathcal{A}| \le z\exp(-\alpha z^2); \quad |\mathcal{B}| \le (1+z^2)\exp(-\alpha z^2); \quad |\mathcal{C}| \le |\mathcal{D}| \le \exp(-\alpha z^2);$$

$$0 \le 1-\sigma^2 \le \mathrm{Const}\cdot\exp(-2\alpha z^2); \quad |\rho| \le \mathrm{Const}\cdot(1+z^2)^2\exp(-2\alpha z^2).$$

All the functions on the l.h.s. are evaluated at $\theta = z/\sqrt{d}$.

Lemma 2 ([2]). *It is easy to show using the definitions that the following limits, as $d \to +\infty$, hold.*

$$\cos^{2d}\left(\frac{z}{\sqrt{d}}\right) \to \exp(-z^2); \quad \mathcal{A} \to -z\exp(-z^2/2);$$

$$\mathcal{B} \to (1-z^2)\exp(-z^2/2); \quad \mathcal{C}, \mathcal{D} \to \exp(-z^2/2);$$

$$\sigma^2\Big(\frac{z}{\sqrt{d}}\Big) \quad \to \quad \frac{1-(1+z^2)\exp(-z^2)}{1-\exp(-z^2)};$$
$$\rho\Big(\frac{z}{\sqrt{d}}\Big) \quad \to \quad \frac{(1-z^2)^2(1-\exp(-z^2))\exp(-z^2)}{1-(1+z^2)\exp(-z^2)}.$$

□

4 Hermite expansion of the Volume

In this part we obtain the Hermite expansion for $\mathcal{V}_{\mathbf{Y}_d}(\mathbf{0})$.

Consider an approximation $\frac{1}{\varepsilon^r}\varphi(\frac{\mathbf{y}}{\varepsilon})$ to the Dirac's delta distribution $\delta_{\mathbf{0}}(\mathbf{y}) = \prod_{j=1}^r \delta_0(y_j)$, for $\mathbf{y} \in \mathbb{R}^r$, we assume that φ is a continuous density function with bounded support. Consider also the following function f^r defined as

$$f^r(\mathbf{y}') = \sqrt{\det(\mathbf{y}'(\mathbf{y}')^T)},$$

where $\mathbf{y}' = (y'_1, \dots, y'_r) = (y'_{11}, \dots, y'_{1m}, \dots, y'_{r1}, \dots, y'_{rm})$. According to convenience we understand $\mathbf{y}'$ as a vector in $\mathbb{R}^{r\times m}$ or as an $r \times m$ matrix. Furthermore, for any $\gamma > 0$ let

$$f^r_\gamma(\mathbf{y}') = f^r(\gamma y'_{11}, y'_{12}, \dots, y'_{1m}, \gamma y'_{21}, \dots, y'_{2m}, \dots, \dots, \gamma y'_{r1}, \dots, y'_{rm}).$$

Thus $f^r = f^r_1$.

Similarly to the case of the number of roots [2], we can obtain the following convergence in $L^2(\Omega)$.

Lemma 3. *Almost surely and in the L^2 sense it holds that*

$$\mathcal{V}_{\mathbf{Y}_d}(\mathbf{0}) = d^{\frac{r}{2}} \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^r} \int_{S^m} \varphi\left(\frac{\mathbf{Y}_d(t)}{\varepsilon}\right) f^r(\overline{Y}'_1(t), \dots, \overline{Y}'_r(t))d\mathbf{t}.$$

Proof. By the coarea formula we have

$$\frac{1}{\varepsilon^r}\int_{\mathbb{R}^r} \varphi\left(\frac{\mathbf{u}}{\varepsilon}\right) \mathcal{V}_{\mathbf{Y}_d}(\mathbf{u})d\mathbf{u} = \frac{1}{\varepsilon^r}\int_{S^m} \varphi\left(\frac{\mathbf{Y}_d(t)}{\varepsilon}\right) f^r(\overline{Y}'_1(t), \dots, \overline{Y}'_r(t))dt,$$

where $\mathcal{V}_{\mathbf{Y}_d}(\mathbf{u})$ stands for the $(m-r)$-volume of the level set $\{t \in S^m : \mathbf{Y}_d = \mathbf{u}\}$. Changing the variable in the left hand side integral we can write

$$\mathcal{Q}_\varepsilon := \int_{\mathbb{R}^r} \varphi(\mathbf{u})\mathcal{V}_{\mathbf{Y}_d}(\varepsilon\mathbf{u})d\mathbf{u} = \frac{1}{\varepsilon^r}\int_{S^m} \varphi\left(\frac{\mathbf{Y}_d(t)}{\varepsilon}\right) f^r(\overline{Y}'_1(t), \dots, \overline{Y}'_r(t)d\mathbf{t},$$

we need to prove the convergence in $L^2(\Omega)$ for this sequence. Let us evaluate

$$\mathbb{E}\left[(\mathcal{Q}_\varepsilon - \mathcal{V}_{\mathbf{Y}_d}(\mathbf{0}))^2\right] = \mathbb{E}\left[\mathcal{Q}_\varepsilon^2\right] - 2\mathbb{E}\left[\mathcal{Q}_\varepsilon \mathcal{V}_{\mathbf{Y}_d}(\mathbf{0})\right] + \mathbb{E}\left[\mathcal{V}^2_{\mathbf{Y}_d}(\mathbf{0})\right]. \tag{4.1}$$

But

$$\mathbb{E}\,[\mathcal{Q}_\varepsilon^2] = \int_{\mathbb{R}^r\times\mathbb{R}^r} \varphi(\mathbf{u}_1)\varphi(\mathbf{u}_2)\mathbb{E}\,[\mathcal{V}_{\mathbf{Y}_d}(\varepsilon\mathbf{u}_1)\mathcal{V}_{\mathbf{Y}_d}(\varepsilon\mathbf{u}_2)]d\mathbf{u}_1 d\mathbf{u}_2$$

and

$$\mathbb{E}\,[\mathcal{Q}_\varepsilon\mathcal{V}_{\mathbf{Y}_d}(\mathbf{0})] = \int_{\mathbb{R}^r} \varphi(\mathbf{u}_1)\mathbb{E}\,[\mathcal{V}_{\mathbf{Y}_d}(\varepsilon\mathbf{u}_1)\mathcal{V}_{\mathbf{Y}_d}(\mathbf{0})]d\mathbf{u}_1.$$

Using the Cauchy-Schwarz inequality we have

$$\mathbb{E}\,[\mathcal{V}_{\mathbf{Y}_d}(\varepsilon\mathbf{u}_1)\mathcal{V}_{\mathbf{Y}_d}(\varepsilon\mathbf{u}_2)] \leq (\mathbb{E}\,[\mathcal{V}_{\mathbf{Y}_d}(\varepsilon\mathbf{u}_1)^2]\mathbb{E}\,[\mathcal{V}_{\mathbf{Y}_d}(\varepsilon\mathbf{u}_2)^2])^{\frac{1}{2}}. \tag{4.2}$$

Furthermore, below we show that the right hand side is a continuous function in the variable $\mathbf{u}$, obtaining

$$\lim_{\varepsilon\to 0}\mathbb{E}\,[\mathcal{Q}_\varepsilon^2] \leq \mathbb{E}\,[\mathcal{V}^2_{\mathbf{Y}_d}(\mathbf{0})].$$

Moreover, given that the process satisfies the hypothesis of Proposition 6.12 of Azaïs & Wschebor book [4] it holds that

$$\mathbb{P}\{\exists t:\ \mathrm{rank}(\mathbf{Y}'_d(t)) < r,\ \mathbf{Y}_d(t) = \mathbf{u}\} = 0.$$

Thus by using the implicit function theorem we have that the function $\mathcal{V}_{\mathbf{Y}_d}(\cdot)$ is a.s. continuous and by a classsical result

$$\mathcal{V}_{\mathbf{Y}_d}(\mathbf{0}) = d^{\frac{r}{2}}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^r}\int_{S^m}\varphi\left(\frac{\mathbf{Y}_d(t)}{\varepsilon}\right) f^r(\overline{Y}'_1(t),\dots,\overline{Y}'_r(t)dt \text{ a.s.}$$

In this form by the Fatou's Lemma we get

$$\mathbb{E}\,[\mathcal{V}^2_{\mathbf{Y}_d}(\mathbf{0})] \leq \lim_{\varepsilon\to 0}\mathbb{E}\,[\mathcal{Q}_\varepsilon^2] \leq \mathbb{E}\,[\mathcal{V}^2_{\mathbf{Y}_d}(\mathbf{0})].$$

The same result can be obtained for the second addend of (4.1), in consequence the convergence in quadratic mean holds.

It remains to prove that the right hand side of (4.2) is a continuous function, we do that in what follows. By Rice formula

$$\begin{aligned}&\mathbb{E}\,[(\mathcal{V}_{\mathbf{Y}_d}(\mathbf{u}))^2]\\ &= d^r\int_{S^m\times S^m}\mathbb{E}\,[f^r(\overline{\mathbf{Y}}'_d(t))f^r(\overline{\mathbf{Y}}'_d(s)) \mid \mathbf{Y}_d(t) = \mathbf{Y}_d(s) = \mathbf{u}]\,p_{\mathbf{Y}(t),\mathbf{Y}(s)}(\mathbf{u},\mathbf{u})dtds.\end{aligned}$$

Clearly the density $p_{\mathbf{Y}(t),\mathbf{Y}(s)}(\mathbf{u},\mathbf{u})$ is continuous as a function of $\mathbf{u}$. We deal now with the conditional expectation.

Recall the notations in (3.6). Let us define the vector $\mathbf{v}(\langle t,s\rangle) = (\mathcal{A},0,\dots,0)^t$, a regression model gives that

$$\begin{aligned}\overline{Y}'_\ell(t) &= \mathbf{v}(\langle t,s\rangle)\tfrac{\langle t,s\rangle^d}{1-\langle t,s\rangle^{2d}}Y_\ell(t) - \mathbf{v}(\langle t,s\rangle)\tfrac{1}{1-\langle t,s\rangle^{2d}}Y_\ell(s) + \xi_{\ell 1}(t,s)\\ \overline{Y}'_\ell(s) &= -\mathbf{v}(\langle t,s\rangle)\tfrac{1}{1-\langle t,s\rangle^{2d}}Y_\ell(t) + \mathbf{v}(\langle t,s\rangle)\tfrac{\langle t,s\rangle^d}{1-\langle t,s\rangle^{2d}}Y_\ell(s) + \xi_{\ell 2}(t,s),\end{aligned}$$

with $\xi_{\ell 1}(t,s),\xi_{\ell 2}(t,s)$ centered Gaussian random variables independent from $Y_\ell(s)$ and $Y_\ell(t)$. In this form we get that the conditional distribution of $\left(\overline{Y}'_\ell(t),\overline{Y}'_\ell(s)\right)$ conditioned to $\mathbf{Y}(t) = \mathbf{Y}(s) = \mathbf{u}$ is normal with mean

$$\begin{pmatrix}\mathbf{v}(\langle t,s\rangle)(\frac{\langle t,s\rangle^d-1}{1-\langle t,s\rangle^{2d}})u_\ell\\ \mathbf{v}(\langle t,s\rangle)(\frac{\langle t,s\rangle^d-1}{1-\langle t,s\rangle^{2d}})u_\ell\end{pmatrix} = \begin{pmatrix}-\mathbf{v}(\langle t,s\rangle)(\frac{1}{1+\langle t,s\rangle^d})u_\ell\\ -\mathbf{v}(\langle t,s\rangle)(\frac{1}{1+\langle t,s\rangle^d})u_\ell\end{pmatrix}$$

and variance

$$\left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{12} & B_{22} \end{array}\right].$$

This result implies that the conditional expectation can be expressed through the following two vectors

$$\zeta_1 := \left(-\mathbf{v}(\langle t,s\rangle)\cdot\frac{1}{1+\langle t,s\rangle^d}\cdot\frac{u_1}{\sigma}+\overline{M}_1,\ldots,-\mathbf{v}(\langle t,s\rangle)\cdot\frac{1}{1+\langle t,s\rangle^d}\cdot\frac{u_r}{\sigma}+\overline{M}_r\right);$$

$$\zeta_2 := \left(-\mathbf{v}(\langle t,s\rangle)\cdot\frac{1}{1+\langle t,s\rangle^d}\cdot\frac{u_1}{\sigma}+\overline{W}_1,\ldots,-\mathbf{v}(\langle t,s\rangle)\cdot\frac{1}{1+\langle t,s\rangle^d}\cdot\frac{u_r}{\sigma}+\overline{W}_r\right),$$

where the $(r\times m)$-dimensional vectors

$$(\overline{M}_1,\ldots,\overline{M}_r) := (M_{11},\ldots,M_{1m},M_{21},\ldots,M_{2m},\ldots,M_{r1},\ldots,M_{rm}),$$

$$(\overline{W}_1,\ldots,\overline{W}_r) := (W_{11},\ldots,W_{1m},W_{21},\ldots,W_{2m},\ldots,W_{r1},\ldots,W_{rm}),$$

are such that the M_{lk} (resp. W_{lk}) are independent standard Gaussian random variables and

$$\mathbb{E}\left[M_{l_1k_1}W_{l_2k_2}\right] = \rho\mathbf{1}_{\{l_1=l_2,\,k_1=k_2=1\}}+\mathcal{D}\mathbf{1}_{\{l_1=l_2,\,k_1=k_2>1\}}.$$

In fact, we have

$$\mathbb{E}\left(f^r(\overline{\mathbf{Y}}'_d(t))f^r(\overline{\mathbf{Y}}'_d(s)) \mid \mathbf{Y}_d(t)=\mathbf{Y}_d(s)=\mathbf{u}\right)=\mathbb{E}\left[f^r_\sigma(\zeta_1)f^r_\sigma(\zeta_2)\right].$$

By the definition of the function f^r_σ and the form of the vectors ζ_1 and ζ_2 this term is evidently a continuous function of the variable $\mathbf{u}$. □

Now, we exhibit the expansion. We introduce the Hermite polynomials $H_n(x)$, $x\in\mathbb{R}$, by $H_0(x)=1$, $H_1(x)=x$ and $H_{n+1}(x)=xH_n(x)-nH_{n-1}(x)$ for $n\geq 1$. A key property of Hermite polynomials is that (Z,W) is a centered Gaussian vector with $\mathrm{Var}(Z)=\mathrm{Var}(W)=1$ and $\mathbb{E}(ZW)=r$, the following bi-dimensional Mehler's formula holds

$$\mathbb{E}\left[H_k(Z)H_l(W)\right]=\delta_{k,l}r^k k!.$$

The tensorial versions are defined for multi-indices $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_r)\in\mathbb{N}^r$ and $\boldsymbol{\beta}=(\beta_{11},\ldots,\beta_{1m},\ldots,\beta_{r1},\ldots,\beta_{rm})$, by

$$\mathbf{H}_\alpha(\mathbf{y})=H_{\alpha_1}(y_1)\ldots H_{\alpha_r}(y_r),$$

$$\widetilde{\mathbf{H}}_{\boldsymbol{\beta}}(\mathbf{y}')=H_{\beta_{11}}(y'_{11})\ldots H_{\beta_{1m}}(y'_{1m})\ldots H_{\beta_{r1}}(y'_{r1})\ldots H_{\beta_{rm}}(y'_{rm}).$$

The idea is to take the Hermite expansions of the r.h.s. in the formula in Lemma 3. We denote the coefficients of the Dirac's delta distribution in the Hermite basis of $L^2(\mathbb{R}^r,\phi_r(\mathbf{x})d\mathbf{x})$ by $b_{\boldsymbol{\alpha}}$. Readily we can show that $b_{\boldsymbol{\alpha}}=0$ if at least one index α_j is odd, otherwise

$$b_{\boldsymbol{\alpha}}=\frac{1}{[\frac{\boldsymbol{\alpha}}{2}]!}\prod_{j=1}^{r}\frac{1}{\sqrt{2\pi}}\left[-\frac{1}{2}\right]^{[\frac{\alpha_j}{2}]}.$$

Since $(f^r)^2$ is a polynomial, $f^r\in L^2(\mathbb{R}^{r\times m},\phi_{r\times m}(\mathbf{y}'))d\mathbf{y}'$. Here and in the sequel ϕ_k stands for the standard normal density function in $\mathbb{R}^k$. For f^r we have

$$f^r(\mathbf{y}')=\sum_{\boldsymbol{\beta}}f^r_{\boldsymbol{\beta}}\widetilde{H}_{\boldsymbol{\beta}}(\mathbf{y}'), \tag{4.3}$$

where $\boldsymbol{\beta}$ and $\widetilde{H}_{\boldsymbol{\beta}}$ are as above and

$$f^r_{\boldsymbol{\beta}} = \frac{1}{\boldsymbol{\beta}!} \int_{\mathbb{R}^{r\times m}} f^r(\mathbf{y}')\widetilde{H}_{\boldsymbol{\beta}}(\mathbf{y}')\phi_{r\times m}(\mathbf{y}')d\mathbf{y}'.$$

Let us introduce the functions

$$g_q(\mathbf{z}) = \sum_{|\boldsymbol{\mu}|=q} a_{\boldsymbol{\mu}} \overline{\mathbf{H}}_{\boldsymbol{\mu}}(\mathbf{z}),$$

where $\mathbf{z} = (\mathbf{y}, \mathbf{y}') \in \mathbb{R}^r \times \mathbb{R}^{r\times m}$, $\boldsymbol{\mu} = (\boldsymbol{\alpha}, \boldsymbol{\beta})$, $|\boldsymbol{\mu}| = |\boldsymbol{\alpha}| + |\boldsymbol{\beta}|$, $a_{\boldsymbol{\mu}} = b_{\boldsymbol{\alpha}} f^r_{\boldsymbol{\beta}}$ and

$$\overline{\mathbf{H}}_{\boldsymbol{\mu}}(\mathbf{z}) = \mathbf{H}_{\alpha}(\mathbf{y})\widetilde{\mathbf{H}}_{\boldsymbol{\beta}}(\mathbf{y}').$$

Thus similarly to [2] we can obtain the expansion.

Proposition 1. *With the same notations as above. We have, in the L^2 sense, that*

$$\bar{\mathcal{V}}_d = \frac{\mathcal{V}_{\mathbf{Y}_d}(\mathbf{0}) - \mathbb{E}\left[\mathcal{V}_{\mathbf{Y}_d}(\mathbf{0})\right]}{d^{\frac{r}{2}-\frac{m}{4}}} = d^{\frac{m}{4}} \sum_{q=1}^{\infty} \int_{S^m} g_q(\mathbf{Z}_d(t))dt.$$

□

Remark 1. *The same type of expansion can be obtained with minor modifications if instead of the volume of the zero set in all the sphere we consider the volume of the set restricted to a Borel set $\mathcal{G} \subset S^m$.*

5 Proof of Theorem 1

Wiener Chaos: Let $\mathbf{B} = \{B(\lambda) : \lambda \geq 0\}$ be a standard Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ being $\mathcal{F}$ the σ-algebra generated by $\mathbf{B}$. The Wiener chaos is an orthogonal decomposition of $L^2(\mathbf{B}) = L^2(\Omega, \mathcal{F}, \mathbb{P})$:

$$L^2(\mathbf{B}) = \bigoplus_{q=0}^{\infty} \mathcal{C}_q,$$

where $\mathcal{C}_0 = \mathbb{R}$ and for $q \geq 1$, $\mathcal{C}_q = \{I^{\mathbf{B}}_q(f_q) : f_q \in L^2_s([0,\infty)^q)\}$ being $I^{\mathbf{B}}_q$ the q-folded multiple integral wrt $\mathbf{B}$ and $L^2_s([0,\infty)^q)$ the space of kernels $f_q : [0,\infty)^q \to \mathbb{R}$ which are square integrable and symmetric, that is, if π is a permutation then $f_q(\lambda_1, \ldots, \lambda_q) = f_q(\lambda_{\pi(1)}, \ldots, \lambda_{\pi(q)})$. Equivalently, each square integrable functional F of the Brownian motion $\mathbf{B}$ can be written as a sum of orthogonal random variables

$$F = \mathbb{E}(F) + \sum_{q=1}^{\infty} I^{\mathbf{B}}_q(f_q),$$

for some uniquely determined kernels $f_q \in L^2_s([0,\infty)^q)$.

Let $f_q, g_q \in L^2_s([0,\infty)^q)$, then for $n = 0, \ldots, q$ we define the contraction by

$$\begin{aligned} f_q \otimes_n g_q(\lambda_1, \ldots, \lambda_{2q-2n}) \\ = \int_{[0,\infty)^n} & f_q(z_1, \ldots, z_n, \lambda_1, \ldots, \lambda_{q-n}) \\ & \cdot g_q(z_1, \ldots, z_n, \lambda_{q-n+1}, \ldots, \lambda_{2q-2n}) dz_1 \ldots dz_n. \quad (5.1) \end{aligned}$$

Now, we can state the generalization of the Fourth Moment Theorem.

Theorem 2 ([10] Theorem 11.8.3)**.** *Let F_d be in $L^2_s(\mathbf{B})$ admit chaotic expansions*

$$F_d = \mathbb{E}\,(F_d) + \sum_{q=1}^{\infty} I_q(f_{d,q})$$

for some kernels $f_{d,q}$. Then, if $\mathbb{E}\,(F_d) = 0$ and

1. *for each fixed $q \geq 1$, $\mathrm{Var}(I_q(f_{d,q})) \to_{d\to\infty} V_q$;*
2. *$V := \sum_{q=1}^{\infty} V_q < \infty$;*
3. *for each $q \geq 2$ and $n = 1, \dots, q-1$,*

$$\|f_{d,q} \otimes_n f_{d,q}\|_{L^2_s([0,\infty)^{2q-2n})} \underset{d\to\infty}{\to} 0;$$

4. *$\lim_{Q\to\infty} \limsup_{d\to\infty} \sum_{q=Q+1}^{\infty} \mathrm{Var}(I_q(f_{d,q})) = 0$.*

Then, F_d converges in distribution towards the $N(0, V)$ distribution. □

We apply this theorem to $F_d = \bar{\mathcal{V}}_d$.

Step 1: Let us compute the variance of the q-term in the expansion

$$\mathbb{E}\left[\left(\int_{S^m} g_q(\mathbf{Z}_d(t))dt\right)^2\right]$$

$$= \sum_{|\boldsymbol{\mu}|=q} \sum_{|\boldsymbol{\mu}'|=q} a_{\boldsymbol{\mu}} a_{\boldsymbol{\mu}'} \int_{S^m\times S^m} \mathbb{E}\,[\mathbf{H}_{\boldsymbol{\alpha}}(\mathbf{Y}(t))\mathbf{H}_{\boldsymbol{\alpha}'}(\mathbf{Y}(s))\widetilde{\mathbf{H}}_{\boldsymbol{\beta}}(\overline{\mathbf{Y}}'(t))\widetilde{\mathbf{H}}_{\boldsymbol{\beta}'}(\overline{\mathbf{Y}}'(s))]dtds.$$

We have viewed that the coefficients $b_{\boldsymbol{\alpha}}$ are zero if one of the α_j is odd. Furthermore, the function $f^r(\mathbf{y})$ is even with respect to each column, thus its Hermite coefficients

$$f^r_{\boldsymbol{\beta}} = f^r_{\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\dots,\boldsymbol{\beta}_r} = \int_{\mathbb{R}^{r\times m}} \sqrt{\det(\mathbf{y}'(\mathbf{y}')^t)}\mathbf{H}_{\boldsymbol{\beta}_1}(y'_1)\dots\mathbf{H}_{\boldsymbol{\beta}_r}(y'_r)\phi_{r\times m}(\mathbf{y}')d\mathbf{y}',$$

are zero if at least one of the $\boldsymbol{\beta}_\ell$ satisfies $|\boldsymbol{\beta}_\ell| = 2k+1$. In this form $|\beta_\ell| = \sum_{j=1}^m \beta_{\ell j}$ is necessarily even. Moreover, $q = |\boldsymbol{\mu}| = |\boldsymbol{\alpha}| + |\boldsymbol{\beta}|$ is also even.

By independence we have

$$\mathbb{E}\,[\mathbf{H}_{\boldsymbol{\alpha}}(\mathbf{Y}(t))\widetilde{\mathbf{H}}_{\boldsymbol{\beta}}(\overline{\mathbf{Y}}(t))\mathbf{H}_{\boldsymbol{\alpha}'}(\mathbf{Y}(s))\widetilde{\mathbf{H}}_{\boldsymbol{\beta}'}(\overline{\mathbf{Y}}'(s))]$$

$$= \prod_{\ell=1}^{r} \mathbb{E}\,[H_{\alpha_\ell}(Y_\ell(t))\mathbf{H}_{\boldsymbol{\beta}_\ell}(\overline{Y}'_\ell(t)H_{\alpha'_\ell}(Y_\ell(t))\mathbf{H}_{\boldsymbol{\beta}'_\ell}(\overline{Y}'_\ell(t)]$$

$$= \prod_{\ell=1}^{r} \mathbb{E}\,[H_{\alpha_\ell}(Y_\ell(s))H_{\alpha'_\ell}(Y_\ell(t))H_{\beta_{\ell 1}}(\overline{Y}'_{\ell 1}(s))H_{\beta'_{\ell 1}}(\overline{Y}'_{\ell 1}(t))]$$

$$\times \prod_{j=2}^{m} \mathbb{E}\,[H_{\beta_{\ell j}}(\overline{Y}'_{\ell j}(s))H_{\beta'_{\ell j}}(\overline{Y}'_{\ell j}(t))]. \tag{5.2}$$

In the second equality we used that the random vectors

$$(Y_\ell(s), Y_\ell(t), \overline{Y}'_{\ell 1}(s), \overline{Y}'_{\ell 1}(t)); \quad (\overline{Y}'_{\ell j}(s), \overline{Y}'_{\ell j}(t)); \quad j \geq 2$$

are independent. Using Mehler's formula, we get

$$\mathbb{E}\,[H_{\beta_{\ell j}}(\overline{Y}'_{\ell j}(s))H_{\beta'_{\ell j}}(\overline{Y}'_{\ell j}(t))] = \delta_{\beta_{\ell j}\beta'_{\ell j}}\beta_{\ell j}!\,(\rho''_{\ell j})^{\beta_{\ell j}},$$

where $\rho''_{\ell k} = \rho''_{\ell k}(\langle s,t\rangle) = \mathbb{E}\,(\overline{Y}'_{\ell j}(s)\overline{Y}'_{\ell j}(t)) = \langle t,s\rangle^{d-1}$. Since $\sum_{j=1}^m \beta_{\ell j}$ is even, we have that either $\beta_{\ell 1}$ is even and then $\sum_{j=2}^m \beta_{\ell j}$ is even too or $\beta_{\ell 1}$ is odd and in this case $\sum_{j=2}^m \beta_{\ell j}$ is also odd.

For the first factor in the r.h.s. of (5.2), using again Mehler's formula we get

$$\mathbb{E}\,[H_{\alpha_\ell}(Y_\ell(s))H_{\alpha'_\ell}(Y_\ell(t))H_{\beta_{\ell 1}}(\overline{Y}'_{\ell 1}(s))H_{\beta'_{\ell 1}}(\overline{Y}'_{\ell 1}(t))] = 0,$$

if $\alpha_\ell + \beta_{\ell 1} \neq \alpha'_\ell + \beta'_{\ell 1}$. Otherwise, consider $\Lambda \subset \mathbb{N}^4$ defined by

$$\Lambda = \{(d_1,d_2,d_3,d_4) : d_1+d_2 = \alpha_\ell,\ d_3+d_4 = \beta_{\ell 1},\ d_1+d_3 = \alpha'_\ell,\ d_2+d_4 = \beta'_{\ell 1}\};$$

then

$$\begin{aligned}&\mathbb{E}\,[H_{\alpha_\ell}(Y_\ell(s))H_{\alpha'_\ell}(Y_\ell(t))H_{\beta_{\ell 1}}(\overline{Y}'_{\ell 1}(s))H_{\beta'_{\ell 1}}(\overline{Y}'_{\ell 1}(t))]\\ &\qquad = \sum_{(d_i)\in\Lambda} \frac{\alpha_\ell!\alpha'_\ell!\beta_{\ell 1}!\beta'_{\ell 1}!}{d_1!d_2!d_3!d_4!}\rho^{d_1}(\rho')^{d_2}(\rho')^{d_3}(\rho'')^{d_4},\end{aligned}$$

where $\rho = \rho(\langle s,t\rangle) = \mathbb{E}\,(Y_\ell(s)Y_\ell(t))$, $\rho' = \mathbb{E}\,(Y_\ell(s)\overline{Y}'_{\ell 1}(t)) = \mathbb{E}\,(\overline{Y}'_{\ell 1}(s)Y_\ell(t))$ and $\rho'' = \mathbb{E}\,(\overline{Y}'_{\ell 1}(s)\overline{Y}'_{\ell 1}(t))$.

Note that the conditions defining the index set Λ implies that the first factor in Equation (5.2) is

$$\prod_{\ell=1}^r \sum_{(d_i)\in\Lambda} \frac{\alpha_\ell!\alpha'_\ell!\beta_{\ell 1}!\beta'_{\ell 1}!}{d_1!d_2!d_3!d_4!}\rho^{d_1}(\rho')^{d_2+d_3}(\rho'')^{d_4}.$$

Hence, if we change $\langle s,t\rangle$ by $-\langle s,t\rangle$, for each ℓ we have the factor

$$\begin{aligned}(-1)^{dd_1}\cdot(-1)^{(d-1)(d_2+d_3)}\cdot(-1)^{dd_4} &= (-1)^{d(d_1+d_4)+(d-1)(d_2+d_3)}\\ &= (-1)^{d\alpha_\ell}(-1)^{d\beta'_{\ell 1}}(-1)^{2(\alpha'_\ell-d_1)} = (-1)^{d\beta'_{\ell 1}}\end{aligned}$$

Remark 2. *Changing $\langle t,s\rangle$ by $-\langle t,s\rangle$ in (5.2) and considering each term for $j = 1\dots,r$ of the product, either β'_{ℓ_1} and $\sum_{j=2}^m \beta'_{\ell_j}$ are even then the sign of this term does not change or the two numbers are odd and then they have a minus in front and the sing neither change. Thus we get that the complete sign of (5.2) does not change.*

Let us now define

$$\tilde{\mathcal{H}}_{qd}(\langle t,s\rangle) = \sum_{|\boldsymbol{\mu}|=q}\sum_{|\boldsymbol{\mu}'|=q} a_{\boldsymbol{\mu}}a_{\boldsymbol{\mu}'}\mathbb{E}\,[\mathbf{H}_{\boldsymbol{\alpha}}(\mathbf{Y}(t))\widetilde{\mathbf{H}}_{\boldsymbol{\beta}}(\overline{\mathbf{Y}}(t))\mathbf{H}_{\boldsymbol{\alpha}'}(\mathbf{Y}(s))\widetilde{\mathbf{H}}_{\boldsymbol{\beta}'}(\overline{\mathbf{Y}}'(s))]. \quad (5.3)$$

In this manner we can write

$$\begin{aligned}d^{\frac{m}{2}}\mathbb{E}\left[\left(\int_{S^m} g_q(\mathbf{Z}_d(t))dt\right)^2\right] &= d^{\frac{m}{2}}\int_{S^m\times S^m}\tilde{\mathcal{H}}_{qd}(\langle t,s\rangle)dtds\\ &= \kappa_m\kappa_{m-1}d^{m/2}\int_0^\pi \sin^{m-1}(\theta)\mathcal{H}_{qd}(\cos(\theta))\,d\theta\\ &= 2\kappa_m\kappa_{m-1}\int_0^{\sqrt{d}\pi/2} d^{(m-1)/2}\sin^{m-1}\left(\frac{z}{\sqrt{d}}\right)\tilde{\mathcal{H}}_{qd}\left(\cos\left(\frac{z}{\sqrt{d}}\right)\right)dz.\end{aligned}$$

For the second equality we used Lemma 3 of [2] about the integration on the sphere of a function invariant by rotations. In the third equality we use (deduced from Remark 2) the invariance of the function with respect to the change of variable $\varphi = \frac{\pi}{2} - \theta$ and finally we made $\theta = \frac{z}{\sqrt{d}}$.

The convergence follows by dominated convergence using for the covariances $\rho_{k,\ell} = \mathbb{E}\left(Z_k(s)Z_\ell(t)\right)$, see (3.5), the bounds in Lemma 1 and the expression for the matrix (3.7). In this manner the integrand can be bounded at most by Const $(1 + z^2)^q \exp\left(-q\alpha z^2\right)$. In conclusion we have

$$V_q^r := \lim_{d\to\infty} d^{\frac{m}{2}}\mathbb{E}\left[\left(\int_{S^m} g_q(\mathbf{Z}_d(t))dt\right)^2\right] = 2\kappa_m\kappa_{m-1}\int_0^\infty z^{m-1}\tilde{\mathcal{H}}_q(z)dz \quad q \geq 1,$$

where

$$\tilde{\mathcal{H}}_q(z) := \lim_{d\to\infty} \tilde{\mathcal{H}}_{qd}\left(\cos\left(\frac{z}{\sqrt{d}}\right)\right).$$

Step 2: Let us prove point 2 of Theorem 2. Recall from (1.1) that

$$V_\infty^r = \lim_{d\to\infty} \operatorname{Var}(\bar{\mathcal{V}}_d) = \lim_{d\to\infty}\sum_{q=0}^{\infty} d^{\frac{m}{2}}\mathbb{E}\left[\left(\int_{S^m} g_q(\mathbf{Z}_d(t))dt\right)^2\right].$$

The second equality follows from Parseval's identity. Thus, by Fatou's Lemma

$$V^r := \sum_{q=0}^{\infty} V_q^r = \sum_{q=0}^{\infty}\lim_{d\to\infty} d^{\frac{m}{2}}\mathbb{E}\left[\left(\int_{S^m} g_q(\mathbf{Z}_d(t))dt\right)^2\right] \leq V_\infty^r < \infty.$$

Actually, equality holds as a consequence of Point 4 and the finiteness of V_∞^r.

Step 3: Let us define

$$I_{q,d} = d^{\frac{m}{4}}\int_{S^m} g_q(\mathbf{Z}_d(t))dt.$$

Lemma 5 below gives a sufficient condition on the covariances of the process $\mathbf{Z}_d$ in order to verify the convergence of the norm of the contractions. Below we write the chaotic components $I_{q,d}$ in Proposition 1 as multiple stochastic integrals wrt a standard Brownian motion $\mathbf{B}$ and use this fact in order to prove Lemma 5.

Let $\mathbf{B} = \{B(\lambda) : \lambda \in [0,\infty)\}$ be a standard Brownian motion on $[0,\infty)$. By the isometric property of stochastic integrals there exist kernels $h_{t,\ell}$ such that the components of the vector $\mathbf{Z}_d$ defined in (3.4) can be written as:

$$Z_\ell(t) = \int_0^\infty h_{t,\ell}(\lambda)dB(\lambda), \ell = 1,\ldots,r(m+1). \tag{5.4}$$

The kernels $h_{t,\ell}$ can be computed explicitly from the definition of Z_ℓ writting the random coefficients as integrals wrt the Brownian motion.

Lemma 4. *With the same notations and assumptions as in Proposition 1. Then, $I_{q,d}$ can be written as a multiple stochastic integral*

$$I_{q,d} = I_q^{\mathbf{B}}(g_{q,d}) = \int_{[0,\infty)^q} g_{q,d}(\boldsymbol{\lambda})dB(\boldsymbol{\lambda});$$

with

$$g_{q,d}(\boldsymbol{\lambda}) = d^{m/4}\sum_{|\boldsymbol{\mu}|=q} a_{\boldsymbol{\mu}}\int_{S^m}(\otimes_{\ell=1}^{r(m+1)} h_{t,\ell}^{\otimes\gamma_\ell})(\boldsymbol{\lambda})dt,$$

where $h_{t,\ell}$ is defined in (5.4) and $I_q^{\mathbf{B}}$ is the q-folded multiple stochastic integral wrt $\mathbf{B}$.

The proof follows the same line as that of the square case.

As $r_d(s,t) = r_d(\langle s,t\rangle)$ then r_d can be seen as a function of one real variable.

Lemma 5. *For $k = 0, 1, 2$, let $r_d^{(k)}$ indicate the k-th derivative of $r_d : [-1, 1] \to \mathbb{R}$. If*

$$d^{m/3} \int_0^{\pi/2} \sin^{m-1}(\theta) |r_d^{(k)}(\cos(\theta))| d\theta \underset{d\to+\infty}{\to} 0, \tag{5.5}$$

then, for $n = 1, \dots, q-1$ and $g_{q,d}$ defined in Lemma 4:

$$\|g_{q,d} \otimes_n g_{q,d}\|_2 \to_{d\to\infty} 0.$$

The proof is the same as the one in the square case.

Therefore, it suffices to verify (5.5). For $k = 0, 1, 2$ we have

$$\begin{aligned} d^{m/3} &\int_0^{\pi/2} \sin^{m-1}(\theta) |r_d^{(k)}(\cos(\theta))| d\theta \\ &= d^{m/3} \int_0^{\sqrt{d}\pi/2} \sin^{m-1}\left(\frac{z}{\sqrt{d}}\right) \left| r_d^{(k)}\left(\cos\left(\frac{z}{\sqrt{d}}\right)\right)\right| \frac{dz}{\sqrt{d}} \\ &= \frac{1}{d^{m/6}} \int_0^{\sqrt{d}\pi/2} d^{\frac{m-1}{2}} \sin^{m-1}\left(\frac{z}{\sqrt{d}}\right) \left| r_d^{(k)}\left(\cos\left(\frac{z}{\sqrt{d}}\right)\right)\right| dz. \end{aligned}$$

Now $d^{\frac{m-1}{2}} \sin^{m-1}\left(z/\sqrt{d}\right) \leq z^{m-1}$ and taking the worst case in Lemma 1 we have $|r_d^{(k)}(z/\sqrt{d})| \leq (1 + z^2)\exp(-\alpha z^2)$. Hence, the last integral is convergent and (5.5) follows. □

Step 4: In what follows we deal with Point 4 in Theorem 2. Let π_q be the projection on the q-th chaos $\mathcal{C}_q$ and $\pi^Q = \sum_{q\geq Q} \pi_q$ be the projection on $\oplus_{q\geq Q}\mathcal{C}_q$. We need to bound the following quantity uniformly in d

$$\frac{d^{m/2}}{4} \mathrm{Var}(\pi^Q(\mathcal{V}_{\mathbf{Y}_d}(\mathbf{0})) = \frac{1}{4} \sum_{q\geq Q} d^{m/2} \int_{S^m\times S^m} \tilde{\mathcal{H}}_{q,d}(\langle s,t\rangle) ds dt, \tag{5.6}$$

where $\tilde{\mathcal{H}}_{qd}$ is defined in (5.3).

In order to bound this quantity we split the integral depending on the (geodesical) distance between $s, t \in S^m$

$$\mathrm{dist}(s,t) = \arccos(\langle s,t\rangle), \tag{5.7}$$

into the integrals over the regions $\{(s,t) : \mathrm{dist}(s,t) < a/\sqrt{d}\}$ and its complement, a will be chosen later. We bound each part in the following two subsections.

5.1 Off-diagonal term

In this subsection we consider the integral in the rhs of (5.6) restricted to the off-diagonal region $\{(s,t) : \mathrm{dist}(s,t) \geq a/\sqrt{d}\}$. That is,

$$\frac{d^{m/2}}{4} \sum_{q\geq Q} \int_{\{(s,t):\,\mathrm{dist}(s,t)\geq a/\sqrt{d}\}} \tilde{\mathcal{H}}_{qd}(\langle s,t\rangle) ds dt.$$

This is the easier case since the covariances of $\mathbf{Z}_d$ are bounded away from 1.

We need the following lemma from Arcones ([1], page 2245). Let X be a standard Gaussian vector on $\mathbb{R}^N$ and $h : \mathbb{R}^N \to \mathbb{R}$ a measurable function such that $\mathbb{E}[h^2(X)] < \infty$ and let us consider its L^2 convergent Hermite's expansion

$$h(x) = \sum_{q=0}^{\infty} \sum_{|\mathbf{k}|=q} h_{\mathbf{k}} \mathbf{H}_{\mathbf{k}}(x).$$

The Hermite rank of h is defined as

$$\operatorname{rank}(h) = \inf\{\tau : \exists\, \mathbf{k}, |\mathbf{k}| = \tau ; \mathbb{E}\left[(h(X) - \mathbb{E}\, h(X))\mathbf{H}_{\mathbf{k}}(X)\right] \neq 0\}.$$

Then, we have

Lemma 6 ([1]). *Let $W = (W_1, \dots, W_N)$ and $Q = (Q_1, \dots, Q_N)$ be two mean-zero Gaussian random vectors on $\mathbb{R}^N$. Assume that*

$$\mathbb{E}[W_j W_k] = \mathbb{E}[Q_j Q_k] = \delta_{j,k},$$

for each $1 \leq j, k \leq N$. We define

$$r^{(j,k)} = \mathbb{E}[W_j Q_k].$$

Let h be a function on $\mathbb{R}^N$ with finite second moment and Hermite rank τ, $1 \leq \tau < \infty$, define

$$\psi := \max\left\{ \max_{1\leq j\leq N} \sum_{k=1}^{N} |r^{(j,k)}|, \max_{1\leq k\leq N} \sum_{j=1}^{N} |r^{(j,k)}| \right\}.$$

Then

$$|\operatorname{Cov}(h(W), h(Q))| \leq \psi^{\tau} \mathbb{E}[h^2(W)].$$

□

We apply this lemma for $N = r \times (1 + m)$, $W = \mathbf{Z}(s)$, $Q = \mathbf{Z}(t)$ and to the function $h(\mathbf{y}, \mathbf{y}') = g_q(\mathbf{y}, \mathbf{y}')$. Recalling that $\rho_{k,\ell}(s, t) = \rho_{k,\ell}(\langle s, t\rangle) = \mathbb{E}\left[Z_k(s) Z_\ell(t)\right]$, the Arcone's coefficient is now

$$\psi(s, t) = \max\left\{ \sum_{1\leq k\leq m+m^2} |\rho_{k,\ell}(s, t)|, \sum_{1\leq \ell\leq m+m^2} |\rho_{k,\ell}(s, t)| \right\}.$$

Thus

$$|\tilde{\mathcal{H}}_{q,d}(\langle s, t\rangle)| \leq \psi(\langle s, t\rangle)^q ||g_q||^2,$$

being $\|g_q\|^2 = \mathbb{E}\left(g_q^2(\zeta)\right)$ for standard normal ζ. The following lemma is an easy consequence of the definition

Lemma 7. *For g_q it holds*

$$||g_q||^2 \leq ||f^r||_2^2.$$

We move to the bound of Arcones' coefficient $\psi(\langle s, t\rangle)$. By the invariance of the distribution of $\mathbf{Y}$ (and $\mathbf{Z}_d$) under isometries we can fix $s = e_0$ and express t in hyperspheric coordinates (3.1), as above we have $\langle e_0, t\rangle = \cos(\theta)$. Direct computation of the covariances $\rho_{k\ell}$ yield that the maximum in the definition of ψ is $|\mathcal{C}| + |\mathcal{A}|$, see (3.6). Lemma 1 entails that $|\mathcal{C}| + |\mathcal{A}| \leq e^{-\alpha z^2}(1 + z)$. For $z = 2$ the bound takes the value $2e^{-4\alpha}$ which is less or equal to one if $\alpha \geq \frac{1}{4} \log 2$, this is always possible because the only restriction that we have is $\alpha < \frac{1}{2}$. Moreover, for δ small enough $e^{-\alpha z^2}(1+z) \geq 1$ if $z < \delta$. This leads to affirm that there exists a $a < 2$ such that for all $z \geq a$ it holds $\mathcal{C} + \mathcal{A} < r_0 < 1$.

These results allow to use the Arcones' result to obtain

$$\begin{aligned}
\sup_d \sum_{q \geq Q} \frac{d^{m/2}}{4} \int_{\left\{(s,t) :\, \mathrm{dist}(s,t) \geq \frac{a}{\sqrt{d}}\right\}} \tilde{\mathcal{H}}_{q,d}(\langle s,t \rangle) ds dt \\
= \sup_d \frac{C_m}{4} \left| \sum_{q \geq Q} d^{\frac{m-1}{2}} \int_a^{\sqrt{d}\pi} \sin^{m-1}\left(\frac{z}{\sqrt{d}}\right) \tilde{\mathcal{H}}_d^q \left(\cos\left(\frac{z}{\sqrt{d}}\right)\right) dz \right| \\
\leq C_m ||f^r||_2^2 \sum_{q \geq Q} r_0^{q-1} \int_a^{\infty} z^{m-1}(1+z) e^{-\alpha z^2} dz \underset{Q \to \infty}{\to} 0.
\end{aligned}$$

5.2 Diagonal term

In this subsection we prove that the integral in the r.h.s. of (5.6) restricted to the diagonal region $\{(s,t) : \mathrm{dist}(s,t) < a/\sqrt{d}\}$ tends to 0 as $Q \to \infty$ uniformly in d, $a < 2$ is fixed. That is, we consider

$$\frac{d^{m/2}}{4} \sum_{q \geq Q} \int_{\{(s,t) :\, \mathrm{dist}(s,t) < a/\sqrt{d}\}} \tilde{\mathcal{H}}_{q,d}(\langle s,t \rangle) ds dt.$$

This is the difficult part, we use an indirect argument.

Next proposition, which proof is presented in [3]., gives a convenient partition of the sphere based on the hyperspheric coordinates (3.1). Define the hyperspherical rectangle (HSR for short) with center $x^{(m)}(\tilde{\theta})$ with $\tilde{\theta} = (\tilde{\theta}_1, \dots, \tilde{\theta}_m)$ and vector radius $\tilde{\eta} = (\tilde{\eta}_1, \dots, \tilde{\eta}_m)$ as

$$HSR(\tilde{\theta}, \tilde{\eta}) = \{x^{(m)}(\theta) : |\theta_i - \tilde{\theta}_i| < \tilde{\eta}_i, i = 1, \dots, m\}.$$

Let $T_t S^m$ be the the tangent space to S^m at t. This space can be identified with $t^\perp \subset \mathbb{R}^{m+1}$. Let $\phi_t : S^m \to t^\perp$ be the orthogonal projection over $t^\perp$, $\mathcal{C}_{\mathbf{Y}_d}(\mathbf{0})$ be the zero set of $\mathbf{Y}_d$ on S^m and $\mathcal{V}$ its volume on S^m.

Proposition 2. *For d large enough, there exists a partition of the unit sphere S^m into HSRs $R_j : j = 1, \dots, k(m,d) = O(d^{m/2})$ and an extra set E such that*

1. $\mathrm{Var}(\mathcal{V}(\mathcal{C}_{\mathbf{Y}_d}(\mathbf{0}) \cap E)) = o(d^{r - \frac{m}{2}})$.
2. *The HSRs R_j have diameter $O(\frac{1}{\sqrt{d}})$ and if R_j and R_ℓ do not share any border point (they are not neighbours), then* $\mathrm{dist}(R_j, R_\ell) \geq \frac{1}{\sqrt{d}}$.
3. *The projection of each of the sets R_j on the tangent space at its center c_j, after normalizing by the multiplicative factor $\sqrt{d}$, converges to the rectangle $[-1/2, 1/2]^m$ in the sense of Hausdorff metric. That is, the Hausdorff metric of*

$$\left[-\frac{1}{2}, \frac{1}{2}\right]^m \setminus \sqrt{d}\, \phi_{c_j}(R_j)$$

tends to 0 as $d \to \infty$.

Set $w = d^{-1/2}$. We can write $S^m = \bigcup_{j=1}^{k(m,w)} R_j \cup E$, and

$$\pi^Q(\mathcal{V}_{\mathbf{Y}_d}(\mathbf{0})) = \sum_{q\geq Q} \int_{S^m} g_q(\mathbf{Z}_d(t))dt = \sum_{q\geq Q} \left[\sum_j \int_{R_j} g_q(\mathbf{Z}_d(t))dt + \int_E g_q(\mathbf{Z}_d(t))dt \right].$$

By the first item in Proposition 2 and the definition of $\tilde{\mathcal{H}}$ see (5.3) we have

$$\mathbb{E}\left[(\pi^Q(\mathcal{V}_{\mathbf{Y}_d}(\mathbf{0}))^2\right] = \sum_j \sum_\ell \sum_{q\geq Q} \int_{R_j} \int_{R_\ell} \tilde{\mathcal{H}}_{q,d}(\langle s,t\rangle)dsdt + o(d^{r-\frac{m}{2}}).$$

Now, we are interested in covering a strip around the diagonal $\{(s,t) \in S^m \times S^m : \operatorname{dist}(s,t) < aw\}$, $a < 2$. Hence, we restrict the sum in the last equation to the set $\{(j,\ell) : |j-\ell| \leq 2\}$. Clearly the number of sets verifyng this condition is $O(w^{-1}) = O(\sqrt{d})$ and below we prove that the tail of the variance of $(\mathcal{V}(\mathcal{C}_{\mathbf{Y}_d}(\mathbf{0}) \cap R_j))/d^{\frac{r}{2}-\frac{m}{4}}$ is $O(d^{-m/2})$, uniformly in j. Therefore, it remains to bound

$$\begin{aligned}
&\sum_{(j,\ell):|j-\ell|<2} \int_{R_j} \int_{R_\ell} \sum_{q\geq Q} \tilde{\mathcal{H}}_{q,d}(\langle s,t\rangle)dsdt \\
&\qquad = \sum_{(j,\ell):|j-\ell|<2} \mathbb{E}\left[\int_{R_j} \sum_{q\geq Q} g_q(\mathbf{Z}_d(s))ds \cdot \int_{R_\ell} \sum_{q\geq Q} g_q(\mathbf{Z}_d(t))dt \right] \\
&\leq \sum_{(j,\ell):|j-\ell|<2} \left[\sum_{q\geq Q} \int_{R_j\times R_j} \tilde{\mathcal{H}}_{q,d}(\langle s,t\rangle)dsdt \right]^{1/2} \left[\sum_{q\geq Q} \int_{R_\ell\times R_\ell} \tilde{\mathcal{H}}_{q,d}(\langle s,t\rangle)dsdt \right]^{1/2}.
\end{aligned}$$

Here we used Cauchy-Schwarz inequality. Fix j, in order to bound

$$\sum_{q\geq Q} \int_{R_j\times R_j} \tilde{\mathcal{H}}_{q,d}(\langle s,t\rangle)dsdt,$$

we note that it coincides with $\operatorname{Var}\left(\pi^Q(\mathcal{V}(\mathcal{C}_{\mathbf{Y}_d}(\mathbf{0}) \cap R_j))\right)$.

At this point it is convenient to work with caps

$$C(s_0, \gamma r) = \{s : d(s, s_0) < \gamma r\}.$$

Note that by the second item in Proposition 2 each HSR R_j is included in a cap of radius γr for some γ depending on m.

By the invariance under isometries of the distribution of $\mathbf{Y}_d$, the distribution of the volume of the zero set intersected with a cap $C(s_0,\gamma r)$ does not depend on its center s_0. Thus, without loss of generality we work with the cap $C(e_0,\gamma r)$ of angle γr centered at the east-pole $e_0 = (1, 0, \ldots, 0)$, see Nazarov-Sodin [9]. We use the local chart $\phi : C(e_0,\gamma r) \to B(0, \sin(\gamma r)) \subset \mathbb{R}^m$ defined by

$$\phi^{-1}(u) = (\sqrt{1 - \|u\|^2}, u), \quad u \in B(0, \sin(\gamma r)),$$

to project this set over the tangent space . Define the random field $\mathcal{Y}_d : B(0,\gamma) \subset \mathbb{R}^m \to \mathbb{R}^r$, as

$$\mathcal{Y}_d(u) = \mathbf{Y}_d(\phi^{-1}(u/r)).$$

Observe that the ℓ coordinates, $\mathcal{Y}_d^{(\ell)}$ say, of $\mathcal{Y}_d$ are independent. Clearly of the zero set of $\mathbf{Y}_d$ on $R \subset C(e_0,\gamma r)$ and the zero set of $\mathcal{Y}_d$ on $\phi(R/r) \subset B(0,\gamma)$ coincide. That is

$$\mathcal{C}_{\mathbf{Y}_d}(\mathbf{0}) \cap R = \mathcal{C}_{\mathcal{Y}_d}(\mathbf{0}) \cap \phi(R/r).$$

Proposition 3. *The sequence of processes* $\mathcal{Y}_d^{(\ell)}(u)$ *and its first and second order derivatives converge in the finite dimensional distribution sense towards the mean zero Gaussian processes* $\mathcal{Y}_\infty$ *with covariance function* $\Gamma(u,v) = e^{-\frac{||u-v||^2}{2}}$ *and its corresponding derivatives.*

The proof of this proposition can be consulted in [9] and also in [3].

Remark 3. *The local limit process* $\mathcal{Y}_\infty$ *has as coordinates* $(\mathcal{Y}_\infty^{(1)}, \dots, \mathcal{Y}_\infty^{(r)})$ *each of one is an independent copy of the random field with covariance* $\Gamma(u) = e^{-\frac{||u||^2}{2}}$, $u \in \mathbb{R}^m$. *Then its covariance matrix writes*

$$\tilde{\Gamma}(u) = \mathrm{diag}(\Gamma(u), \dots, \Gamma(u)).$$

The second derivative matrix $\tilde{\Gamma}''(u)$ *can be written in a similar way, but here the blocks are equal to the matrix* $\Gamma''(u) = (a_{ij})$ *where* $a_{ij} = e^{-\frac{||u||^2}{2}} H_1(u_i)H_1(u_j)$ *if* $i \neq j$, *and* $a_{ii} = e^{-\frac{||u||^2}{2}} H_2(u_i)$. *We can adapt the Estrade and Fournier [5] result that says that the second moment of the roots in a compact set of such a process exists if for some* $\delta > 0$ *we have*

$$\int_{B(0,\delta)} \frac{||\tilde{\Gamma}''(u) - \tilde{\Gamma}''(0)||}{||u||^m} du = m \int_{B(0,\delta)} \frac{||\Gamma''(u) - I||}{||u||^m} du < \infty.$$

Since Γ *is* C^∞ *we have*

$$||\Gamma''(0) - \Gamma''(u)|| = o(||u||) \text{ as } u \to 0,$$

The convergence of the above integral follows easily using hyperspherical coordinates.

A key fact is that the local limit process, though it can not be defined globally, has the same distribution regardless j. Thus, we bound $\mathrm{Var}\left(\pi^Q(\mathcal{V}(\mathcal{C}_{\mathbf{Y}_d}(\mathbf{0}) \cap R_j)\right)$ uniformly in j by approximating it with the tail of the variance of the volume of the zero set of the limit process $\mathcal{Y}_\infty$ on the limit set $[-1/2, 1/2]^m$.

Proposition 4. *For all* $j \leq k(m,d)$ *and* $\varepsilon > 0$ *there exist* Q_0 *and* d_0 *such that for* $Q \geq Q_0$

$$\sup_{d > d_0} \mathbb{E}\left[(\pi^Q(\mathcal{V}(\mathcal{C}_{\mathbf{Y}_d}(\mathbf{0}) \cap R_j))^2\right] < \varepsilon.$$

Proof. Let $R = R_j \subset C(e_0, \gamma r)$, By Remark 1, the Hermite expansion holds true also for the volume of the zero set of $\mathbf{Y}_d$ on any subset of S^m. Hence,

$$\mathcal{V}(\mathcal{C}_{\mathbf{Y}_d}(\mathbf{0}) \cap R) = \sum_{q=0}^{\infty} d^{\frac{r}{2}} \int_R g_q(\mathbf{Z}_d(t)) dt.$$

Let us define $\tilde{R} = \phi(R) \subset B(0, \sin\frac{a}{\sqrt{d}}) \subset \mathbb{R}^m$. It follows that

$$\mathcal{V}(\mathcal{C}_{\mathcal{Y}_d}(\mathbf{0}) \cap \tilde{R}) = \mathcal{V}(\mathcal{C}_{\mathbf{Y}_d}(\mathbf{0}) \cap R) = \sum_{q=0}^{\infty} d^{\frac{r}{2}} \int_{\tilde{R}} g_q(\mathcal{Y}_d(u), \mathcal{Y}_d'(u)) J_\phi(u) du,$$

being $J_\phi(u) = (1 - \|u\|^2)^{-1/2}$ the jacobian. Rescaling $u = v/\sqrt{d}$ we get

$$\frac{\mathcal{V}(\mathcal{C}_{\mathcal{Y}_d}(\mathbf{0}) \cap \tilde{R})}{d^{\frac{r}{2} - \frac{m}{4}}} = \sum_{q=0}^{\infty} \int_{\sqrt{d}\tilde{R}} g_q\left(\mathcal{Y}_d\left(\frac{v}{\sqrt{d}}\right), \mathcal{Y}_d'\left(\frac{v}{\sqrt{d}}\right)\right) J_\phi\left(\frac{v}{\sqrt{d}}\right) dv.$$

Besides, Rice formula, the domination for $\mathcal{H}_{q,d}$ previously obtained, the convergence of $\mathcal{Y}_d$ to $\mathcal{Y}_\infty$ in Proposition 3 and the convergence, after normalization, of $\bar{R}$ to $[-1/2,1/2]^m$ in Proposition 2 yield

$$\operatorname{Var}\left(\frac{\mathcal{V}(\mathcal{C}_{\mathcal{Y}_d}(\mathbf{0})\cap\tilde{R})}{d^{\frac{r}{2}-\frac{m}{4}}}\right)\underset{d\to\infty}{\to}\operatorname{Var}\left(\mathcal{V}\left(\mathcal{C}_{\mathcal{Y}_\infty}(\mathbf{0})\cap\left[-\frac{1}{2},\frac{1}{2}\right]^m\right)\right). \tag{5.8}$$

In fact, for the second moment we have

$$\begin{aligned}
&\mathbb{E}\left[\left(\frac{\mathcal{V}(\mathcal{C}_{\mathcal{Y}_d}(\mathbf{0})\cap\tilde{R})}{d^{\frac{r}{2}-\frac{m}{4}}}\right)^2\right]\\
&= d^m\int_{\tilde{R}\times\tilde{R}}\mathbb{E}\left[f^r(\mathcal{Y}'_d(u))f^r(\mathcal{Y}'_d(v))\,|\mathcal{Y}_d(u)=\mathcal{Y}_d(v)=0\right]p_{u,v}(0,0)J_\phi(u)J_\phi(v)dudv\\
&=\int_{\sqrt{d}\tilde{R}\times\sqrt{d}\tilde{R}}\mathbb{E}\left[f^r\left(\mathcal{Y}'_d\Big(\frac{u}{\sqrt{d}}\Big)\right)f^r\left(\mathcal{Y}'_d\Big(\frac{v}{\sqrt{d}}\Big)\right)|\mathcal{Y}_d\Big(\frac{u}{\sqrt{d}}\Big)=\mathcal{Y}_d\Big(\frac{v}{\sqrt{d}}\Big)=0\right]\\
&\qquad\qquad\times p_{u,v}(0,0)J_\phi\Big(\frac{u}{\sqrt{d}}\Big)J_\phi\Big(\frac{v}{\sqrt{d}}\Big)dudv\\
&\underset{d\to\infty}{\to}\int_{([\frac{1}{2},\frac{1}{2}]^m)^2}\mathbb{E}\left[f^r(\mathcal{Y}'_\infty(u))f^r(\mathcal{Y}'_\infty(v))\mid\mathcal{Y}_\infty(u)=\mathcal{Y}_\infty(v)=0\right]p_{\mathcal{Y}_\infty(u),\mathcal{Y}_\infty(v)}(0,0)dudv\\
&\qquad=\mathbb{E}\left[\left(\mathcal{V}\left(\mathcal{C}_{\mathcal{Y}_\infty}(\mathbf{0})\cap\left[-\frac{1}{2},\frac{1}{2}\right]^m\right)\right)^2\right]<\infty.
\end{aligned}$$

The term of the square of the expectation is easier.

The same arguments show that for all q we have

$$\begin{aligned}
V_{q,d}^{loc}&:=\operatorname{Var}\Big(\pi_q\Big(\frac{\mathcal{V}(\mathcal{C}_{\mathcal{Y}_d}(\mathbf{0})\cap\tilde{R})}{d^{\frac{r}{2}-\frac{m}{4}}}\Big)\Big)\\
&\underset{d\to\infty}{\to}\operatorname{Var}\left(\pi_q\left(\mathcal{V}\left((\mathcal{C}_{\mathcal{Y}_\infty}(\mathbf{0}))\cap\left[-\frac{1}{2},\frac{1}{2}\right]^m\right)\right)\right)=:V_q^{loc}.
\end{aligned}$$

Thus, for all Q it follows that $\sum_{q=0}^Q V_{q,d}^{loc}\to_{d\to\infty}\sum_{q=0}^Q V_q^{loc}$. By Parseval's identity, (5.8) can be written as

$$\sum_{q=0}^\infty V_{q,d}^{loc}\underset{d\to\infty}{\to}\sum_{q=0}^\infty V_q^{loc}.$$

Thus, by taking the difference we get

$$\sum_{q>Q}V_{q,d}^{loc}\underset{d\to\infty}{\to}\sum_{q>Q}V_q^{loc}. \tag{5.9}$$

Given that the series $\sum_{q=0}^\infty V_q^{loc}$ is convergent, we can choose Q_0 such that for $Q\geq Q_0$ it holds $\sum_{q>Q}^\infty V_q^{loc}\leq\varepsilon/2$. Hence, for this Q_0 and by using (5.9) we can choose d_0 such that for all $d>d_0$ and $Q\geq Q_0$

$$\sum_{q>Q}V_{d,q}^{loc}\leq\varepsilon.$$

Namely, there exists d_0 such that for $Q\geq Q_0$

$$\sup_{d>d_0}\mathbb{E}\left[\left(\pi^Q\left(\frac{\mathcal{V}(\mathcal{C}_{\mathcal{Y}_d}(\mathbf{0})\cap\tilde{R})}{d^{\frac{r}{2}-\frac{m}{4}}}\right)\right)^2\right]<\varepsilon.$$

□

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