AIP Journal of Mathematical Physics

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Citation: J. Math. Phys. **53**, 012501 (2012); doi: 10.1063/1.3668045
View online: http://dx.doi.org/10.1063/1.3668045
View Table of Contents: http://jmp.aip.org/resource/1/JMAPAQ/v53/i1
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JOURNAL OF MATHEMATICAL PHYSICS **53**, 012501 (2012)

Static solutions from the point of view of comparison geometry

Martin Reiris[a]
Max Planck Institute für Gravitationasphysik, Albert Einstein Institute, Germany

(Received 27 April 2011; accepted 2 November 2011; published online 5 January 2012)

We analyze (the harmonic map representation of) static solutions of the Einstein equations in dimension three from the point of view of comparison geometry. We find simple monotonic quantities capturing sharply the influence of the Lapse function on the focussing of geodesics. This allows, in particular, a sharp estimation of the Laplacian of the distance function to a given (hyper)-surface. We apply the technique to asymptotically flat solutions with regular and connected horizons and, after a detailed analysis of the distance function to the horizon, we recover the Penrose inequality and the uniqueness of the Schwarzschild solution. The proof of this last result does not require proving conformal flatness at any intermediate step. © *2012 American Institute of Physics*. [doi:10.1063/1.3668045]

I. INTRODUCTION

In this article, we introduce a family of quantities, denoted by $\mathcal{M}_a$ (where a, an arbitrary real number, is the parameter of the family) naturally attached to (integrable) geodesic congruences $\mathcal{F}$, of static solutions of the Einstein equations in dimension three. The invariants (which can be seen as a real functions over the range of the congruence) are shown to be monotonic along each of the geodesics of $\mathcal{F}$. Moreover, whenever $\mathcal{M}_a$ is stationary along a geodesic γ of $\mathcal{F}$, then the local geometry along γ can be seen to be of Schwarzschild form. In this sense, $\mathcal{M}_a$ measures a certain departure of the given static solution to the Schwarzschild solution. The framework that we will develop out of these invariants is a natural extension of the standard comparison techniques of Riemannian spaces of non-negative Ricci curvature. However, as we incorporate into $\mathcal{M}_a$ the influence that the lapse exerts on the Ricci curvature and, as a result, the monotonicity of $\mathcal{M}_a$ sharply captures the departure from the Schwarzschild solution (not from the Euclidean space), the framework here developed can be best described as one that compares static solutions to the Schwarzschild solution. It is thus not peculiar that when the technique is applied to asymptotically flat static solutions with regular and connected horizons, the uniqueness of the Schwarzschild solution is achieved with remarkable naturalness. It is worth noting that the novel proof of this central result in general relativity that we shall provide does not require the intermediate step of proving conformal flatness of previous proofs. The ideas that we will describe can be interpreted as partial results on the bigger proposal of developing a more complex comparison theory for static solutions in arbitrary dimensions.

Before continuing with the description of the contents, we briefly introduce static solutions of the Einstein equations and summarize some properties that would place the contents into an adequate perspective.

A. Elements of static solutions

A static solution of the Einstein equations in dimension three is given by a triple (Σ, g, N) where Σ is a smooth Riemannian three manifold possibly with boundary, g is a smooth Riemannian metric

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[a] Electronic mail: martin@aei.mpg.de.

and N, the *Lapse Function*, is a smooth function, strictly positive in int(Σ), and satisfying

$$N Ric = \nabla\nabla N, \tag{1}$$

$$\Delta N = 0. \tag{2}$$

These equations, note, are invariant under simultaneous but independent scalings on g and N.

Remark 1: In this article, we will restrict to dimension three. Our most important invariant, the quantity $\mathcal{M}$ (see later), is monotonic only in dimension three and we do not know, at the moment, a replacement of it to higher dimensions. The static Einstein equations (1) and (2) are valid in any dimension.

The description of static solutions is better separated into local and global properties. From the local point of view, the geometry of static solutions is controlled in C^∞ by two weak invariants. This is a direct consequence of Anderson's curvature estimates[1] (applying in dimension three) which are described as follows. Let (Ω, g, N) be a static solution of the Einstein equations, where (Ω, g) is a complete Riemannian manifold with or without boundary. Then there is a universal constant $K > 0$ such that for any $p \in \Omega$ we have

$$|Rm| + |\nabla \ln N|^2 \leq \frac{K}{dist(p, \partial\Omega)^2}, \tag{3}$$

where if $\partial\Omega = \emptyset$ we set $dist(p, \partial\Omega) = \infty$. Note that this shows, in particular, that the only complete and boundary-less static solution in dimension three is covered (after normalizing N to one) by the trivial solution ($\mathbb{R}^3$, $g_{\mathbb{R}^3}$, $N = 1$). Anderson's curvature estimates together with the Bishop-Gromov volume comparison and standard elliptic estimates imply the following *interior estimates* for static solutions in dimension three.

Lemma 1 (Interior's estimates (Anderson)): Let Ω be a closed three-dimensional manifold with non-empty boundary $\partial\Omega$. Suppose that (Ω, g, N) *is a static solution of the Einstein equations. Let $p \in \Omega$, let $d = dist(\partial\Omega)$, and let $V_1 = Vol(B(p, d_1))$ for $d_1 < d$. Then there is $d_2(d, d_1, V_1) >$ 0, and for any $i \geq 0$ there are $\Lambda(d, d_1, V_1, i) > 0$, $I(i, d_1, V_1) > 0$, such that in $j(p) \geq I$ and $\|\nabla^i Rm\|_{L^\infty_g(B(p,d_2))} \leq \Lambda$.*

These interior estimates, in turn imply, as are well known, the control of the $C^i_{\{x_j\}}$ *norm of the entrances g_{ij}* of g, *in suitable harmonic coordinates* $\{x_j\}$ covering $B(p, d_2)$, *and from them pre-compactness statements can be obtained.*

The global geometry of static solutions instead is greatly influenced by boundary conditions and, in many cases, boundary conditions provide uniqueness. This occurs when, for instance, one assumes that $\partial\Sigma$ consist of a finite set of *regular horizons* plus further hypothesis on the asymptotic of (Σ, g) at infinity. We will adopt the following definition (see Ref. 1).

Definition 1: The boundary $\partial\Sigma$ of the smooth manifold Σ is a regular horizon iff $\partial\Sigma$ is a finite union of compact (boundary-less) surfaces H_i, $i = 1, \ldots n$, $\partial\Sigma = \{q/N(q) = 0\}$ and at each H_i we have $|\nabla N||_{H_i} > 0$.

It follows easily from the static equations (1) and (2) that every regular horizon $\partial\Sigma$ is totally geodesic and $|\nabla N|$ is constant and different from zero on each component.

Perhaps the easiest examples of complete solutions with regular horizons are the *Flat solutions* that we will denote by the triple (Σ_F, g_F, N_F). They have the presentation

$$\Sigma_F = [0, \infty) \times T^2, \; N_F = r, \; g_F = dr^2 + h_F, \tag{4}$$

where h_F is a flat metric in T^2. The family is parameterized by the set of flat metrics in T^2 (non-isometric). Note that we have demanded that N grows linearly with respect to arc length and with slope one. Of course any N that grows linearly can be scaled to have growth of slope one.

Yet, the prototypical and central examples of static metrics are the Schwarzschild solutions. Recall, the Schwarzschild solution (Σ_N, g_S, N_S) of mass $m \geq 0$ has the presentation

$$\Sigma_S = [2m, \infty) \times S^2,\ N_S = \sqrt{1 - \frac{2m}{r}},\ g_S = dr^2 + r^2(1 - \frac{2m}{r})d\Omega^2, \tag{5}$$

while if $m < 0$ the presentation

$$\Sigma_S = (0, \infty) \times S^2,\ N_S = \sqrt{1 - \frac{2m}{r}},\ g_S = dr^2 + r^2(1 - \frac{2m}{r})d\Omega^2. \tag{6}$$

The "uniqueness of the Schwarzschild solution," as in known today and in the form presented below, came as the result of several efforts, starting from the seminal work of Israel in 1967. For the history of the developments which lead to the proof of this important result as well as accurate references we refer to the article.[8]

Theorem 1: (Schwarzschild's uniqueness, Refs. 5, 7, and 4): *Let* (Σ, g, N) *be a static solution of the Einstein equations of dimension three. Suppose it is asymptotically flat (with one end) and with regular, possibly empty, and possibly disconnected horizon* $\partial\Sigma$. *Then the solution is a Schwarzschild solution of non-negative mass.*

Several hypothesis of this theorem can be relaxed still obtaining the same uniqueness outcome. For instance, suppose there is one end but the hypothesis of asymptotic flatness, or even the topological nature of the end, is withdrawn, then results exist showing that the solution is still one of the Schwarzschild families of positive mass. In particular, when $N \leq N_0 < \infty$ but nothing of the end is known, not even the *a priori* topology, then it can be shown that the solution is indeed a Schwarzschild solution. This follows from a combination of results. First, observe that N cannot go uniformly to zero over the end, for in such case, as N is harmonic and is zero over the horizon, we would violate the maximum principle. Using the notation in Ref. 1 denotes by $t(p)$ the g-distance from a point p to the horizon H. Denote also by $B(H, \bar{t})$ the ball of center H and radius $\bar{t}$, namely, $B(H, \bar{t}) = \{p/t(p) < \bar{t}\}$. Now, from Theorem 0.3 (ii) in Ref. 1, either the end is asymptotically flat or small in the sense that $\int^{\infty} \frac{1}{A(\partial B(H,\bar{t}))} d\bar{t} = \infty$. Assume $N \leq N_0$. Consider $f = N_0 + 1 - N$. Then $\Delta f = 0$ and $\Delta \ln f = -|\nabla \ln f|^2$. Define $F(t) := \int_{B(H,t)\setminus B(H,\bar{t}_1)} |\nabla \ln f|^2 dV$. For t_1 small, we have $\int_{\partial B(H,t_1)} g(\nabla \ln f, n_{in})dA > 0$, where n_{in} is the unit normal to $\partial B(H, t_1)$ pointing inwards to the ball. Using this fact, integrating $\Delta \ln f = -|\nabla \ln f|^2$ over $B(H, t) \setminus B(H, \bar{t}_1)$ and using Cauchy-Schwarz one easily deduce the inequality $f'/F^2 \geq 1/A$. From it one gets $1/F(t) \leq 1/F(t_2) - \int_{t_2}^{t} \frac{1}{A} d\bar{t}$, where $t_2 > t_1$. Thus if the end is small, one would get $F = \infty$ at a finite distance form H, which is not possible. The same occurs when it is known that outside a compact set, each end is homeomorphic to $\mathbb{R}^3$ minus a ball[9] and over there the metric $N^2 g$ is complete, which occurs, for example, when $N \geq N_0 > 0$. In all these generalizations, which are important for deeper understanding of Einstein's theory, it is assumed that the space (Σ, g), as a metric space, is complete.

We feel that the following broader conjecture may be accessible.

Conjecture 1: Let (Σ, g, N) *be a complete solution of the static Einstein equations with regular but possibly disconnected (non-empty) horizon* $\partial\Sigma$. *Suppose that the conformal metrics* $N^2 g$ *and* $N^{-2} g$ *are complete outside given domains of compact closure on each end of* (Σ, g). *Then the solution is either a Schwarzschild solution or a flat solution.*

Observe that no assumption is made on the topology of the ends.

When boundary data are prescribed, and are not the data of a regular horizon, and the hypothesis of asymptotic flatness is kept, then much less is known about the existence of solutions although a conjecture[2] and partial results do exist[10] under some hypothesis. In whatever case, Dirichlet-type of problems for the Einstein equations are interesting from physical and mathematical reasons. A theory, a highly necessary task, is still lacking.

The Schwarzschild family is unique but why? Are the present proofs satisfactory as an answer to this question? Do we need to place the problem of the uniqueness of the Schwarzschild family into a larger one to understand it better? Which one would be that bigger perspective? Could it be a Dirichlet-type of theory for the static Einstein equations? Despite all the accumulated knowledge, some aspects of the uniqueness of the Schwarzschild solutions remain (to us) somehow mysterious. The present work would try to clarify the phenomenon from the perspective of comparison geometry. Finally, it is worth remarking that there are yet further reasons of why it is important to have different proofs and points of view regarding Theorem 1. Just mention that the elusive and yet inconclusive notions of localized energy or the even more conjectural notion of entropy may have to do and could be better clarified with different understandings of the Schwarzschild uniqueness.

B. $\mathcal{M}_a$ and comparison geometry

The idea underlying the technique that we will describe is rather simple. First, and most important, we will work in the *harmonic map representation* of static solutions. Namely, instead of working with the variables (g, N), we will work with the variables $(\mathtt{g}, \mathtt{N}) = (N^2 g, \ln N)$. The Einstein equations (1) and (2) now become

$$\mathtt{Ric} = 2d\mathtt{N} \otimes d\mathtt{N}, \tag{7}$$

$$\Delta \mathtt{N} = 0. \tag{8}$$

It is apparent from here that $\mathtt{Ric} \geq 0$, which is a quite central property. Consider now a congruence of geodesics (or geodesic segments) $\mathcal{F}$ for the metric g minimizing the distance from any of their points to a (hyper)-surface $\mathcal{S}$. Thus, any geodesic in $\mathcal{F}$ has an initial point in $\mathcal{S}$. We will assume the geodesics (or geodesic segments) are inextensible beyond their last point or that the last point is the point on γ where γ stops to be length minimizing to $\mathcal{S}$. It can be that such last point does not exist in which case the geodesic "ends" at "infinity." It is known that the *Cut locus* $\mathcal{C}$, namely, the set of last points of the geodesics in the congruence is a closed set of measure zero. Outside $\mathcal{C}$ the distance function to $\mathcal{S}$ is a smooth function with gradient of norm one. Given a point p in Σ, we will denote by $\mathtt{s}(p)$ the distance from p to $\mathcal{S}$. Consider now a point p, not in $\mathcal{C}$ and not in $\mathcal{S}$ and around it consider the smooth surface formed by the set of points which have the same distance to $\mathcal{S}$ than p (the equidistant surface or the level set of the distance function). The second fundamental form of such surface in the outgoing direction (from $\mathcal{S}$) at p will be denoted by $\Theta(p)$ or simply Θ. The mean curvature will be $\theta(p) = tr_{h(p)}\Theta(p)$ where $tr_{h(p)}\Theta(p)$ means the trace of $\Theta(p)$ with respect to the induced two-metric in the surface or level set. Thus, we can think θ as a function along geodesics γ in $\mathcal{F}$. The mean curvature satisfies the important *focussing equation* or *Riccati equation* along the geodesics γ,

$$\theta' = -|\Theta|^2 - \mathtt{Ric}(\gamma', \gamma') = -\frac{\theta^2}{2} - \mathtt{Ric}(\gamma', \gamma') - |\hat{\Theta}|^2. \tag{9}$$

Above, $'$ denotes derivative with respect to arc length and $\hat{\Theta}$ is the traceless part of Θ. Recall that

$$\Delta \mathtt{s} = \theta.$$

Thus any estimate on θ obtained out of the focussing equation serves as an estimate on the Laplacian of the distance function.

For instance, if $\mathtt{Ric} \geq 0$, then standard estimates in comparison theory follow by discarding the last two terms in Eq. (9) and integrating the inequality $\theta' \leq -\theta^2/2$. If $\mathtt{s}$ is the distance function to a point or, the same, the distance function to the boundary of a small geodesic ball plus the radius of the ball, one gets (Calabi 1958,[11]), $\theta \leq 2/\mathtt{s}$ and

$$\Delta \mathtt{s} \leq 2/\mathtt{s},$$

everywhere and in the barer sense (Ref. 6, p. 262). Comparison estimates on areas and volumes of geodesic balls are obtained from

$$\frac{dA'}{dA} = \theta, \quad dV' = dA,$$

where dA is the element of area of the equidistant surfaces to $\mathcal{S}$ and dV is the element of volume enclosed by dA.

The situation we face is similar in that the Ricci curvature is non-negative, but this time the structure of the Ricci curvature is explicitly given. By incorporating `Ric` as part of the focussing inequality, namely, considering

$$\theta' \leq -\frac{\theta^2}{2} - 2\mathtt{N}'^2,$$

we will obtain a sharp estimate for θ. We will show that for any real number a the quantity

$$\mathcal{M}_a = (\frac{\theta}{2}(\mathtt{s}+a)^2 - (\mathtt{s}+a))N^2$$

is monotonically decreasing (Proposition 1) along any geodesic of the congruence and is stationary if and only if the geometry along the geodesic is of Schwarzschild form (Proposition 2). Thus, we get the estimate

$$\theta \leq \frac{2}{\mathtt{s}+a}(1 + \frac{\mathcal{M}_0}{(\mathtt{s}+a)^2 N^2}),$$

where $\mathcal{M}_0$ is the value of $\mathcal{M}_a$ at the start, on $\mathcal{S}$, of the geodesic. The fundamental set of equations out of which comparison estimates can be obtained is therefore

$$\theta \leq \frac{2}{\mathtt{s}+a}(1 + \frac{\mathcal{M}_0}{(\mathtt{s}+a)^2 N^2}), \tag{10}$$

$$\Delta \mathtt{s} = \theta,\ \frac{dA'}{dA} = \theta,\ dV' = dA, \tag{11}$$

$$\Delta \ln N = 0. \tag{12}$$

To use these set of equations efficiently one must first use the system

$$\Delta \mathtt{s} \leq \frac{2}{\mathtt{s}+a}(1 + \frac{\mathcal{M}_0}{(\mathtt{s}+a)^2 N^2}),$$

$$\Delta \ln N = 0,$$

together with additional boundary data on N and $\mathcal{M}_0$. For the case of the application to the uniqueness of the Schwarzschild solutions, that we carry out later, the substantial information that is extracted out of this system is, in a sense, concentrated in Theorem 2, where a distance comparison result is established between $\mathtt{s}$ and $\hat{\mathtt{s}} = 2mN^2/(1-N^2)$.

From the point of view of areas and volumes comparisons, we note that, by using Eqs. (10)–(12), the expression

$$\frac{dA}{dA_0} exp(\int_{\mathtt{s}_0}^{\mathtt{s}} \frac{2}{\bar{\mathtt{s}}+a}(1 + \frac{\mathcal{M}_0}{\bar{\mathtt{s}}+a)^2 N^2}) d\bar{\mathtt{s}})$$

is seen to be monotonically decreasing too. From it and $dV' = dA$ suitable information on the growth of areas and volumes of geodesic balls (with center $\mathcal{S}$) can be obtained. These types of estimates will play an important role in the proof of the uniqueness of the Schwarzschild solutions in Sec. II.

Yet, the structure of the harmonic-map representation of the Einstein equations is richer than the information contained in the system (10)–(12). Indeed, Weitzenböck's formula for the static equations

$$\frac{1}{2}\Delta|\nabla f|^2 = |\nabla\nabla f|^2 + < \nabla\Delta f, \nabla f > +2 < \frac{\nabla N}{N}, \nabla f >^2,$$

valid for any function f, together with Eq. (12) can provide useful estimates on functions of the form $f = f(N)$. They, in turn, provide useful information on N. These estimates, are worth remarking, have nothing to do with the distance function. The most obvious consequence of Weitzenböck's formula comes out when we chose $f = \ln N$. In this case, we obtain

$$\frac{1}{2}\Delta|\nabla \ln N|^2 = |\nabla\nabla \ln N|^2 + 2|\nabla \ln N|^2.$$

In applications to the uniqueness of the Schwarzschild solutions, we will use, however, the Weitzenböck formula with the choice $f = \hat{\mathtt{s}} = 2mN^2/(1-N^2)$. This will provide the important estimate $|\nabla\hat{\mathtt{s}}| \leq 1$ in Sec. I, which, as we will see, it is necessary to close up the proof of the uniqueness of the Schwarzschild solutions.

It is worth remarking at this point that many of the techniques here developed carry over the much bigger family of metrics and potentials satisfying

$$\mathtt{Ric} \geq 2d\mathtt{N} \otimes d\mathtt{N},$$

$$\Delta\mathtt{N} \geq 0, \quad 0 < N < 1.$$

To show the applicability of Eqs. (10) and (11), as we said before, we will fully analyze from this perspective asymptotically flat static solutions with regular and connected horizons and recover Theorem 1. It is worth remarking the naturalness from which the uniqueness of the Schwarzschild solutions will come out of these comparison techniques. Despite of that, the required analysis will be somewhat extensive. To prove Theorem 1, we carefully compare the distance function to the horizon, $\mathtt{s}$, to the function $\hat{\mathtt{s}}$, through the set of Eqs. (10) and (11). The final goal to achieve is to show the equality $\mathtt{s} = \hat{\mathtt{s}} = 2mN^2/(1-N^2)$ from which it will follow that $\mathcal{M}_{2m}$ has to be stationary along any length-minimizing geodesic to H (in this case the integral lines of $\nabla\hat{\mathtt{s}}$) and equal to m. It then follows, from the sharpness of the monotonicity of $\mathcal{M}$ that the solution has to be a Schwarzschild solution (of positive mass). Note a technical aspect however. As $N = 0$ over the horizon H, the metric $\mathtt{g} = N^2 g$ is singular there. Although this will make the analysis technically delicate, a satisfactory remedy is found if one replaces H by a sequence $H_{\Gamma_i} = \{N = \Gamma_i\}$ of the Γ_i-level set of N ($\Gamma_i \downarrow 0$), approaching H, and perform then a limit analysis. This circumvention of the singularity at the horizon will appear often in the reasonings.

It is worth noting that, at the moment, we do not know how to obtain Theorem 1, when the horizon is not connected. The exact reproduction of the arguments that leads to the proof of Theorem 1 for connected horizons, applied to the case of non-connected horizons, give interesting results, which are not difficult to obtain but that will not be given here.

We will now give guidelines of the structure of the article. In Sec. II, we introduce and discuss the main monotonic quantity $\mathcal{M}$, give explicit examples of the monotonicity and discuss the stationary case. This section is the core of the article. The other sections discuss further properties of $\mathcal{M}$ and applications. In particular, in Sec. III, we start the discussion of asymptotically flat solutions with regular and connected horizons. In Sec. III A, we study $\mathcal{M}$ over regular horizons. In Sec. III B, we recall the notion of asymptotic flatness and cite a classical result[3] on the possibility to chose special coordinates at infinity in static solutions displaying precisely the Schwarzschild-type of fall off. The existence of such coordinate system $\{\bar{x}\}$ will be central. In Sec. III C, we introduce the important notion of *coordinate-distance* lag, measuring a mismatch between the distance from a point p to the horizon, $\mathtt{s}(p)$, and the coordinate distance $|\bar{x}(p)|$. In Sec. III D, we discuss our first substantial result. We prove a *distance comparison* result (Theorem 2) between $\mathtt{s}$ and $\hat{\mathtt{s}} = 2mN^2/(1-N^2)$. To achieve it, we must show first that the inequality

$$\Delta\mathtt{s} \leq \frac{2}{\mathtt{s}+a}\left(1 + \frac{\mathcal{M}_0}{(\mathtt{s}+a)^2N^2}\right)$$

holds in a barer sense all over the manifold Σ. This is done in Proposition 10. Without that tool, the comparison result would not be possible to achieve. Using that we show in Sec. III E that the Penrose inequality $A \leq 16\pi m^2$, where A is the area of the connected horizon and m is the add (ADM) mass, must hold. In Sec. III F we show, again using the distance comparison, that the opposite Penrose inequality must hold, namely, $A \geq 16\pi m^2$. Thus, after Sec. III F we would have proved that $A = 16\pi m^2$. Despite the strong implications of this inequality, the uniqueness of the Schwarzschild solutions requires further analysis. This is carried out in Sec. III G Indeed it is in this section that it is proved that $\mathtt{s} = \hat{\mathtt{s}}$. This follows from a further study of the coordinate-distance lag in Sec. I where it is shown that it must be zero. An elaboration of the area and volume comparison in Sec. II finishes the analysis of all the elements of the proof which is summarized in Theorem 3. Further explanations on the contents and strategies are given at the beginning of the each section.

We will use alternatively the notation (Σ, g, N) or $(\Sigma, \mathrm{g}, \mathtt{N}) = (\Sigma, \mathrm{g}, \ln N)$, used according to which representation is best suited to describe a claim or a statement. When we say that $(\Sigma, \mathrm{g}, \ln N)$ is an asymptotically flat static solution with regular and connected horizons, we mean that (Σ, g, N) is an asymptotically flat static solution with regular horizon as was described before, but that we will working in its harmonic map representation.

II. A COMPARISON APPROACH TO STATIC SOLUTIONS IN THE HARMONIC MAP REPRESENTATION

Let $(\Sigma, \mathrm{g}, \ln N)$ be a static solution in the harmonic representation. To every oriented integrable congruence $\mathcal{F}$ of g-geodesics, we will associate a family of real functions $\{\mathcal{M}_a, a \in \mathbb{R}\}$ defined over the range of $\mathcal{F}$. We will show that, fixed a, $\mathcal{M}_a(\gamma(\mathtt{s}))$ is monotonically decreasing for any $\gamma \in \mathcal{F}$ ($\mathtt{s}$ is the g-arc length, increasing in the positive direction). This central fact will follow by making use of the focusing equation (9). The definition of $\mathcal{M}_a$ and the proof of its monotonicity are given in the proposition below. To avoid excessive notation we will use the following convention in the notation: for every function f defined over the range of $\mathcal{F}$ (for example, $f = \theta$ or $f = N$) and $\gamma \in \mathcal{F}$ we will write $f := f(\gamma(s))$ and $df(\gamma(s))/ds := df/ds := f'$. Also, for the same reason of economy and simplicity, we will suppress the sub-index a and write simply $\mathcal{M}$.

Proposition 1: Let $\mathcal{F}$ be an oriented integrable congruence of geodesics. Let $\gamma(\mathtt{s})$, $\mathtt{s} \in [\mathtt{s}_0, \mathtt{s}_1]$ be a geodesic in $\mathcal{F}$. Let a be a real number and let $\tilde{\mathtt{s}} = a + \mathtt{s}$. Then we have

$$((\frac{\theta}{2}\tilde{\mathtt{s}}^2 - \tilde{\mathtt{s}})N^2)' = -\tilde{\mathtt{s}}^2 N^2 \frac{|\hat{\Theta}|^2}{2} - (\tilde{\mathtt{s}}\frac{\theta}{2} - 1 - \tilde{\mathtt{s}}\frac{N'}{N})^2 N^2. \tag{13}$$

Therefore, fixed any real number a, the quantity $\mathcal{M} = (\frac{\theta}{2}\tilde{\mathtt{s}}^2 - \tilde{\mathtt{s}})N^2$ is monotonically decreasing along any $\gamma \in \mathcal{F}$ (the notation $\mathcal{M}$ accounts for "mass").

Proof: We compute

$$((\frac{\theta}{2}\tilde{\mathtt{s}}^2 - \tilde{\mathtt{s}})N^2)' = \frac{\theta'}{2}\tilde{\mathtt{s}}^2 N^2 + \theta\tilde{\mathtt{s}}N^2 - N^2 + \theta\tilde{\mathtt{s}}^2 NN' - \tilde{\mathtt{s}}2NN'.$$

We use now the focusing equation (9) to get

$$((\frac{\theta}{2}\tilde{\mathtt{s}}^2 - \tilde{\mathtt{s}})N^2)' = -\frac{|\hat{\Theta}|^2}{2}\tilde{\mathtt{s}}^2 N^2 - \frac{\theta^2}{4}\tilde{\mathtt{s}}^2 N^2 - \tilde{\mathtt{s}}N'^2 + \theta\tilde{\mathtt{s}}N^2 - N^2 + \theta\tilde{\mathtt{s}}^2 NN' - \tilde{\mathtt{s}}2NN'.$$

The six terms following the first on the right-hand side of this expression can be arranged as $-(\tilde{\mathtt{s}}\theta/2 - 1 - N'/N)^2 N^2$, thus obtaining (13). □

Example 1 (The Schwarzschild case): Consider a Schwarzschild metric of mass m, of arbitrary sign, in the presentations, according to the sign of the mass, of Eqs. (5) or (6). Note that $\mathrm{g} = dr^2 + r^2(1 - 2m/r)d\Omega^2$ and $N^2 = (1 - 2m/r)$. For any given point q in S^2 considers the ray $[2m, \infty) \times \{q\}$ (if $m \geq 0$) or $(o, \infty) \times \{q\}$ (if $m < 0$) parameterized by the arc length $\mathtt{s} = r - 2m$ of $\mathtt{s} = r$ (respectively, to the sign of the mass). In either case we compute the mean curvature θ as

$$\theta = \frac{2}{r} + \frac{2m}{r(r - 2m)} = \frac{2}{r}\frac{(r - m)}{r - 2m}. \tag{14}$$

Let $b = a - 2m$ if $m \geq 0$ and $b = a$ if $m < 0$, then the quantity $\mathcal{M}$ has the following form:

$$\mathcal{M} = (\frac{1}{r}\frac{r - m}{r - 2m}(r + b)^2 - (r + b))(1 - \frac{2m}{r}) = ((m + b) - \frac{mb}{r})(1 + \frac{b}{r})$$

independently of the sign of the mass. Taking the derivative with respect to arc length and rearranging terms we obtain

$$\mathcal{M}' = \frac{b^2}{r}(\frac{2m}{r} - 1),$$

which is explicitly non-positive independently of the sign of the mass. This shows the monotonicity of $\mathcal{M}$ for any value of b. Note that $lim_{\mathrm{s}\to\infty}\mathcal{M} = m + b$. Observe too that when $b = 0$, i.e., $a = 2m$, then $\mathcal{M}$ is constant and equal to m. □

Example 2 (*The flat solutions*): For a flat solution (Σ_F, g_F, N_F), we have $\mathrm{g} = r^2dr^2 + r^2h_F$. Making $r^2/2 = s$ we get $\mathrm{g} = ds^2 + 2sh_F$ and $N^2 = 2s$, where $s > 0$. For any point p in T^2 consider the ray $[0, \infty) \times p$. Consider the congruence of geodesics conformed by all these rays. The mean curvature is calculated as $\theta = 1/s$. Thus for any real number a we have

$$\mathcal{M} = (\frac{1}{2s}(s+a)^2 - (s+a))2s = -s^2 + a^2, \tag{15}$$

which is monotonically decreasing in the domain of s, namely, $(0, \infty)$.

Note that for the "dual" solution $(\Sigma_F, g_F, 1/N_F)$ we have, for any real number a, the expression $\mathcal{M} = -1/4 + a^2/s^2$ which is monotonically decreasing in the domain of s, namely, $(0, \infty)$. Note that when $a = 0$ then $\mathcal{M}$ is stationary and equal to $-1/4$. □

The next proposition discusses the case when $\mathcal{M}_a$ is stationary.

Proposition 2: Let $\mathcal{F}$ be an oriented and integrable congruence of geodesics. When, for a given a, $\mathcal{M}$ is constant along a geodesic segment $\gamma(\mathrm{s})$, $\mathrm{s} \in [\mathrm{s}_1, \mathrm{s}_2]$, then along γ we have

$$\hat{\Theta} = 0 \tag{16}$$

and

$$N^2 = N_0^2 + 2\frac{\mathcal{M}_0}{\mathrm{s}_0 + a} - 2\frac{\mathcal{M}_0}{\mathrm{s} + a}, \tag{17}$$

where N_0 and $\mathcal{M}_0$ are the values of N and $\mathcal{M}$ at $\mathrm{s} = \mathrm{s}_0 \in (\mathrm{s}_1, \mathrm{s}_2)$. We also obtain

$$\theta = \frac{2}{\mathrm{s} + a} + 2\frac{\mathcal{M}_0}{(\mathrm{s} + a)^2N^2}. \tag{18}$$

Proof: If along a geodesic γ the value of $\mathcal{M}$ remains constant, then the right-hand side of (13) must be identically zero. This implies that

$$\hat{\Theta} = 0,$$

which shows (16) and also implies that

$$\tilde{\mathrm{s}}\frac{\theta}{2} - 1 - \tilde{\mathrm{s}}\frac{N'}{N} = 0.$$

Multiply now this expression by $\tilde{\mathrm{s}}$ and rearrange it as

$$\tilde{\mathrm{s}}^2\frac{\theta}{2} - \tilde{\mathrm{s}} = \tilde{\mathrm{s}}^2\frac{N'}{N}. \tag{19}$$

Recall that $\mathcal{M} = (\tilde{\mathrm{s}}^2\frac{\theta}{2} - \tilde{\mathrm{s}})N^2$. Using this expression, Eq. (19), and (because we are assuming that $\mathcal{M}$ is constant) writing $\mathcal{M} = \mathcal{M}_0 = \mathcal{M}(\mathrm{s}_0)$, we obtain

$$\mathcal{M}_0 = \tilde{\mathrm{s}}^2NN' = \tilde{\mathrm{s}}^2\frac{(N^2)'}{2}.$$

Moving $\tilde{\mathtt{s}}^2$ to the denominator of the left-hand side and integrating (in $\mathtt{s}$) from $\mathtt{s} = \mathtt{s}_0$ to $\mathtt{s}$ we obtain (17). To obtain (18) solve for θ in $(\tilde{\mathtt{s}}^2\frac{\theta}{2} - \tilde{\mathtt{s}})N^2 = \mathcal{M}_0$. □

Remark 2 (Further remarks to Proposition 2): Observe form Proposition (2) that (if for some number a) $\mathcal{M}$ is constant along a geodesic γ of infinite length and $lim_{\mathtt{s}\to\infty} N(\gamma(\mathtt{s})) = 1$, then making the change of variables $r = \mathtt{s} + a$ in (17) and (18) we obtain, along γ, the expressions

$$N^2(r) = 1 - \frac{2\mathcal{M}_0}{r},$$

$$\theta = \frac{2}{r} + \frac{2\mathcal{M}_0}{r(r - 2\mathcal{M}_0)} = \frac{2}{r}\frac{(r - \mathcal{M}_0)}{r - 2\mathcal{M}_0},$$

and including (16)

$$\hat{\Theta} = 0,$$

which, comparing with Example 1, are exactly of Schwarzschild form if we identify $\mathcal{M}_0$ with "a" ADM mass m. Moreover, if γ is defined on $(\mathtt{s}_0 = 0, \infty)$ and $\gamma(\mathtt{s}_0)$ "lies" on a "horizon" ($\lim_{\mathtt{s}\to 0} N(\gamma(\mathtt{s})) = 0$), then $N_0 = 0$ and $1 = N_0^2 + 2\mathcal{M}_0/(\mathtt{s}_0 + a) = 2m/a$. Therefore, $a = 2m$ and $\mathtt{s} = r - 2m$. Note that $m = \mathcal{M}_0$ cannot be negative otherwise θ reaches infinity for an $\mathtt{s} \in (0, \infty)$ (thus the whole γ cannot belong to $\mathcal{F}$). Thus, we establish the same relation $\mathtt{s} = r - 2m$ as in a Schwarzschild solution of positive mass. On the other hand, if γ is defined on $(\mathtt{s}_0 = 0, \infty)$ and $\gamma(\mathtt{s}_0)$ "lies" on a "naked singularity" ($\lim_{\mathtt{s}\to 0} N(\gamma(\mathtt{s})) = +\infty$), then $a = 0$ and $\mathtt{s} = r$. Note that in this case $m = \mathcal{M}_0$ must be negative otherwise $N^2 = 1 - 2m/r = 1 - 2m/\mathtt{s}$ gets negative for small $\mathtt{s} > 0$. Thus, we establish the same relation $\mathtt{s} = r$ as in a Schwarzschild solution of negative mass.

Remark 3: There are several ways to include the summand $-2(\dot{N}/N)^2$ to obtain an estimation on the growth of θ. The following proposition, whose proof is left to the reader, is one such instance. Although we will not use it for the rest of the article, it illustrates very well, the many ways in which the focussing equation can be used to extract geometric information.

Proposition 3: Let θ be the mean curvature of the integrable congruence $\mathcal{F}$. Let $\gamma(\mathtt{s})$, $\mathtt{s} \in [\mathtt{s}_0, \mathtt{s}_1]$ be in $\mathcal{F}$. Then we have

1. *θN^2 is monotonically decreasing, namely, $(\theta N^2)^{\dot{}} \leq -(\frac{\theta N}{\sqrt{2}} - \sqrt{2}\dot{N})^2$. Therefore, we have $\theta \leq \theta_0(N_0/N)^2$, where $\theta_0 = \theta(\mathtt{s}_0)$ and $N_0 = N(\mathtt{s}_0)$.*
2. *Suppose that $\theta(\mathtt{s}) > 0$ for all $\mathtt{s}$ in $[\mathtt{s}_0, \mathtt{s}_1]$. Then we have*

$$\theta(\mathtt{s}) \leq \frac{1}{\frac{1}{\theta_0} + \frac{\mathtt{s}-\mathtt{s}_0}{2} + \frac{1}{2\theta_0^2 N_0^4}\frac{(N^2-N_0^2)^2}{(\mathtt{s}-\mathtt{s}_0)}}. \tag{20}$$

As θ is monotonically decreasing the same formula holds for all $\mathtt{s}$ in the domain where γ is length minimizing provided only $\theta_0 > 0$.

□

Equation (20) clearly displays the influence of the Lapse N in the focussing of geodesics beyond the natural focussing that comes out of the non-negativity of the Ricci curvature. Equation (20) can serve, in particular, to obtain information on the relationship between volume growth of tubular neighborhoods of a horizon and the growth of N from it.

III. APPLICATIONS TO ASYMPTOTICALLY FLAT STATIC SOLUTIONS WITH REGULAR AND CONNECTED HORIZONS

In this section, we show that any asymptotically flat static solution with regular and connected horizon must satisfy the Penrose inequality. This is proved in Sec. III E Separately, in Sec. III F, we will prove that one such solution must satisfy the opposite Penrose inequality and that the horizon must be geometrically round. This will lead us into the verge of proving Theorem 1 which is carried out in Sec. III G To achieve the inequalities some preliminary material is introduced in Secs. III A – III D. In Sec. III A, we compute "the value of $\mathcal{M}$" for the "congruence of geodesics emanating perpendicularly to H" (note that g is singular on H) which we will be used crucially in the other sections. Technically, we will elude the fact that g is singular on H by considering instead of H suitable sequences $\{H_{\Gamma_i}\}$ of two-surfaces approaching H as $i \to \infty$. In this way, "the value of $\mathcal{M}$ over H" will be defined as a limit. Similarly, we will define $\mathtt{s}(p) := dist_{\mathrm{g}}(p, H) := \lim_{i\to\infty} dist_{\mathrm{g}}(p, H_{\Gamma_i})$. In Sec. III A, we recall the notion of *Asymptotic Flatness* and introduce, following Ref. 3, a coordinate system adapted to asymptotically flat static solutions that will be very useful later. In Sec. III C, we introduce the notion of *Coordinate-Distance Lag* which is necessary to prove, in Theorem 2 of Sec. III D, a central *Distance Comparison* where we establish a lower bound for the g-distance function to the horizon (the function $\mathtt{s}$) in terms of a certain function of N, m, and A (the function $\hat{\hat{\mathtt{s}}}$). For any divergent sequence of points $\{p_i\}$ the coordinate-distance lag associated with $\{p_i\}$ is defined as $\bar{\delta}(\{p_i\}) = \limsup \mathtt{s}(p_i) - r(p_i) + 2m$, where $r = |\bar{x}|$ and $\{\bar{x} = (x_1, x_2, x_3)\}$ is the coordinate system introduced in Sec. III B and it will be seen to be $\bar{\delta}(\{p_i\}) = \limsup \mathtt{s}(p_i) - \hat{\mathtt{s}}(p_i)$. The Penrose inequality in Sec. III E is then proved by showing first, using a standard comparison of mean curvatures, that if $P := A/(16\pi m) > 1$ (i.e., the Penrose inequality does not hold), then there is a divergent sequence whose coordinate-distance lag is non-negative (Corollary 2) and on the other hand proving, using the distance comparison of Sec. III D, that if $P > 1$, then the coordinate-distance lag must be negative for any divergent sequence (Proposition 12). This reaches a contradiction. To prove the opposite Penrose inequality it is shown that the Gaussian curvature κ of H must satisfy $\kappa \geq 4(4\pi m/A)^2$ to prevent a violation of the distance comparison near the horizon integrating this inequality over H and using Gauss-Bonnet the opposite Penrose inequality is achieved. As a by-product of both inequalities, one obtains that the horizon must be geometrically round, namely, that $\kappa = 4\pi/A$.

A. The value of $\mathcal{M}$ over regular horizons

Let $(\Sigma, \mathrm{g}, \ln N)$ be a static solution and let H be a regular and connected horizon. Consider an embedded (orientable) surface $\mathcal{S} \subset \Sigma \setminus H$. Let n_1 and n_2 be the two unit-normal vector fields to $\mathcal{S}$. As we noted before if $\mathcal{F}$ is the congruence of geodesics emanating perpendicularly to $\mathcal{S}$ and following one of the perpendicular directions to $\mathcal{S}$, say n_1, then the mean curvature θ of the congruence $\mathcal{F}$ over $\mathcal{S}$ is equal to the mean curvature of the surface $\mathcal{S}$ in the direction of n_1. Now to define $\mathcal{M}$ over H (where g is singular) for the "congruence of geodesics emanating perpendicular to H" we will calculate $\mathcal{M}$ over a suitable sequence of surfaces and then take the limit as the surfaces approach H. Such calculation is performed in the paragraphs below. The following notation will be used in this section and those that follow.

Notation 1: Let Γ_0 be a number sufficiently small in such a way that for any $\Gamma \leq \Gamma_0$, Γ is a regular value for the lapse N and the set $H_\Gamma := \{N = \Gamma\}$ is isotopic to H (note that $|\nabla N| \neq 0$ over a regular horizon H). One such Γ_0 will be called regular. For any two $\Gamma < \bar{\Gamma}$ denote by $\Omega_{\Gamma,\bar{\Gamma}}$ the closed region enclosed by H_Γ and $H_{\bar{\Gamma}}$. The region enclosed by H_Γ and H will be denoted by Ω_{H,H_Γ}.

Let $\{\Gamma_i\}_{i=1}^{i=\infty}$ be a sequence such that $\Gamma_i \downarrow 0$ and $\Gamma_i \leq \Gamma_0$ with Γ_0 as in Notation 1. Define

$$\mathcal{M}_H := \lim_{\Gamma_i \to 0} (\frac{\theta}{2} a^2 - a) N^2 |_{H_{\Gamma_i}}.$$

The next proposition shows the limit above exists (so it is well defined) and is always constant over H. Define $|\nabla N|_H = |\nabla N|_g|_H$.

Proposition 4: Let (Σ, g, N) be a static solution with regular horizon $\partial\Sigma$. Let H be a connected component of $\partial\Sigma$. Then, we have

$$\mathcal{M}_H = |\nabla N|_H a^2. \tag{21}$$

Proof: Denote (as we have done before) by θ the mean curvature of H_Γ with respect to g and θ_g the mean curvature with respect to g. From the conformal relation $\mathrm{g} = N^2 g$ we know that

$$\theta = \frac{\theta_g}{N} + 2\frac{n(N)}{N^2},$$

where $n(N)$ is the normal derivative of N in the outgoing direction (outgoing to $\partial\Omega_{H,H_{\Gamma_i}}$ and n a unit vector with respect to g). Thus, we get

$$(\frac{\theta}{2}a^2 - a)N^2 = a^2 n(N) + \frac{a^2\theta_g N}{2} - aN^2.$$

We get Eq. (21) in the limit when $\Gamma_i \to 0$. □

B. Asymptotically flat static solutions

We will use a useful characterization of asymptotically flat static solutions $(\Sigma, \mathrm{g}, \ln N)$ due to Beig and Simon.[3] Following Ref. 3, we say that $(\Sigma, \mathrm{g}, \ln N)$ is *asymptotically flat* iff there is a coordinate system $\{\bar{x} = (x_1, x_2, x_3)$ with $x_1^2 + x_2^2 + x_3^2 = |\bar{x}|^2 \geq |\bar{x}|_0^2\}$ outside a compact set in Σ such that

1. $\ln N = O^2(\frac{1}{|\bar{x}|})$ and $\mathrm{g}_{ij} - \delta_{ij} = O^2(\frac{1}{|\bar{x}|^2})$; where we use the notation $\phi(\bar{x}) = O^2(f(|\bar{x}|))$ to mean that for some positive numbers c_1, c_2, and c_3 we have
$$|\phi| \leq c_1|f(|\bar{x}|)|,\ |\partial_i\phi| \leq c_2|\partial_{|\bar{x}|}f(|\bar{x}|)| \text{ and } |\partial_i\partial_j\phi| \leq c_3|\partial^2_{|\bar{x}|}f(|\bar{x}|)|.$$
2. The second derivatives of $\ln N$ and $\mathrm{g}_{ij} - \delta_{ij}$ have bounded C^α-norm (defined with respect to the coordinate system $\{\bar{x}\}$) bounded; namely, if $\phi = \partial_k\partial_l \ln N$ or $\phi = \partial_k\partial_l(\mathrm{g}_{ij} - \delta_{ij})$ for all $1 \leq k, l, i, j \leq 3$, then
$$\|\phi\|_{C^\alpha} = \sup_{|\bar{x}-\bar{x}'|\leq 1} \frac{|\phi(\bar{x}) - \phi(\bar{x}')|}{|\bar{x} - \bar{x}'|^\alpha} < \infty.$$

Proposition 5 (Beig-Simon[3]): Let $(\Sigma, \mathrm{g}, \ln N)$ be an asymptotically flat static solution. Then, there is a coordinate system $\{\bar{x} = (x_1, x_2, x_3),\ |\bar{x}| \geq |\bar{x}|_1\}$ (not necessarily equal to the one defining asymptotic flatness), such that

$$\ln N^2 = -\frac{2m}{|\bar{x}|} + O^2(\frac{1}{|\bar{x}|^3}), \tag{22}$$

$$\mathrm{g}_{ij} = \delta_{ij} - \frac{m^2}{|\bar{x}|^4}(\delta_{ij}|\bar{x}|^2 - x_i x_j) + O^2(\frac{1}{|\bar{x}|^3}), \tag{23}$$

where $|\bar{x}|^2 = x_1^2 + x_2^2 + x_3^2$ and m is the ADM mass of the solution.

Note that the remainders are $O^2(1/|\bar{x}|^3)$, in particular, $\ln N$ has zero dipole moment. This fact will be important later. Note too that $|\bar{x}|^2 d\Omega^2 = |\bar{x}|^2(d\theta^2 + \sin^2\theta d\varphi^2) = (\delta_{ij} - (x_i x_j)/|\bar{x}|^2)dx_i dx_j$ therefore we have

$$\mathrm{g} = \delta_{ij}dx^i dx^j - m^2 d\Omega^2 + O^2(\frac{1}{|\bar{x}|^3}) = (d|\bar{x}|)^2 + (|\bar{x}|^2 - m^2)d\Omega^2 + O^2(\frac{1}{|\bar{x}|^3}).$$

To make contact with the representation (5) of the Schwarzschild solution proceed as follows. Let $(|\bar{x}|, \theta, \varphi)$ be the spherical coordinate system associated with the coordinate system $\{\bar{x}\}$. Make the change of variables $(|\bar{x}|, \theta, \varphi) \to (r, \theta, \varphi)$ with $r = |\bar{x}| + m$. Then, for the metric g, we obtain

$$\mathrm{g} = dr^2 + r^2(1 - \frac{2m}{r})d\Omega^2 + O^2(\frac{1}{r^3}) = \mathrm{g}_S + O^2(\frac{1}{r^3}).$$

For the Lapse N instead, we obtain the following expansion. From (22) we have

$$N^2 = 1 - \frac{2m}{|\bar{x}|} + \frac{2m^2}{|\bar{x}|^2} + O^2(\frac{1}{|\bar{x}|^3}).$$

Now use

$$\frac{1}{|\bar{x}|} = \frac{1}{r-m} = \frac{1}{r} + \frac{m}{r^2} + \frac{m^2}{r^3} + O^2(\frac{1}{r^4})$$

to get

$$N^2 = 1 - \frac{2m}{r} + O^2(\frac{1}{r^3}).$$

We can thus rephrase Proposition 5 in the following form.

Proposition 6: Let $(\Sigma, \mathrm{g}, \ln N)$ *be an asymptotically flat static solution. Then, there is a coordinate system* $\{\bar{x} = (x_1, x_2, x_3),\ (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}} = r \geq r_1\}$ *(not necessarily equal to the one defining asymptotic flatness), such that*

$$N^2 = 1 - \frac{2m}{r} + O^2(\frac{1}{r^3}), \tag{24}$$

$$\mathrm{g} = dr^2 + r^2(1 - \frac{2m}{r})d\Omega^2 + O^2(\frac{1}{r^3}), \tag{25}$$

where m is the ADM mass of the solution.

The following proposition on the asymptotic of the mean curvatures of the coordinate spheres $S_r = \{p/r(p) = r\}$ is now direct.

Proposition 7: Let (Σ, g, N) *be an asymptotically flat static solution and consider a coordinate system as in Proposition 6. Then, the mean curvature* θ_r *of the level surfaces* $S_r = \{p/r(p) = r\}$ *satisfy, at every point in* S_r*, the estimate*

$$\theta_r = \frac{2}{r} + \frac{2m}{r^2} + O(\frac{1}{r^3}). \tag{26}$$

C. The coordinate-distance lag

Let $(\Sigma, \mathrm{g}, \ln N)$ be an asymptotically flat static solution with regular and connected horizon H. We would first like to introduce the *distance function* to H, the definition of which is more or less evident. We will follow Notation 1.

Let $p \in \Sigma \backslash H$ and let $\{\Gamma_i\}_{i=1}^{i=\infty}$ be a strictly decreasing sequence such that, $\Gamma_i \leq \Gamma_0$, $\lim \Gamma_i = 0$, and $p \notin \Omega_{H_1, H}$. We note that if $j > i$, then

$$dist(p, H_{\Gamma_i}) < dist(p, H_{\Gamma_j}),$$

and we have

$$dist(p, H_{\Gamma_i}) \leq dist(p, H_{\Gamma_j}) \leq dist(p, H_{\Gamma_i}) + diam(\Omega_{H_{\Gamma_j}, H_{\Gamma_i}}), \tag{27}$$

where the diameter of $\Omega_{H_{\Gamma_j}, H_{\Gamma_i}}$, $diam(\Omega_{H_{\Gamma_j}, H_{\Gamma_i}})$, tends to zero as $i(< j) \to \infty$. Denote $s_\Gamma(p) := dist(p, H_\Gamma)$. The inequality (27) shows that

$$s(p) := \lim_{i \to \infty} s_{\Gamma_i}(p),$$

for any sequence $\{\Gamma_i\}$ as above, is well defined and independent on $\{\Gamma_i\}$. We thus define the distance from p to H in that way. Note that given a point p in $\Sigma \backslash H$, one can always construct a length minimizing geodesic from p to H by taking the limit of length minimizing geodesics from p to H_{Γ_i}. This fact will be used later without further mention.

Now consider the Schwarzschild solution $\bar{g}_S = dr^2 + (1 - \frac{2m}{r})r^2 d\Omega^2$ and consider a ray $\gamma(r) = (r, \theta_0, \varphi_0)$, $r \in [2m, \infty)$, which is, naturally, length minimizing between any two of its points. Let $\mathrm{s}(\gamma(r))$ be the length of γ between $r = 2m$ and r. Then $\mathrm{s}(\gamma(r)) = r - 2m$ and, therefore, the limit $lim_{r\to\infty}\mathrm{s}(\gamma(r)) - r + 2m = 0$. Now consider a ray $\gamma(\tau)$ on an (another) asymptotically flat static solution with regular and connected horizon H, joining H to infinity. Then in this different scenario, instead, the limit $\lim s(\gamma(\tau)) - r(\gamma(\tau)) + 2m$ may be different from zero. We advocate now to define *the coordinate distance lag* measuring precisely this *a priori* mismatch.

Definition 2: Let $(\Sigma, \mathrm{g}, \ln N)$ *be an asymptotically flat static solution with connected and regular horizon. Let* $\{\bar{x} = (x_1, x_2, x_3),\ |\bar{x}| = r \geq r_1\}$ *be a coordinate system as in Proposition 6. Let* $\{p_i\}$ *be a diverging sequence of points (i.e.,* $s(p_i) \to \infty$*) (lying inside the range of* $\{\bar{x}\}$*). Then, the coordinate distance lag,* $\bar{\delta}$*, associated with the sequence* $\{p_i\}$ *is defined as*

$$\bar{\delta} = \limsup_{i\to\infty}\ \mathrm{s}(p_i) - r(p_i) + 2m.$$

Note that coordinate-distance lags are always zero in the Schwarzschild solution. From the next proposition it will follow that coordinate-distance lags are always finite.

Proposition 8: Let $(\Sigma, \mathrm{g}, \ln N)$ *be an asymptotically flat static solution with connected and regular horizon H. Let* $\{\bar{x} = (x_1, x_2, x_3),\ |\bar{x}| = r \geq r_1\}$ *be a coordinate system as in Proposition 6. Then there are finite* $c_1 > c_2$*, depending on* $(\Sigma, \mathrm{g}, \ln N)$*, with the following property: for every divergent sequence of points* $\{p_i\}$ *(lying inside the range of* $\{\bar{x}\}$*) we have*

$$\mathrm{s}(p_i) - c_2 \leq r(p_i) \leq \mathrm{s}(p_i) - c_1. \tag{28}$$

Proof: We start showing the first inequality in Eq. (28). Let us first consider r_2 such that for every $\bar{x}$ such that $r(|x|) \geq r_2$ and a tangent vector v at $\bar{x}$ we have

$$\frac{|R(v, v)|}{|\mathrm{g}_S(v, v)|} \leq \frac{R_0}{r^3} \leq 1,$$

where R is the remainder tensor $R := \mathrm{g} - \mathrm{g}_S$, g_S is the Schwarzschild metric (5) and R_0 is a positive constant. It is clear that we do not lose anything in assuming that $r_2 = r_1$.

Let $d_0 = sup_{q\in S_{r_2}}\{dist(q, H)\}$ and for each $i \geq 0$ consider the curve $\alpha(r) = (r, \theta(p_i)), \varphi(p_i))$ starting at S_{r_2} and ending at p_i (namely, the range of r is $[r_2, r(p_i)]$. We will make use of the inequality

$$\sqrt{1+x} \leq 1 + |x|, \text{ if } |x| < 1, \tag{29}$$

to estimate the distance $\mathrm{s}(p_i)$ from above. We have

$$\mathrm{s}(p_i) \leq d_0 + \int_{r_2}^{r(p_i)} \sqrt{\mathrm{g}_S(\alpha', \alpha') + R(\alpha', \alpha')}dr. \tag{30}$$

As the integration is on $[r_2, r(p_i)]$ we have, by the definition of r_2, $|R(\alpha', \alpha')|/|\mathrm{g}_S(\alpha', \alpha')| \leq R_0/r^3 \leq 1$ (note that $\alpha' = \partial_r$). Thus by inequality (29) we have

$$\sqrt{\mathrm{g}_S(\alpha', \alpha') + R(\alpha', \alpha')} \leq \sqrt{\mathrm{g}(\alpha', \alpha')} + \frac{R_0}{r^3}.$$

Putting this into Eq. (30) and integrating we have

$$\mathrm{s}(p_i) \leq r(p_i) + (d_0 + \frac{R_0}{2r_2^2} - r_2).$$

This proves the first inequality.

To show the second inequality on the right-hand side of Eq. (28) we proceed as follows. Consider now an arbitrary curve $\alpha(\tau)$ joining S_{r_2} to p_i, lying inside the region enclosed by S_{r_2} and $S_{r(p_i)}$ and parameterized by the arc length, with respect to g_S, τ. Then, for the length of α, $l(\alpha)$, we have

$$l(\alpha) = \int \sqrt{\mathrm{g}_S(\alpha', \alpha') + R(\alpha', \alpha')}d\tau.$$

We are going to make use of the inequality

$$1 - |x| \leq \sqrt{1+x}, \text{ if } |x| \leq 1. \tag{31}$$

Note that because $\mathrm{g}_S(\alpha', \alpha') = 1$ we have $|R(\alpha', \alpha')| \leq R_0/r^3$. Therefore, from the inequality (31) we have

$$l(\alpha) \geq \int (1 - \frac{R_0}{r^3})d\tau. \tag{32}$$

Now note that $|dr/d\tau| \leq 1$. To see this consider an arbitrary parameterization of α by, say t. Then $d\tau/dt = \sqrt{\mathrm{g}_S(\partial_t\alpha, \partial_t\alpha)} \geq |dr/dt|$. Thus, noting that the integrand in Eq. (32) is positive, we can write

$$l(\alpha) \geq \int (1 - \frac{R_0}{r^3})d\tau \geq \int (1 - \frac{R_0}{r^3})|\frac{dr}{d\tau}|d\tau \geq \int (1 - \frac{R_0}{r^3})\frac{dr}{d\tau}d\tau.$$

Integrating we get

$$l(\alpha) \geq r_i - r_2 - \frac{R_0}{2r_2^2}. \tag{33}$$

Now clearly we have $\mathtt{s}(p_i)$ is greater or equal than the infimum of the lengths of all the curves α joining p_i to S_{r_2} and lying inside the region enclosed by S_{r_2} and $S_{r(p_i)}$. By the estimation in Eq. (33) above we have thus

$$\mathtt{s}(p_i) \geq r(p_i) - (r_2 + \frac{R_0}{2r_2}),$$

which proves the inequality on the right-hand side of Eq. (28). □

Corollary 1: Let $(\Sigma, \mathrm{g}, \ln N)$ *be an asymptotically flat static solution with connected and regular horizon H. Let* $\{\bar{x} = (x_1, x_2, x_3),\ |\bar{x}| = r \geq r_1\}$ *be a coordinate system as in Proposition 6. There are* $c_1 > c_2$ *depending on* $(\Sigma, \mathrm{g}, \ln N)$ *with the following property: for every diverging sequence of points* $\{p_i\}$ *(lying inside the range of* $\{\bar{x}\}$*) we have*

$$c_2 \leq \bar{\bar{\delta}}(\{p_i\}) \leq c_1.$$

D. Distance comparison

Consider an asymptotically flat static solution with regular and connected horizon, $(\Sigma, \mathrm{g}, \ln N)$. Let $\mathtt{s}(p) = dist(p, H)$. If the solution $(\Sigma, \mathrm{g}, \ln N)$ were the Schwarzschild solution, then we would have

$$\mathtt{s}(p) = r(p) - 2m = \frac{2m}{1 - N(p)^2} - 2m.$$

As it turns out, given an arbitrary solution $(\Sigma, \mathrm{g}, \ln N)$, the function $\hat{\mathtt{s}}$ defined exactly by

$$\hat{\mathtt{s}}(p) := \frac{2m}{1 - N(p)^2} - 2m$$

provides, via a *comparison of Laplacians*, a lower bound for the distance function $\mathtt{s}$. The next proposition computes the expression of the Laplacian of $\hat{\mathtt{s}}$.

Proposition 9: Let $(\Sigma, \mathrm{g}, \ln N)$ *be a static solution of the Einstein equations. Then, the Laplacian of* $\hat{\mathtt{s}}$ *has the following expression:*

$$\Delta\hat{\mathtt{s}} = \frac{2}{\bar{s}+2m}\Big(1+\frac{m}{(\hat{\mathtt{s}}+2m)N^2}\Big)|\nabla\hat{\mathtt{s}}|^2. \tag{34}$$

Proof: Note first the identities

$$\hat{\mathtt{s}} = 2m\frac{N^2}{1-N^2}, \tag{35}$$

$$N^2 = \frac{\hat{\mathtt{s}}}{\hat{\mathtt{s}}+2m},\ N^2+1 = 2\frac{\hat{\mathtt{s}}+m}{\hat{\mathtt{s}}+2m}. \tag{36}$$

We calculate

$$\nabla\frac{1}{1-N^2} = 2\frac{N\nabla N}{(1-N^2)^2} = 2\frac{N^2}{(1-N^2)^2}\nabla\ln N.$$

Next, we compute the divergence of this expression to get

$$\Delta\frac{1}{1-N^2} = 4\frac{|\nabla N|^2}{(1-N^2)^2} + 8\frac{N^2|\nabla N|^2}{(1-N^2)^3} = 4\frac{|\nabla N|^2}{(1-N^2)^3}(1+N^2),$$

where we have used the fact that $\Delta\ln N = 0$. This expression is equal to

$$\Delta\frac{1}{1-N^2} = \Big|\nabla\frac{1}{1-N^2}\Big|^2(1+N^2)\Big(\frac{1-N^2}{N^2}\Big).$$

After inserting back the coefficient $2m$ and using the identity (35) we get

$$\Delta\hat{\mathtt{s}} = \frac{1+N^2}{\hat{\mathtt{s}}}|\nabla\hat{\mathtt{s}}|^2.$$

Finally, using the identity (36) we have

$$\frac{N^2+1}{\hat{\mathtt{s}}} = 2\frac{\hat{\mathtt{s}}+m}{\hat{\mathtt{s}}+2m}\frac{1}{\hat{\mathtt{s}}} = \frac{2}{\hat{\mathtt{s}}+2m}\Big(1+\frac{m}{(\hat{\mathtt{s}}+2m)N^2}\Big).$$

□

The asymptotic behavior of $\hat{\mathtt{s}}(p)$, when $r(p)\to\infty$ is deduced from Proposition 6 and we have

$$\hat{\mathtt{s}}(p) = \frac{2m}{\frac{2m}{r(p)}+O(\frac{1}{r(p)^3})} - 2m = r(p) - 2m + O\Big(\frac{1}{r(p)}\Big), \tag{37}$$

if $r(p)$ is big enough. This asymptotic expression will be important and will be used many times later.

The reason why we have expressed the Laplacian of $\hat{\mathtt{s}}$ in the form (34) was to make it comparable with the Laplacian of $\mathtt{s}$, that satisfies the inequality

$$\Delta\mathtt{s} \le \frac{2}{\mathtt{s}+2Pm}\Big(1+\frac{Pm}{(\mathtt{s}+2Pm)N^2}\Big)|\nabla\mathtt{s}|^2 \tag{38}$$

in a certain *barer sense* as is explained in Proposition 10. In the above equation P is equal to the expression

$$P = \frac{A}{16\pi m^2}$$

and will be called *the Penrose quotient*. Note that the Penrose inequality $A \leq 16\pi m^2$ holds iff $P \leq 1$. Note too that wherever s is smooth we have $|\nabla s|^2 = 1$. We have included such factor in (38) to make the comparison to (34) more evident.

The fact that the inequality (38) holds in a barer sense will allow us to assume, when comparing s to $\hat{\mathrm{s}}$, that s is a smooth function. This fact will be further explained in Theorem 2. We now introduce a proposition describing the *sense* in which inequality (38) holds.

Proposition 10: Let $(\Sigma, \mathrm{g}, \ln N)$ *be an asymptotically flat static solution with regular and connected horizon. Let* $\{p_i\}_{i=1}^{i=\infty}$ *be a sequence of points in* Σ *converging to p in* $\Sigma \backslash H$. *Let* $\{\Gamma_i\}_{i=1}^{i=\infty}$ *be a sequence such that* $\lim_{i\to 0}\Gamma_i \downarrow 0$, $\Gamma_1 \leq \Gamma_0$ *with* Γ_0 *regular (Notation 1) and* $\{p_i, i = 1, \ldots, i = \infty\} \subset \Sigma \setminus \Omega_{H,H_{\Gamma_0}}$. *Consider the sequence of distance functions* $\{\mathrm{s}_{\Gamma_i}(p) = dist(p, H_{\Gamma_i})\}_{i=1}^{i=\infty}$. *Then, there is sequence of continuous functions* $\tilde{\mathrm{s}}_{\Gamma_i}$ *such that for each* Γ_i:

1. $\tilde{\mathrm{s}}_{\Gamma_i}$ *is defined on the domain* $\Sigma \setminus \Omega_{H,H_{\Gamma_i}}$,
2. $\tilde{\mathrm{s}}_{\Gamma_i}$ *is smooth at* p_i,
3. $\tilde{\mathrm{s}}_{\Gamma_i} \geq \mathrm{s}_{\Gamma_i}$, $\tilde{\mathrm{s}}_{\Gamma_i}(p_i) = \mathrm{s}_{\Gamma_i}(p_i)$ *and* $|\nabla \tilde{\mathrm{s}}|^2(p_i) = 1$,
4. $$\Delta \tilde{\mathrm{s}}_{\Gamma_i}(p_i) \leq 2\frac{1}{\tilde{\mathrm{s}}_{\Gamma_i}(p_i) + \tilde{a}_i}\Big(1 + \frac{\tilde{a}_i}{2(\tilde{\mathrm{s}}_{\Gamma_i}(p_i) + \tilde{a}_i)N^2(p_i)}\Big)|\nabla \tilde{\mathrm{s}}_{\Gamma_i}|^2(p_i),$$
   where $\{\tilde{a}_i\}$ *is a sequence such that* $\lim_{i\to\infty}\tilde{a}_i = 2mP$.
5. *Moreover,* $\{\tilde{\mathrm{s}}_{\Gamma_i}\}$ *converges uniformly in* C^0 *to* $\mathrm{s}(p) = dist(p, H)$ *in the sense that*
   $$\lim_{i\to\infty} \sup_{q\in\Sigma\setminus\Omega_{H,H_{\Gamma_i}}} |\tilde{\mathrm{s}}_{\Gamma_i}(q) - \mathrm{s}(q)| = 0.$$

The proof of this proposition will be a direct consequence of the following proposition in Riemannian geometry. We will use the following notation and terminology.

Notation 2: Let (Σ, g) *be a complete Riemannian manifold with non-empty and connected boundary* $\partial\Sigma$. *The inner-normal bundle* $\mathcal{N}(\partial\Sigma)$ *of* Σ *at* $\partial\Sigma$ *is defined as the set of vectors* $v(q)$, *normal to* $\partial\Sigma$ *at q, and pointing inwards to* Σ. *We will consider the exponential map* $exp : \mathcal{N}(\partial\Sigma) \to \Sigma$ *such that to every* $v(q) \in \mathcal{N}(\partial\Sigma)$ *assigns the end point of the geodesic segment of length* $|v(q)|$ *that start at q with velocity* $v(q)/|v(q)|$.

Proposition 11: Let (Σ, g) *be a complete Riemannian three-manifold, not necessarily compact. Let* $\mathcal{S}_1$ *be an immersed smooth surface separating* Σ *into two connected (open) components* Σ_1 *and* Σ_2. *Let p be a point in* Σ_1 *and* $\gamma_{q,p}$ *be a geodesic segment minimizing the distance between p and* $\partial\Sigma_1 = \mathcal{S}_1$, *starting at* $q \in \partial\Sigma_1$ *and ending at p. We can write* $\gamma_{q,p}(\tau) = exp(\tau v(q))$, $\tau \in [0, 1]$, *with* $v(q) = l(\gamma_{q,p})n(q)$ *where n(q) is the inward unit-normal vector to* $\partial\Sigma_1$ *at q. If the differential of the exponential map* $exp : \mathcal{N}(\partial\Sigma_1) \to \Sigma_1$ *is not injective at* $v(q)$, *then for every smooth surface* $\mathcal{S}_2$ *immersed in* $\Sigma_1 \cup \mathcal{S}_1$ *such that*

1. $\mathcal{S}_2$ *touches* $\mathcal{S}_1$ *only at q.*
2. *The second fundamental forms* $\Theta_1(q)$ *and* $\Theta_2(q)$ *of* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *(respectively) at q and defined with respect to n(q) satisfy*
   $$\Theta_2(q) > \Theta_1(q),$$

we have

1. $\gamma_{q,p}$ *is the only geodesic segment minimizing the distance between p and* $\mathcal{S}_2$.
2. *The exponential map* $exp : \mathcal{N}(\partial\tilde{\Sigma}_1) \to \tilde{\Sigma}_1$ *is injective at* $v(q)$, *where* $\tilde{\Sigma}_1$ *is the connected component of* $\Sigma \setminus \mathcal{S}_2$ *containing* $\mathcal{S}_1$.

Proof: First, it is clear that $\gamma_{p,q}$ is the only geodesic segment minimizing the distance between p and $\mathcal{S}_2$ for $\mathcal{S}_2$ touches $\mathcal{S}_1$ only at q. This proves the first *item* of the claim.

To prove the second suppose on the contrary that the exponential map $exp : \mathcal{N}(\partial\tilde{\Sigma}_1) \rightarrow \tilde{\Sigma}_1$ is not injective at $v(q)$. Then there is a curve $w(\lambda)$, $\lambda \in [0, \lambda_1]$ of vectors in $\mathcal{N}(\tilde{\Sigma})$ of norm (for all λ) equal to $l(\gamma_{p,q})$, such that $w(0) = v(q)$ and such that $d\,exp(w'(0)) = 0$. Therefore, $J(s) = d\,exp(\frac{s}{l(\gamma_{p,q})} w'(0)$ is a Jacobi field such that $J(s) \neq 0$ for any $s \in [0, l(\gamma_{p,q}))$. Let $\alpha(s, \lambda)$, $(s, \lambda) \in [0, l(\gamma_{p,q})] \times [0, \lambda_1]$ be a smooth one-parameter family of curves such that $\partial_\lambda \alpha(s, 0) = J(s)$ and such that $\partial_s \alpha(0, \lambda) \in \mathcal{N}(\tilde{\Sigma}_1)$. Then because $J(s)$ is a Jacobi field we have that the second variation of the length of the curves $\alpha_\lambda(s) = \alpha(s, \lambda)$ (variation with respect to λ) is equal to zero (although it is a standard fact in Riemannian geometry, the reader can check this fact in pp. 227–228 of Ref. 12, the proof there is for Jacobi fields vanishing at the two extreme points, but it is simply adapted to this situation as well). On the other hand consider the curves $\bar{\alpha}(s, \lambda) = \alpha(s, \lambda)$, with $(s, \lambda) \in [0, s(\lambda)] \times [0, l(\gamma_{p,q})]$ where the point $\alpha(s(\lambda), \lambda)$ is the intersection of $\alpha(s, \lambda)$ (a curve as a function of s) and $\mathcal{S}_1$. Now, because of the condition in *item 2*, $\Theta_2(q) > \Theta_1(q)$, the second variation (with respect to λ) of $\bar{\alpha}$ is positive. Thus, the second variation (with respect to λ) of the length of the curves $\tilde{\alpha}(s, \lambda) = \alpha(s, \lambda)$, $(s, \lambda) \in [s(\lambda), l(\gamma_{p,q})] \times [0, \lambda_1]$ is negative, which is a contradiction as $\gamma_{p,q}$ is length minimizing between p and $\mathcal{S}_1$. □

Proof (of Proposition 10): Let γ_{p_i,q_i}, $q_i \in H_{\Gamma_i}$ be a length minimizing geodesic joining p_i and H_{Γ_i}. Suppose first that s_{Γ_i} is smooth at p_i for each i. Then, we claim that taking $\tilde{\mathsf{s}}_{\Gamma_i} = \mathsf{s}_{\Gamma_i}$ is enough. It is clear that the *items 1,2,3,* and *5* of the claim are satisfied with this choice. We need, therefore, to check that there is sequence $\tilde{a}_i$ for which the equation in *item 4* is satisfied and $\lim_{i\rightarrow\infty} \tilde{a}_i = 2mP$. For this, we are going to use the monotonicity, for every a of $\mathcal{M} = \mathcal{M}_a$ an over γ_{p_i,q_i}, and then we will chose a conveniently (which will be our choice of $\tilde{a}_i$). Of course $\mathcal{M}$ is defined, for each i, for the congruences $\mathcal{F}_i$ of length minimizing geodesics segments to H_{Γ_i}. Thus, we have

$$\frac{\theta(p_i)}{2}(\mathsf{s}_{\Gamma_i}(p_i) + a)^2 N^2(p_i) - (\mathsf{s}_{\Gamma_i}(p_i) + a)N^2(p_i) = \mathcal{M}_a(p_i) \leq \mathcal{M}_a(q_i).$$

Solving for $\theta(p_i) = \Delta\mathsf{s}_{\Gamma_i}(p_i)$ we get

$$\Delta\mathsf{s}_{\Gamma_i}(p_i) \leq \frac{2}{(\mathsf{s}_{\Gamma_i}(p_i) + a)}(1 + \frac{\mathcal{M}_{\Gamma_i}(q_i)}{(\mathsf{s}_{\Gamma_i}(p_i) + a)N^2(p_i)}).$$

We need now to show that we can chose a for each i (thus having $a = \tilde{a}_i$) in such a way that $\mathcal{M}_{\Gamma_i}(q_i) \leq \tilde{a}_i/2$. Therefore, we need to have

$$\mathcal{M}_{\Gamma_i}(q_i) = \frac{\theta_{\Gamma_i}(q_i)}{2} a^2 N^2(q_i) - aN(q_i)^2 \leq \frac{a}{2}.$$

Thus we chose

$$a = \sup_{q\in H_{\Gamma_i}} \{\frac{2(\frac{1}{2} + N(q)^2)}{\theta(q)N^2(q)}\}. \tag{39}$$

Now, the numerator tends to one and the denominator, because of Eq. (21), tends to $2|\nabla N|_H = 8\pi m/A = 1/(2mP)$. The claim in this case follows.

If on the contrary, the functions s_{Γ_i} are not smooth at p_i, then we know by Proposition 11 that the distance functions $\tilde{\mathsf{s}}_{\Gamma_i}$ to a hypersurface $\tilde{H}_{\Gamma_i}$ included in $\Omega_{H,H_{\Gamma_i}}$ will be smooth at p_i provided they touch H_{Γ_i} only at q_i and have strictly greater second fundamental form at q_i. Besides these last two conditions nothing else is required on the hypersurfaces $\tilde{H}_{\Gamma_i}$ for $\tilde{\mathsf{s}}_{\Gamma_i}$ to be smooth at p_i. Thus, it is clear that if we chose the hypersurfaces $\tilde{H}_{\Gamma_i}$ close enough to H_{Γ_i} (but satisfying the two requirements) and $\tilde{a}_i$ using the same formula as in Eq. (39) (but with q varying on $\tilde{H}_{\Gamma_i}$), then $\tilde{\mathsf{s}}_{\Gamma_i}$ will satisfy *items 1–5* of the claim. □

Theorem 2 (Distance comparison): *Let* $(\Sigma, \mathrm{g}, \ln N)$ *be an asymptotically flat static solution with regular and connected horizon. Then, we have*

$$\frac{2m}{1 - N^2(p)} - 2m = \hat{\mathsf{s}}(p) \leq \max\{1, \frac{1}{P}\}\mathsf{s}(p) = \max\{1, \frac{16\pi m^2}{A}\}dist(p, H), \tag{40}$$

for all p in Σ, *where P is the Penrose quotient. Moreover,*

$$\lim_{\mathtt{s}(p)\to\infty}\frac{\hat{\mathtt{s}}(p)}{\mathtt{s}(p)}=1 \text{ and } \lim_{\mathtt{s}(p)\to 0}\frac{\hat{\mathtt{s}}(p)}{\mathtt{s}(p)}=\frac{1}{P}.$$

Proof: We will consider the quotient $\hat{\mathtt{s}}/\mathtt{s}$ as a function on $\Sigma\backslash H$. Let us first find the boundary conditions, namely, $\lim\hat{\mathtt{s}}(p)/\mathtt{s}(p)$ when $\mathtt{s}(p)\to\infty$ and $\mathtt{s}(p)\to 0$ (at infinity and at the horizon, respectively). From Proposition 8 and the estimation (37), we deduce

$$\lim_{\mathtt{s}(p)\to\infty}\frac{\hat{\mathtt{s}}(p)}{\mathtt{s}(p)}=1.$$

To calculate the quotient at the horizon we proceed like this. Consider the congruence of geodesics with respect to g, emanating perpendicularly to H and parameterized by the arc length τ which is measured from the initial point of the geodesic at H. Any given coordinate system $\{\bar{x}=(x_1,x_2)\}$ on an open set of H can be propagated along the congruence to the level sets of the distance function with respect to g, namely, the τ_0-level sets $\{\tau=\tau_0\}$ and we can write

$$g=d\tau^2+h_{ij}(\bar{x},\tau)dx_idx_j$$

and

$$\hat{\mathtt{s}}(\tau,\bar{x})=\frac{2m}{1-N^2(\tau,\bar{x})}-2m=2m|\nabla N|^2_H\tau^2+O(\tau^3). \tag{41}$$

We note then that because H is totally geodesic, the second fundamental form is zero and we have

$$\partial_\tau h_{ij}(\tau,\bar{x})\Big|_{\tau=0}=0.$$

Thus

$$g=d\tau^2+h_{ij}(0,\bar{x})dx_idx_j+O(\tau^2). \tag{42}$$

Combining (41) and (42), we get

$$\mathtt{g}=N^2g=|\nabla N|^2_H\tau^2(d\tau^2+h_{ij}(0,\bar{x})dx_idx_j)+O(\tau^3)d\tau^2+O(\tau^4)h_{ij}dx_idx_j.$$

From this expression it is simple that if $\{p_i\}$ is a sequence in $\Sigma\backslash H$ converging to a point in H we have

$$\mathtt{s}(p_i)=|\nabla N|_H\frac{\tau(p_i)^2}{2}+O(\tau(p_i)^3). \tag{43}$$

We can combine (41) and (43) to conclude that for any sequence $\{p_i\}$ in $\Sigma\backslash H$ converging to a point in H we have

$$\lim\frac{\hat{\mathtt{s}}(p_i)}{\mathtt{s}(p_i)}=4m|\nabla N|_H. \tag{44}$$

Now, $|\nabla N|_H$ is equal to $4\pi m/A$ as can be seen by integrating $\Delta N=0$ between $S_r=\{p/r(p)=r\}$ and H and taking the limit when $r\to\infty$. With this value of $|\nabla N|_H$ we get from (44),

$$\lim_{\mathtt{s}(p)\to 0}\frac{\hat{\mathtt{s}}(p)}{\mathtt{s}(p)}=\frac{16\pi m^2}{A}=\frac{1}{P}.$$

We would now like to compare $\hat{\mathtt{s}}$ to $\mathtt{s}$ using (34) and (38). For this purpose, it is simpler to consider the dimensionless quantities $\hat{u}=\hat{\mathtt{s}}/2m$ and $u=\mathtt{s}/2mP$. In terms of them (34) and (38) become

$$\Delta\hat{u}=\frac{2}{\hat{u}+1}(1+\frac{1}{2(\hat{u}+1)N^2})|\nabla\hat{u}|^2, \tag{45}$$

$$\Delta u\leq\frac{2}{u+1}(1+\frac{1}{2(u+1)N^2})|\nabla u|^2. \tag{46}$$

We will consider now the quotient $\phi = \hat{u}/u$ and note that the boundary conditions at H and at infinity become, respectively, $\lim_{\mathsf{s}(p)\to 0}\hat{u}(p)/u(p) = 1$ and $\lim_{\mathsf{s}(p)\to\infty}\hat{u}(p)/u(p) = P$. If we prove that $\hat{u}/u \leq \max\{1, P\}$, then we will be proving (40). Thus, we will proceed by contradiction and assume that there is a point $\bar{p} \in \Sigma \setminus H$ such that $\hat{u}(\bar{p}) > \max\{1, P\}u(\bar{p})$ and that such point is an absolute maximum for $\hat{u}/u$ (note the boundary conditions). We will assume below that the function s is smooth at $\bar{p}$ or equivalently that u is smooth at $\bar{p}$. Otherwise use the fact that s satisfies Eq. (38) in a barer sense as follows. Replace s by s_Γ for Γ sufficiently small in such a way that $\hat{u}/u_\Gamma$, with $u_\Gamma = \mathsf{s}_\Gamma/2mP$ still has a maximum greater than $\max\{1, P\}$, say at $\bar{\bar{p}}$. Then substitute once more s_Γ by $\tilde{\mathsf{s}}_\Gamma \geq \mathsf{s}_\Gamma$ as in Proposition 10 and consider thus the quotient $\hat{u}/\tilde{u}_\Gamma$, with $\tilde{u}_\Gamma = \tilde{\mathsf{s}}_\Gamma/2mP$, which still has a maximum greater than $\max\{1, P\}$ at $\bar{\bar{p}}$. If Γ is sufficiently small we would reach a contradiction following the same argument as below.

We compute

$$\Delta\frac{\hat{u}}{u} = \frac{\Delta\hat{u}}{u} - 2\frac{<\nabla\hat{u}, \nabla u>}{u^2} - \frac{\hat{u}}{u^2}\Delta u + 2\frac{\hat{u}}{u^3}|\nabla u|^2. \tag{47}$$

Because $\hat{u}/u$ reaches an absolute maximum at $\bar{p}$ we have $\nabla(\hat{u}/u|_{\bar{p}}) = 0$ and thus

$$\left.\frac{\nabla\hat{u}}{\hat{u}}\right|_{\bar{p}} = \left.\frac{\nabla u}{u}\right|_{\bar{p}}, \tag{48}$$

with $|\nabla u|^2(\bar{p}) = 1/2mP \neq 0$. If we use (48) in (47), we note that the second and fourth terms on the right-hand side cancel out at $\bar{p}$. Thus, we will get a contradiction of the fact that $\hat{u}/u$ reaches an absolute maximum at $\bar{p}$ if we can prove that the sum of the first and third terms on the right-hand side of (47) is positive at $\bar{p}$ (the Maximum Principle). We will prove that in what follows.

We compute

$$\left.\Delta\frac{\hat{u}}{u}\right|_{\bar{p}} = \left.\frac{1}{u^2(\bar{p})}(u\Delta\hat{u} - \hat{u}\Delta u)\right|_{\bar{p}}$$

and using (45) and (46), we get the inequality

$$\left.\Delta\frac{\hat{u}}{u}\right|_{\bar{p}} \geq \left.\frac{2}{u^2}(\frac{u}{(1+\hat{u})}(1 + \frac{1}{2(1+\hat{u})N^2}))\frac{\hat{u}^2}{u^2}|\nabla u|^2 - \frac{\hat{u}}{(1+u)}(1 + \frac{1}{2(1+u)N^2})|\nabla u|^2)\right|_{\bar{p}}.$$

Thus, we would like to prove that

$$\left.\frac{\hat{u}}{1+\hat{u}}(1 + \frac{1}{2(1+\hat{u})N^2})\right|_{\bar{p}} > \left.\frac{u}{1+u}(1 + \frac{1}{2(1+u)N^2})\right|_{\bar{p}}. \tag{49}$$

Recalling from (36) that $N^2 = \hat{u}/(1+\hat{u})$ and substituting that into (49) we deduce that we would like to show that

$$\left.\frac{\hat{u}}{(1+\hat{u})}(1 + \frac{1}{2\hat{u}})\right|_{\bar{p}} > \left.\frac{u}{1+u}(1 + \frac{1+\hat{u}}{2(1+u)\hat{u}})\right|_{\bar{p}}.$$

We will arrange now this equation in a different form. To this, right hand term $u/(1 + u)$ is moved to the left-hand side, while the left hand term $1/(2(1+\hat{u}))$ is moved to the right-hand side. In this way, we obtain a new inequality where the left-hand side is

$$\left.\frac{\hat{u}}{1+\hat{u}} - \frac{u}{1+u}\right|_{\bar{p}} = \left.\frac{\hat{u} - u}{(1+u)(1+\hat{u})}\right|_{\bar{p}}$$

and where the right-hand side is

$$\left.\frac{u(1+\hat{u})}{2\hat{u}(1+u)^2} - \frac{1}{2(1+\hat{u})}\right|_{\bar{p}} = \left.\frac{1}{2\hat{u}(1+\hat{u})(1+u)^2}(u(1+\hat{u})^2 - \hat{u}(1+u)^2)\right|_{\bar{p}}.$$

This last expression can be further arranged into

$$\left.\frac{1}{2\hat{u}(1+\hat{u})(1+u)^2}(\hat{u} - u)(\hat{u}u - 1)\right|_{\bar{p}}.$$

Thus combining the results on the left and right hands we conclude that we would like the inequality

$$\frac{\hat{u}-u}{(1+u)(1+\hat{u})}\bigg|_{\bar{p}} > \frac{1}{2\hat{u}(1+\hat{u})(1+u)^2}(\hat{u}-u)(\hat{u}u-1)\bigg|_{\bar{p}}$$

to be satisfied. Thus we would like to have

$$2(\hat{u}-u)\hat{u}(1+u)\bigg|_{\bar{p}} > (\hat{u}-u)(\hat{u}u-1)\bigg|_{\bar{p}},$$

but because we are assuming $\hat{u}(\bar{p}) > \max\{1, P\}u(\bar{p}) \geq u(\bar{p})$ the inequality above is clearly satisfied. □

E. The Penrose inequality

In this section, we will prove the Penrose inequality for asymptotically flat static solutions with regular and connected horizon. We start by observing and interesting corollary to Theorem 2.

Corollary 2 (To Theorem 2): Let $(\Sigma, \mathrm{g}, \ln N)$ *be an asymptotically flat static solution with connected and regular horizon. Suppose that the Penrose inequality does not hold, namely, assume that the Penrose quotient* $P = \frac{A}{16\pi m^2}$ *is greater than one. Then, for any divergent sequence of points* $\{p_i\}$*, the associated coordinate-distance lag is greater or equal than zero, namely,* $\bar{\delta}(\{p_i\}) \geq 0$.

Proof: If $P > 1$, then $\max\{1, \frac{1}{P}\} = 1$ and from Theorem 2 we have then

$$\hat{\mathsf{s}}(p) = \frac{2m}{1-N^2(p)} - 2m \leq \mathsf{s}(p), \text{ for all } p \in \Sigma.$$

Evaluating this inequality at $\{p_i\}$ and using the asymptotic of $\hat{\mathsf{s}}$ described in Eq. (37) we get

$$0 \leq \mathsf{s}(p_i) - r(p_i) + 2m + O(\frac{1}{r(p_i)}).$$

Therefore,

$$0 \leq \limsup_{i\to\infty} \mathsf{s}(p_i) - r(p_i) + 2m = \bar{\delta}(\{p_i\})$$

as desired. □

The following proposition, however, shows (in particular) that if the Penrose inequality does not hold, then there is a divergent sequence $\{p_i\}$ whose coordinate-distance lag is negative, namely, $\bar{\delta}(\{p_i\}) < 0$. The two results thus show the Penrose inequality on asymptotically flat static solutions with regular and connected horizon.

Proposition 12: Let $(\Sigma, \mathrm{g}, \ln N)$ *be an asymptotically flat static solution with regular and connected horizon H. Then, there is a divergent sequence* $\{p_i\}$ *such that*

$$\bar{\delta}(\{p_i\}) \leq m(1-P).$$

In particular if $P > 1$*, then* $\bar{\delta}(\{p_i\}) < 0$.

Proof: Let $\{\Gamma_i\}_{i=1}^{i=\infty}$ be a sequence such that $\Gamma_i \downarrow 0$ (with $\Gamma_1 \leq \Gamma_0$ and Γ_0 regular as in Notation 1), and let $\{r_i\}_{i=1}^{i=\infty}$ be a sequence such that $r_i \uparrow \infty$ (and r_1 as in Proposition 6). Consider the congruence of length minimizing geodesics $\mathcal{F}$ emanating perpendicularly to H_{Γ_i}. The geodesic segment, γ_i, minimizing the length between H_{Γ_i} and S_{r_i} is clearly in $\mathcal{F}$. Let p_i be the point of γ_i at S_{r_i}, let q_i be the initial point at H_{Γ_i}, and let $v(q_i)$ be the (unit) velocity of γ_i at q_i. γ_i is naturally perpendicular to S_{r_i} at p_i and to H_{Γ_i} at q_i. Consider now the exponential map $exp : \mathcal{N}_i \to \Sigma$, where $\mathcal{N}_i$ is the inner-normal bundle of $\Sigma \setminus \Omega_{H,H_{\Gamma_i}}$ at H_{Γ_i} as in Notation 2. Assume that the differential of the exponential map is smooth at the point $l(\gamma_i)v(q_i)$ in $\mathcal{N}_i$, if not, work instead with a suitable function $\tilde{\mathsf{s}}_{\Gamma_i}$ as in Proposition 10. Note that, in the notation of Proposition 10, we have $l(\gamma_i) = \mathsf{s}_{\Gamma_i}(p_i)$. Then, there is ϵ_i such that the surface defined by $\bar{S}_i = \{\exp(l(\gamma_i)v(q)),\ q \in B_{H_{\Gamma_i}}(q_i, \epsilon_i)\}$ is smooth. Moreover, $\bar{S}_i$ is tangent to S_{r_i} at p_i, its mean curvature is equal to the mean curvature θ of $\mathcal{F}$ restricted to it and, because γ_i is length minimizing between S_{r_i} and H_{Γ_i}, it lies inside the region enclosed by H_{Γ_i} and S_{r_i}. Therefore, from the standard comparison of mean curvatures we have

$$\theta(p_i) \geq \theta_{r_i}(p_i),$$

where θ_{r_i} is the mean curvature of S_{r_i}. Consider now $\mathcal{M}$ with $a = A/8\pi m$ and over γ_i. As $\mathcal{M}$ is monotonic we have

$$\theta(p_i) \leq \frac{2}{\mathrm{s}_{\Gamma_i} + \frac{A}{8\pi m}} + \frac{2\mathcal{M}(q_i)}{(\mathrm{s}_{\Gamma_i} + \frac{A}{8\pi m})^2 N^2(p_i)}.$$

Now, to use this equation we need several facts. First, from Proposition 7 we have $\theta_{r_i} = 2/r_i + 2m/(r_i^2) + O(1/r_i^3)$. Therefore, we have

$$\frac{2}{r_i} + \frac{2m}{r_i(r_i - 2m)} + O(1/r_i^3) \leq \frac{2}{\mathrm{s}_{\Gamma_i} + \frac{A}{8\pi m}} + \frac{2\mathcal{M}(q_i)}{(\mathrm{s}_{\Gamma_i} + \frac{A}{8\pi m})^2 N^2(p_i)}. \tag{50}$$

We can arrange this better as

$$\frac{2(\mathrm{s}_{\Gamma_i} + \frac{A}{8\pi m} - r_i)}{r_i(\mathrm{s}_{\Gamma_i} + \frac{A}{8\pi m})} + \frac{2m}{r_i(r_i - 2m)} - \frac{2\mathcal{M}(q_i)}{(\mathrm{s}_{\Gamma_i} + \frac{A}{8\pi m})^2 N^2(p_i)} \leq O(1/r_i^3). \tag{51}$$

Second, from Proposition (21) we have $\lim \mathcal{M}(q_i) = |\nabla N|_H (\frac{A}{8\pi m})^2 = \frac{A}{16\pi m}$. Finally, we have $\lim \mathrm{s}(p_i) - \mathrm{s}_{\Gamma_i}(p_i) = 0$ and from Proposition 8 it is $\lim r_i/\mathrm{s}_{\Gamma_i} = 1$. Multiplying Eq. (51) by $\mathrm{s}^2_{\Gamma_i}$, taking the limsup while using the facts described above gives finally

$$\bar{\delta}(\{p_i\}) = \limsup \mathrm{s}(p_i) - r_i + 2m \leq m(1 - P)$$

as desired. □

Using Corollary 2 and Proposition 12 we deduce the Penrose inequality.

Proposition 13 (The Penrose inequality): Let (Σ, g, N) *be an asymptotically flat static solution with a regular and connected horizon H. Let A be the area of H and m be the ADM mass of the solution. Then*

$$A \leq 16\pi m^2. \tag{52}$$

F. The opposite Penrose inequality

In this section, we prove the *opposite Penrose inequality,* namely, that $A \geq 16\pi m^2$. The proof will follow after carefully studying the behavior of the quotient $\hat{\mathrm{s}}/\mathrm{s}$ at the singularity of g, namely, the (unique) horizon H, and using then the distance comparison in Theorem 2. We will denote by κ the Gaussian curvature of the two-metric on H inherited from g.

Proposition 14: Let (Σ, g, N) *be an asymptotically flat static solution with regular and connected horizon. Consider a g-geodesic* γ *starting perpendicularly from H at q, and parameterized with respect to the g-arc length of* γ *from q,* τ. *Define* $\hat{\mathrm{s}}(\gamma(\tau)) = \int_0^\tau N(\gamma(\tau))d\tau$. *Then, we have*

$$\frac{d}{d\hat{\hat{\mathrm{s}}}}\frac{\hat{\mathrm{s}}}{\hat{\hat{\mathrm{s}}}}\bigg|_q = 8m(\frac{4\pi m}{A})^2 - 2m\kappa\bigg|_q. \tag{53}$$

Proof: Note that, as is written in the statement of the proposition, we will work in the natural representation (Σ, g, N) of the static solution.

Now first we note that $d\hat{\hat{\mathrm{s}}}(\tau)/d\tau = N(\alpha(\tau))$. Derivatives with respect to τ will be denoted by a prima, i.e., $f'(\alpha(\tau))' = df(\alpha(\tau))/d\tau$. We compute (when $\tau \neq 0$)

$$\frac{d}{d\hat{\hat{\mathrm{s}}}}\frac{\hat{\mathrm{s}}}{\hat{\hat{\mathrm{s}}}} = \frac{2m((2\frac{N'}{1-N^2} + 2\frac{N^2N'}{(1-N^2)^2})\hat{\mathrm{s}} - 2m\frac{N^2}{(1-N^2)})}{\hat{\hat{\mathrm{s}}}^2}. \tag{54}$$

We want to calculate now the limit of this expression when $\tau \to 0$. We will separate the right-hand side of (54) into two terms and calculate the limit for each one of them separately. The first limit we will calculate is

$$\lim_{\tau\to 0}\frac{4mN^2N'}{(1-N^2)^2\hat{\hat{\mathrm{s}}}} = 4m|\nabla N|_H \lim_{\tau\to 0}\frac{N^2}{\hat{\hat{\mathrm{s}}}}, \tag{55}$$

which arises from the middle term on the right-hand side of Eq. (54). The right-hand side of (55) was obtained using that $N'(\tau) \to |\nabla N|_H$ and $(1 - N^2)^2 \to 1$. We calculate now the limit on the right-hand side of (55) using L'Hôpital rule and we have

$$\lim_{\tau\to 0}\frac{N^2}{\hat{\hat{\mathrm{s}}}} = \lim_{\tau\to 0} 2N' = 2|\nabla N|_H.$$

Thus we get

$$\lim_{\tau\to 0}\frac{4mN^2N'}{(1-N^2)^2\hat{\hat{\mathrm{s}}}} = 8m(|\nabla N|_H^2) = 8m(\frac{4\pi m}{A})^2. \tag{56}$$

The second limit that we will calculate is

$$\lim_{\tau\to 0}\frac{2m}{1-N^2}\frac{(2N'\hat{\hat{\mathrm{s}}} - N^2)}{\hat{\hat{\mathrm{s}}}^2} = 2m\lim_{\tau\to 0}\frac{(2N'\hat{\hat{\mathrm{s}}} - N^2)}{\hat{\hat{\mathrm{s}}}^2}, \tag{57}$$

which arises from the combination of the first and third terms on the right-hand side of (54). Again, to obtain the right-hand side of (57), we use the fact that the factor $2m/(1 - N^2)$ would be, in the limit, $2m$. We calculate the limit on the right-hand side of (57) by L'Hôpital rule and obtain

$$2m\lim_{\tau\to 0}\frac{(2N' - 2N' + 2\hat{\hat{\mathrm{s}}}\frac{N''}{N})}{2\hat{\hat{\mathrm{s}}}} = 2m\ Ric(n,n), \tag{58}$$

where $n = \alpha'(0)$ is the outward g-unit normal vector to H at $\alpha(0)$. To obtain the right-hand side above we used the static equation (2), namely, $N''(\alpha(0)) = Ric(\alpha'(0), \alpha'(0))N(\alpha(0))$ (note that $\alpha(\tau)$ is a g-geodesic).

Recall now the structure equation $2\kappa(q) + |\Theta|^2(q) - \theta^2(q) = R(q) - 2Ric(n(q), n(q))$, where q is a point in H. Again, n is the outward g-unit normal vector to H at q. $\Theta(q)$ and $\theta(q)$ are the second fundamental forms of H, calculated using g, and evaluated at q. For a regular horizon we know that $\Theta = 0$, $\theta = 0$. R and Ric are the scalar and Ricci curvatures of g, respectively. For a static solution (Σ, g, N) it is $R = 0$ everywhere. κ, as said above is the Gaussian curvature of H with the two-metric inherited from g. Thus, from the structure equation we get that for all q in H we have $\kappa(q) = -Ric(n, n)$. Using this fact in (58) and combining (58) and (56) to complete the limit (54), we obtain (53). □

Proposition 15: Let (Σ, g, N) *be an asymptotically flat static solution with regular and connected horizon. If there is a point* q *at* H *for which*

$$\kappa(q) < 4(\frac{4\pi m}{A})^2, \tag{59}$$

then there is a point p *in* $\Sigma\backslash H$ *such that* $\hat{\mathrm{s}}(p)/\mathrm{s}(p) > 1/P$, *where* P *is the Penrose quotient.*

Proof: Suppose there is a point q in H for which inequality (59) hods. By Proposition 14, there is a g-geodesic emanating perpendicularly to H for which

$$\frac{d}{d\hat{\hat{\mathrm{s}}}}\frac{\hat{\mathrm{s}}}{\hat{\hat{\mathrm{s}}}} > 0.$$

Also applying L'hôpital rule we get

$$\lim_{\tau\to 0}\frac{\hat{\mathsf{s}}}{\hat{\hat{\mathsf{s}}}}=\lim_{\tau\to 0}\frac{\frac{4mNN'}{(1-N^2)^2}}{N}=4m|\nabla N|_H=\frac{1}{P}.$$

Therefore, we have $\hat{\mathsf{s}}(\gamma(\tau))/\hat{\hat{\mathsf{s}}}(\gamma(\tau))>1/P$ for τ small. Now we observe that $\hat{\hat{\mathsf{s}}}(\gamma(\tau))\geq \mathsf{s}(\gamma(\tau))$ because s is the g-distance function to H and $\hat{\hat{\mathsf{s}}}(\gamma(\tau))$ is the g-length of γ between $\gamma(0)$ and $\gamma(\tau)$. Thus, for τ small we have

$$\frac{\hat{\mathsf{s}}(\gamma(\tau))}{\mathsf{s}(\gamma(\tau))}=\frac{\hat{\mathsf{s}}(\gamma(\tau))}{\hat{\hat{\mathsf{s}}}(\gamma(\tau))}\frac{\hat{\hat{\mathsf{s}}}(\gamma(\tau))}{\mathsf{s}(\gamma(\tau))}\geq\frac{\hat{\mathsf{s}}(\gamma(\tau))}{\hat{\hat{\mathsf{s}}}(\gamma(\tau))}>\frac{1}{P}.$$

□

Corollary 3: Let (Σ, g, N) *be an asymptotically flat static solution with regular and connected horizon H. Then, H is homeomorphic to a two-sphere and the inverse Penrose inequality holds, A* $\geq 16\pi m^2$. *Moreover, if the Penrose inequality holds, namely,* $A\leq 16\pi m^2$, *then* $\kappa = 4\pi/A$ *and the horizon is round.*

Proof: By Proposition 15 if there is a point q in H for which $\kappa(q)<4(4\pi m/A)^2$, then there is point p in $\Sigma\backslash H$ such that $\hat{s}(p)/s(p)>1/P$ but this contradicts the distance comparison of Theorem 2. Therefore, $\kappa\geq 4(4\pi m/A)^2$ and, by Gauss-Bonnet, H must be homeomorphic to a two sphere. Moreover,

$$\int_H \kappa dA=4\pi\geq 4(\frac{4\pi m}{A})^2.$$

Thus

$$A\geq 16\pi m^2,$$

which finishes the first part of the claim. Suppose now that $A\leq 16\pi m^2$ then, as $\kappa\geq 4(4\pi m/A)^2$ we must have $k=4(4\pi m/A)^2=4\pi/A$ which finishes the claim. □

G. The uniqueness of the Schwarzschild solution

1. Further properties of the coordinate-distance lag

The proof of the uniqueness of the Schwarzschild solutions does not follow directly in our setting from the equality $A = 16\pi m^2$. Indeed it is required first to prove that for any divergence sequence $\{p_i\}$ the associated coordinate-distance lag $\bar{\delta}(\{p_i\})$ is zero. We advocate now to prove this intermediate step. We need two preliminary propositions. We start showing that $|\nabla\hat{\mathsf{s}}|\leq 1$.

Proposition 16: Let $(\Sigma, \mathrm{g}, \ln N)$ *be an asymptotically flat static solution with regular and connected horizon. Then,* $|\nabla\hat{\mathsf{s}}|_{\mathrm{g}}\leq 1$.

Proof: We observe first that $\lim_{\mathsf{s}(p)\to\infty}|\nabla\hat{\mathsf{s}}|_{\mathrm{g}}(p)=1$. But we also have $\lim_{\mathsf{s}(p)\to 0}|\nabla\hat{\mathsf{s}}|_{\mathrm{g}}=1$. To see this last claim we compute

$$|\nabla\hat{\mathsf{s}}|_{\mathrm{g}}(p)=\frac{4m}{(1-N^2(p))^2}|\nabla N(p)|_g\to 4m|\nabla N|_H.$$

But we already know from Corollary 3 that $P = 1$ and thus $|\nabla N|_H = 4\pi m/A = 1/4m$. The claim follows.

We show now that there cannot exist a point p in $\Sigma\backslash H$ for which $|\nabla\hat{\mathsf{s}}|(p)>1$. We will assume without loss of generality that $m = 1$. The assumption simplifies the writing. Define

$$\hat{\mathsf{s}}_\alpha=\frac{1}{1-N^{2\alpha}}-1$$

and thus

$$N^{2\alpha} = \frac{\hat{\mathsf{s}}_\alpha}{\hat{\mathsf{s}}_\alpha + 1}.$$

Then, we compute

$$2\alpha N^{2\alpha-1}\nabla N = \frac{1}{(\hat{\mathsf{s}}_\alpha + 1)^2}\nabla \hat{\mathsf{s}}_\alpha$$

and thus

$$\frac{\nabla N}{N} = \frac{1}{2\alpha \hat{\mathsf{s}}_\alpha(\hat{\mathsf{s}}_\alpha + 1)}\nabla \hat{\mathsf{s}}_\alpha.$$

But $\Delta \ln N = 0$ and then $\nabla(1/(\hat{\mathsf{s}}_\alpha(\hat{\mathsf{s}}_\alpha + 1))\nabla \hat{\mathsf{s}}_\alpha) = 0$ which can be written as

$$\Delta \hat{\mathsf{s}}_\alpha = \frac{2\hat{\mathsf{s}}_\alpha + 1}{\hat{\mathsf{s}}_\alpha(\hat{\mathsf{s}}_\alpha + 1)}|\nabla \hat{\mathsf{s}}_\alpha|^2. \tag{60}$$

The interesting thing about this expression is that it does not depend explicitly on α. We note too that we have

$$< \frac{\nabla N}{N}, \nabla \hat{\mathsf{s}}_\alpha >= \frac{1}{2\alpha \hat{\mathsf{s}}_\alpha(\hat{\mathsf{s}}_\alpha + 1)}|\nabla \hat{\mathsf{s}}_\alpha|^2. \tag{61}$$

The crucial and obvious observation about the family $\{\hat{\mathsf{s}}_\alpha\}$ is that given any open set Ω of compact closure $\bar{\Omega} \subset \Sigma \setminus H$ then $\hat{\mathsf{s}}_\alpha$ converges uniformly in C^2 to s over $\bar{\Omega}$ as $\alpha \to 1$. Thus it follows from the limits of s at H and infinity observed at the beginning that if $\max\{|\nabla \mathsf{s}|(q), q \in \Sigma\} > 1$, then there is an $\epsilon > 0$ such that for every α with $|\alpha - 1| < \epsilon$ the function $|\nabla \hat{\mathsf{s}}_\alpha|$ posses at least one local maximum greater than one. For a given α we will denote by p_α a point at which a local maximum of $\hat{\mathsf{s}}_\alpha$ greater than one takes place.

We will use Weitzenböck's formula

$$\frac{1}{2}\Delta|\nabla \hat{\mathsf{s}}_\alpha|^2 = |\nabla\nabla \hat{\mathsf{s}}_\alpha|^2 + < \nabla \Delta \hat{\mathsf{s}}_\alpha, \nabla \hat{\mathsf{s}}_\alpha > + 2 < \frac{\nabla N}{N}, \nabla \hat{\mathsf{s}}_\alpha >^2, \tag{62}$$

and we will use it evaluated at p_α. We note first that for every vector $w \in T_{p_\alpha}\Sigma$ we have $< \nabla_w \nabla \hat{\mathsf{s}}_\alpha, \nabla \hat{\mathsf{s}}_\alpha >= 0$. Because of this we have $|\nabla\nabla \hat{\mathsf{s}}_\alpha|^2 = |\nabla\nabla \hat{\mathsf{s}}_\alpha|^2_{T_{p_\alpha}\Sigma} = |\nabla\nabla \hat{\mathsf{s}}_\alpha|_{\nabla \hat{\mathsf{s}}_\alpha(p_\alpha)^\perp}$ where $\nabla \hat{\mathsf{s}}_\alpha(p_\alpha)^\perp$ is the perpendicular subspace to $\nabla \hat{\mathsf{s}}_\alpha$ in $T_{p_\alpha}\Sigma$. Thus, we have

$$|\nabla\nabla \hat{\mathsf{s}}_\alpha|^2(p_\alpha) \geq \frac{1}{2} tr_{\nabla \hat{\mathsf{s}}_\alpha(p_\alpha)^\perp}\nabla\nabla \hat{\mathsf{s}}_\alpha = \frac{1}{2}\Delta \hat{\mathsf{s}}_\alpha(p_\alpha).$$

This expression will be used in the first term on the right-hand side of Eq. (62). For the second instead we note from Eq. (60) that

$$\nabla \Delta \hat{\mathsf{s}}_\alpha \Big|_{p_\alpha} = -\left(\frac{1}{\hat{\mathsf{s}}_\alpha^2} + \frac{1}{(\hat{\mathsf{s}}_\alpha + 1)^2}\right)|\nabla \hat{\mathsf{s}}_\alpha|^2 \Big|_{p_\alpha}.$$

For the third term on the right-hand side of Eq. (62), we will use Eq. (61). All together gives for Eq. (62) the expression

$$0 \geq \frac{1}{2}\Delta|\nabla \hat{\mathsf{s}}_\alpha|^2 \Big|_{p_\alpha} \geq |\nabla \hat{\mathsf{s}}_\alpha|^2\left(\frac{(2\hat{\mathsf{s}}_\alpha + 1)^2}{2(\hat{\mathsf{s}}_\alpha^2(\hat{\mathsf{s}}_\alpha + 1)^2} - \frac{\hat{\mathsf{s}}_\alpha^2 + (\hat{\mathsf{s}}_\alpha + 1)^2}{\hat{\mathsf{s}}_\alpha^2(\hat{\mathsf{s}}_\alpha + 1)^2} + \frac{2}{4\alpha^2}\frac{1}{\hat{\mathsf{s}}_\alpha^2(\hat{\mathsf{s}}_\alpha + 1)^2}\right)\Big|_{p_\alpha}.$$

Further expanding the term in parenthesis we obtain

$$0 \geq \frac{1}{2}\Delta|\nabla \hat{\mathsf{s}}_\alpha|^2 \Big|_{p_\alpha} \geq \frac{|\nabla \hat{\mathsf{s}}_\alpha|^2}{2\hat{\mathsf{s}}_\alpha^2(\hat{\mathsf{s}}_\alpha + 1)^2}\left(-1 + \frac{1}{\alpha}\right)\Big|_{p_\alpha}.$$

Choosing α such that $1 - \epsilon < \alpha < 1$ we get a contradiction. This finishes the proof of the proposition. □

Define now $\delta = \mathsf{s} - \hat{\mathsf{s}}$. We will study δ, and it will be shown that it has asymptotically positive Laplacian (in a barer sense).

Proposition 17: Let $(\Sigma, \mathsf{g}, \ln N)$ *be an asymptotically flat static solution with regular and connected horizon H. The Laplacian of* δ *has the following asymptotic expression:*

$$\Delta\delta \leq \frac{-\delta}{(s+2m)^2} + O(\frac{1}{s^3}),$$

in the barer sense.

Note that $\delta \geq 0$. However, note too that because there are sequences $\{p_i\}$ for which $\delta(p_i) \to 0$, it cannot be said that $\Delta\delta$ becomes negative outside a sufficiently big compact set. The asymptotic expression is, however, still valid.

Proof: Recall first the expression for $\Delta\hat{\mathsf{s}}$ in Eq. (34). We find first the asymptotic expression for $|\nabla\hat{\mathsf{s}}|^2$. But observing that $\hat{\mathsf{s}} = 2m(\frac{1}{1-N^2} - 1)$ it is easily deduced from the asymptotic expression of N that $\nabla\hat{\mathsf{s}} = \nabla r + O(1/r^2)$. Thus $|\nabla\hat{\mathsf{s}}|^2 = 1 + (1/r^2) = 1 + O(1/s^2)$.

Now subtract to the expression (38) with $P = 1$ and $|\nabla\mathsf{s}|^2 = 1$, the expression (34). That gives

$$\begin{aligned}\Delta\delta &\leq \frac{2}{\mathsf{s}+2m}(1+\frac{m}{\mathsf{s}+2m}) - \frac{2}{\hat{\mathsf{s}}+2m}(1+\frac{m}{\hat{\mathsf{s}}+2m}) + O(\frac{1}{\mathsf{s}^3})\\ &= \frac{-2\delta}{(\mathsf{s}+2m)(\hat{\mathsf{s}}+2m)} + 2m\frac{(\hat{\mathsf{s}}^2-\mathsf{s}^2)}{(\mathsf{s}+2m)^2(\hat{\mathsf{s}}^2+2m)^2} + O(\frac{1}{\mathsf{s}^3}).\end{aligned}$$

Thus

$$\Delta\delta \leq \frac{-\delta}{(\mathsf{s}+2m)^2} + O(\frac{1}{\mathsf{s}^3})$$

as claimed. □

We prove now a crucial property of δ, namely, that it is Lipschitz "at large scales." To explain the concept we need to introduce some terminology. Let $\{(r, \theta, \varphi)\}$ be a coordinate system as in Proposition 6. Let D be the annulus in $\mathbb{R}^3$, $D = \{(r, \theta, \varphi), 1 \leq r \leq 2\}$. For any $\lambda > 0$ sufficiently small consider the map from D into Σ given by $\bar{x} \to \bar{x}/\lambda$. Denote by δ_λ the pull-back of δ to D, namely, $\delta_\lambda(\bar{x}) = \delta(\bar{x}/\lambda)$. Let $\bar{x}_1$ and $\bar{x}_2$ be two points in D. Denote by $\phi(\bar{x}_s, \bar{x}_2)$ the angle formed by $\bar{x}_1$ and $\bar{x}_2$, namely, $<\bar{x}_1, \bar{x}_2> = |\bar{x}_1||\bar{x}_2|\cos\phi(\bar{x}_1, \bar{x}_2)$. We would like to show that there is $\lambda_0 > 0$ and $K > 0$ such that δ_λ is Lipschitz with constant K for any $0 < \lambda < \lambda_0$. The next Proposition explains this property and two further that will also be needed later. It is perhaps the most technical, but otherwise straightforward Proposition of the article.

Proposition 18: Let $\delta = \mathsf{s} - \hat{\mathsf{s}}$. *Then,*

1. *there exists* $K > 0$ *and* $\lambda_0 > 0$ *such that for any* $\bar{x}_1, \bar{x}_2$ *in D and* $0 < \lambda < \lambda_0$ *we have*

$$|\delta_\lambda(\bar{x}_1) - \delta_\lambda(\bar{x}_2)| \leq K|\bar{x}_1 - \bar{x}_2|.$$

2. *Let* $\bar{x}_1$ *and* $\bar{x}_2$ *be two points in D belonging to the same radial line, namely,* $\bar{x}_1 = \beta\bar{x}_2$. *Then for any sequence* $\{\lambda_i\} \downarrow 0$ *we have* $|\delta_{\lambda_1}(\bar{x}_1) - \delta_{\lambda_i}(\bar{x}_2)| \to 0$.

Proof: In Σ consider a coordinate sphere $S_{r_0} = \{\bar{x}/r(\bar{x}) = r_0\}$ (where $\{\bar{x}\}$ is a coordinate system as in Proposition 6). The distance function from S_{r_0} to H is Lipschitz, say with constant K_1, namely, for any q_0, q_1 in S_{r_0} we have $|\mathsf{s}(q_0) - \mathsf{s}(q_1)| \leq K_1|\phi(q_0, q_1)|$.

Let now $\bar{x}_1$ be a point in D. Let λ such that $|\bar{x}_1|/\lambda \gg r_0$. Denote $p_1 = \bar{x}_1/\lambda$. Let γ_1 be the length minimizing geodesic joining $\bar{x}_1/\lambda$ to H. Let q_1 be the point of intersection of γ_1 with S_{r_0}.

Consider a rotation of angle ϕ_0 in $\mathbb{R}^3$, denote it by R_{ϕ_0}. Also denote by $p_2 = R_{\phi_0}(p_1)$, $\gamma_2 = R_{\phi_0}(\gamma_1)$, and $q_2 = R_{\phi_0}(q_1)$. Let l_1 be the length of γ_1 between p_1 and q_1 and let l_2 be the length between p_2 and q_2 of γ_2.

We will show first that there is a constant $K_2 > 0$ independent on λ such that $|l_1 - l_2| \leq K_2|\phi_0|$. Note that in the coordinate system $\{\bar{x}\}$ we have $\mathrm{g} = \mathrm{g}_S + O(1/r^3)$. Suppose γ_1 is parameterized with respect to the arc-length, $\bar{\mathtt{s}}$, provided by the Schwarzschild metric g_S. Let $l(\phi) = l(R_\phi(\gamma_1))$, where $0 < \phi < \phi_0$. Then, we have

$$|\partial_\phi l| = |\int_{\bar{\mathtt{s}}_0=0}^{\bar{\mathtt{s}}_1} \frac{\mathrm{g}(\nabla_{\partial_\phi}\gamma', \gamma')}{\mathrm{g}(\gamma', \gamma')^{\frac{1}{2}}} d\bar{\mathtt{s}}|. \tag{63}$$

Moreover,

$$\mathrm{g}(\nabla_{\partial_\phi}\gamma', \gamma') = \mathrm{g}((\nabla_{\partial_\phi} - \nabla^S_{\partial_\phi})\gamma', \gamma') + (\mathrm{g} - \mathrm{g}_S)(\nabla^S_{\partial_\phi}\gamma', \gamma') + \mathrm{g}_S(\nabla^S_{\partial_\phi}\gamma', \gamma').$$

We note now that the last term on the right-hand side of the previous equation is zero, and the first two terms on the right-hand side are $O(1/\bar{\mathtt{s}}^2)$. Using this in Eq. (63) we get that $|l_1 - l_2| \leq K_2|\phi_0|$ as desired.

We have now

$$\mathtt{s}(p_2) \leq l_2 + \mathtt{s}(q_2) \leq l_1 + \mathtt{s}(q_1) + K_1|\phi_0| + K_2|\phi_0| = \mathtt{s}(p_1) + (K_1 + K_2)|\phi_0|.$$

Because p_1 and ϕ are arbitrary we have

$$|\mathtt{s}(p_1) - \mathtt{s}(p_2)| \leq K|\phi_0|.$$

Thus for any $\bar{x}_1$ and $\bar{x}_2$ in D of equal norm, $|\bar{x}_1| = |\bar{x}_2|$, and λ (sufficiently small), we have

$$|\delta_\lambda(\bar{x}_1) - \delta_\lambda(\bar{x}_2)| = |\mathtt{s}(\frac{\bar{x}_1}{\lambda}) - \frac{|\bar{x}_1|}{\lambda} + 2m - \mathtt{s}(\frac{\bar{x}_2}{\lambda}) + \frac{|\bar{x}_2|}{\lambda} - 2m| = |\mathtt{s}(\frac{\bar{x}_1}{\lambda}) - \mathtt{s}(\frac{\bar{x}_2}{\lambda})| \leq K|\phi(\bar{x}_1, \bar{x}_2)|. \tag{64}$$

We continue with an observation. Recall that the Ricci curvature of g decays, in r, as $O(1/r^3)$ (in facts it decays as $1/r^4$). Consider the annulus $D_\lambda = \{\bar{x},\ \lambda^{1/12} \leq |\bar{x}| \leq 2\}$ and consider the map from D_λ into Σ given by $\bar{x} \to \bar{x}/\lambda$. Let g_λ be the pull-back of the metric g under this map. From the fact that $|\mathtt{Ric}|$ decays as $O(1/r^3)$ we get $\sup\{|\mathtt{Ric}_{\mathrm{g}_\lambda}(\bar{x})|_{\mathrm{g}_\lambda}/\bar{x} \in D_\lambda\} = O(\lambda^{\frac{1}{4}})$. From this it follows that, as λ tends to zero, and therefore as D_λ tends to the closed ball of radius two minus the origin, the metrics g_λ converge in $C^{1,\beta}$ (for any $0 < \beta < 1$) to the flat metric over any fixed annulus D_{λ_1}, $0 < \lambda_1 < 2$. Thus for any $\bar{x} \in D$ and sequence $\{\lambda_i\} \downarrow 0$, length minimizing geodesics, γ_p, joining $p = \bar{x}/\lambda$ to H converge in C^1 over any D_{λ_1} to the radial line passing through $\bar{x}$.

What we would like to know now is the "rate" at which the geodesics approach the radial lines. More precisely, we will study the g_S-angle ξ, formed by ∂_r and γ' at any point along γ. To this respect we proceed as follows. Consider the rotational killing fields X of the Schwarzschild solution. For every X, we have $|X|_{\mathrm{g}} = r(1 + O(1/r))$. Given one of the X's, we compute, along the geodesic γ_p (again $p = \bar{x}/\lambda$),

$$\mathrm{g}(\gamma', X)' = \mathrm{g}(\gamma', \nabla_{\gamma'}X) = \mathrm{g}(\gamma', (\nabla_{\gamma'} - \nabla^S_{\gamma'})X) + \mathrm{g}_S(\gamma', \nabla^S_{\gamma'}X) + (\mathrm{g} - \mathrm{g}_S)(\gamma', \nabla^S_{\gamma'}X).$$

The second term on the right-hand side of the previous equation is zero, while the other two are of the order $O(1/r^2) = O(1/\mathtt{s}^2)$. Let q be the first point where γ_p reaches the radial sphere S_{r_0} (r_0 is fixed) and let p_1 be any intermediate point between p and q. Integrate now $\mathrm{g}(\gamma', X)'$ (with respect to the g arc-length, $\mathtt{s}$) between $\mathtt{s}(p_1)$ and the value of $\mathtt{s}(q)$ using the estimate we have found before for $\mathrm{g}(\gamma', X)$ to get

$$|\mathrm{g}(\gamma', X)(p_1) - \mathrm{g}(\gamma', X)(q)| \leq c_1,$$

where c_1 is a constant independent on p_1 and q. Note that this inequality is valid for any rotational killing field X. Observing that rotational killing fields at S_{r_0} have bounded norm, we get

$$|\mathrm{g}(\gamma', X)(p_1)| \leq c_2,$$

where c_2 is a constant. Moreover,

$$\mathrm{g}_S(\gamma', X) = \mathrm{g}(\gamma', X) + (\mathrm{g}_S - \mathrm{g})(\gamma', X) = \mathrm{g}(\gamma', X) + O(1/r).$$

Thus, we have

$$|\mathrm{g}_S(\gamma', X)| \leq c_3,$$

where c_3 is a constant. Pick now the rotational killing field X which is collinear, at p_1, to the component of γ', g_S-perpendicular to ∂_r. Let ξ be the g_S-angle formed by ∂_r and γ'. We have

$$|\mathrm{g}_S(\gamma', X)(p_1)| = |X|_{\mathrm{g}_S}(p_1)||\gamma'|_{\mathrm{g}_S}|\sin\xi(p_1)| \leq c_4,$$

where c_4 is a constant. So we get

$$|\sin\xi| \leq \frac{c_5}{r},$$

where c_5 is a constant. We have

$$\frac{dr}{d\mathtt{s}} = \mathrm{g}_S(\nabla^S r, \gamma') = 1 + O(1/r^2) = 1 + O(1/\mathtt{s}^2). \tag{65}$$

We will use this inequality in what follows. Let $\bar{x}_1$ be a point in D. Let $p_1 = \bar{x}_1/\lambda$ and let γ be a geodesic minimizing the length between p_1 and H. Let p_2 be a point in γ such that $p_2 = \bar{x}_2/\lambda$ with $\bar{x}_2$ in D. Integrating (65) between $\mathtt{s}(p_1)$ and $\mathtt{s}(p_2)$ we get

$$r(p_1) - \mathtt{s}(p_1) = r(p_2) - \mathtt{s}(p_2) + |\bar{x}_1 - \bar{x}_2|O(\lambda).$$

Therefore,

$$|\delta_\lambda(\bar{x}_1) - \delta_\lambda(\bar{x}_2)| = |\bar{x}_1 - \bar{x}_2|O(\lambda). \tag{66}$$

We are ready to prove the proposition. Let $\bar{x}_1$ and $\bar{x}_2$ be two points in D. Let $p_1 = \bar{x}_1/\lambda$ and $p_2 = \bar{x}_2/\lambda$. Let $p_3 = \bar{x}_3/\lambda$ be the point of intersection of the length minimizing geodesic joining p_1 to H and the coordinate sphere $S_{|\bar{x}_2/\lambda|}$. From (64) and (66) we get

$$|\delta_\lambda(\bar{x}_1) - \delta_\lambda(\bar{x}_2| \leq |\delta_\lambda(\bar{x}_1) - \delta_\lambda(\bar{x}_3)| + |\delta_\lambda(\bar{x}_3 - \delta_\lambda(x_2)| \leq |\bar{x}_1 - \bar{x}_3|O(\lambda) + K\phi(\bar{x}_3, \bar{x}_2).$$

As $|\bar{x}_1 - \bar{x}_3| \leq c_6 d_D(\bar{x}_1, \bar{x}_3)$, for some constant c_6, the *item 1* of the proposition follows. *Item 2* follows from the fact that $O(\lambda) \to 0$ as $\lambda \to 0$. □

The following direct implication will be crucial for the discussion that follows.

Corollary 4: For any sequence $\{\lambda_i\}$ *such that* $\lambda_i \downarrow 0$, *there exists a subsequence* $\{\lambda_{i_k}\} \downarrow 0$ *and a Lipschitz function* δ_0 *(depending on* $\{\lambda_{i_k}\}$*) for which* $\delta_{\lambda_{i_k}}$ *converges uniformly to* δ_0 *on D. The function* δ_0 *is constant on radial lines.*

We would now like to prove that the coordinate-distance lag $\bar{\delta}(\{p_i\})$ of any divergent sequence $\{p_i\}$ is zero. Naturally, this is the same as saying that δ converges uniformly to zero at infinity. If this is not the case, then it is simple to see, arguing by contradiction, that we would be in the following situation. There would exist $\{\lambda_i\}$ with $\lambda_i \downarrow 0$ such that δ_{λ_i} converges uniformly to a Liptschitz function δ_0 and there would exist points x, y in D for which $\delta_0(x) = 0$, $|x| = 3/2$ and $\delta_0(y) > 0$, $|y| = 3/2$, and $|x - y| < 1/2$. Assume we are in such situation. Define in D the Euclidean balls $B_x = B(x, |x - y|)$ and $B_y = B(y, \xi)$, where ξ is small enough to have $\delta_0|_{B_y} > c_1 > 0$, where c_1 is a constant. Following Ref. 6 (p. 258) we can find a function h on $\bar{B}_x$ such that

1. $h\Big|_{(\partial(B_x)\setminus B_y)} < c_2 < 0$, where c_2 is a constant,
2. $h(x) = 0$,
3. $\Delta_{g_{\lambda_i}} h\Big|_{\bar{B}_x} > c_3 > 0$, where c_3 is a constant and g_{λ_i} is the scaled metric $\lambda_i^2 g$.

Note that the scaled metrics $\lambda_i^2 g$ converge (in C^∞) to the flat Euclidean metric. As δ_{λ_i} converges uniformly to δ_0 we deduce that there is $\mu_0 > 0$ such that for any $0 < \mu \le \mu_0$ (and $i \ge i_0(\mu_0)$) we have $(-\delta_{\lambda_i} + \mu h)|_{\partial B_x} < \mu c_4 < 0$, where c_4 is a constant. We also have $\lim(-\delta_{\lambda_i}(x) + \mu h(x)) \to 0$. It follows that having chosen i_1 big enough, the function $-\delta_{\lambda_i} + \mu h$, $(\mu \le \mu_0)$, for $i \ge i_1$ has a maximum on B_x. Denote it by z_i. If the function s were to be smooth at z_i/λ_i and, therefore, $-\delta_{\lambda_i} + \mu h$ were smooth at z_i, then one would get a contradiction to the maximum principle, as for i sufficiently big, one would have

$$\Delta_{g_{\lambda_i}}(-\tilde{\delta}_{\lambda_i} + \mu h)(z_i) \ge \frac{\mu c_3}{2} > 0.$$

We explain now how to use Proposition 10 to overcome the case when z_i are not smooth points of s. One can replace s by s_{Γ_i}, for a suitable $\{\Gamma_i\} \downarrow 0$, in the expression $\delta_{\lambda_i}(x) = (\mathsf{s} - \hat{\mathsf{s}})(x/\lambda_i)$ in such a way that the new expression $(-(\mathsf{s}_{\Gamma_i} - \hat{\mathsf{s}}) + \mu h)(x/\lambda_i)$, has a maximum $\tilde{z}_i$ on B_x. Further, by Proposition 10 one can replace s_{Γ_i} by $\tilde{\mathsf{s}}_{\Gamma_i}$ in such a way that the new expression $\tilde{\delta}_{\lambda_i}(x) = (\tilde{\mathsf{s}}_{\Gamma_i} - \hat{\mathsf{s}})(x/\lambda_i)$ satisfies

1. $-\tilde{\delta}_{\lambda_i}(x) = -(\tilde{\mathsf{s}}_{\Gamma_i} - \hat{\mathsf{s}})(x/\lambda_i) \le -(\mathsf{s}_{\Gamma_i} - \hat{\mathsf{s}})(x/\lambda_i)$,
2. $-\tilde{\delta}_{\lambda_i}(\tilde{z}_i) = (\mathsf{s}_{\Gamma_i} - \hat{\mathsf{s}})(\tilde{z}_i/\lambda_i)$, and thus $-\tilde{\delta}_{\lambda_i} + \mu h$ has a maximum at $\tilde{z}_i$ on B_x.
3. $\Delta_{g_{\lambda_i}}(-\tilde{\delta}_{\lambda_i} + \mu h)(\tilde{z}_i) \ge \frac{\mu c_3}{2}$.

These three facts now contradict the maximum principle. □

We have thus proved in the following.

Proposition 19: Let $(\Sigma, \mathrm{g}, \ln N)$ *be an asymptotically flat static solution with regular and connected horizon. Then for any divergent sequence* $\{p_i\}$, *the coordinate-distance lag* $\bar{\delta}(\{p_i\})$ *is zero.*

2. Area and volume comparison

Proposition 20: Let $(\Sigma, \mathrm{g}, \ln N)$ *be an asymptotically flat static solution with regular and connected horizon. Consider a sequence* $\{\Gamma_i\} \downarrow 0$. *Let* $\mathcal{F}_{\Gamma_i}$ *be the congruence of length minimizing geodesics to* H_{Γ_i}. *Then, for every* $L > 0$ *we have*

$$Vol(\cup_{\gamma \in \mathcal{F}_{\Gamma_i}, l(\gamma) \le L}\{\gamma\}) \to 0$$

as $\Gamma_i \downarrow 0$. *Above* $\{\gamma\}$ *means the set of points in* γ.

Proof: The first goal to achieve is to make the monotonicity of $\mathcal{M}$ to look like a *comparison of areas* and consequently a *comparison of volumes*. Let $\{\Gamma_i\} \downarrow 0$. Consider for each Γ_i the congruence $\mathcal{F}_{\Gamma_i}$ of length minimizing geodesics to H_{Γ_i}. We will work outside the locus at all times. Let dA be the element of area of the level sets of the congruence. Let s_{Γ_i} be the distance function to H_{Γ_i}. Then,

$$\theta = \frac{1}{A}\frac{dA}{d\mathsf{s}_{\Gamma_i}}.$$

Let γ be a geodesic in $\mathcal{F}_{\Gamma_i}$. Consider $\mathcal{M}_a$ with $a = 2m$ over γ. Denote by $\mathcal{M}_{\Gamma_i}$ the value of $\mathcal{M}$ at the initial point of γ in H_{Γ_i}. Then from the monotonicity of $\mathcal{M}$ we have

$$(\frac{1}{2A}\frac{dA}{d\mathsf{s}_{\Gamma_i}}(\mathsf{s}_{\Gamma_i} + 2m)^2 - (\mathsf{s}_{\Gamma_i} + 2m))N^2 \le \mathcal{M}_{\Gamma_i}.$$

Rearranging terms we get

$$\frac{d}{d\mathsf{s}_{\Gamma_i}}(\frac{dA}{(\mathsf{s}_{\Gamma_i} + 2m)^2)}) \le \frac{2\mathcal{M}_{\Gamma_i}}{N^2(\mathsf{s}_{\Gamma_i} + 2m)^2}dA.$$

We thus get

$$\frac{d}{d\mathsf{s}_{\Gamma_i}}\ln\frac{dA}{(\mathsf{s}_{\Gamma_i} + 2m)^2} \le \frac{2\mathcal{M}_{\Gamma_i}}{N^2(\mathsf{s}_{\Gamma_i} + 2m)^2}.$$

Integrating we obtain

$$\frac{dA}{(\mathrm{s}_{\Gamma_i}+2m)^2} \leq \frac{dA_0}{(2m)^2}\exp\Big(\int_0^{\mathrm{s}_{\Gamma_i}} \frac{2\mathcal{M}_{\Gamma_i}}{N^2(\mathrm{s}_{\Gamma_i}+2m)^2}d\mathrm{s}_{\Gamma_i}\Big), \tag{67}$$

where dA_0 is the element of area of H_{Γ_i}. Recalling that $N^2 = \hat{\mathrm{s}}/(\hat{\mathrm{s}}+2m)$ it is clear that we need an estimation of $\hat{\mathrm{s}}$ in terms of s_{Γ_i} to have an inequality in terms of s_{Γ_i} only. We advocate to that in the following lines. We explain first how to get a relation between s and s_{Γ_i} and then we explain how to obtain one in terms of $\hat{\mathrm{s}}$ and s_{Γ_i}.

First, recall from (43) that for any point q in H_{Γ_i} we have (for Γ_i small enough) that $\mathrm{s}(q) = \hat{\mathrm{s}}(q) + O(\hat{\mathrm{s}}^{\frac{3}{2}})$. Now let p be a point in γ. Then we have $\mathrm{s}(p) \leq \mathrm{s}_{\Gamma_i}(p) + \mathrm{s}(q)$, where q is the initial point of γ at H_{Γ_i}. Thus $\mathrm{s}(p) \leq \mathrm{s}_{\Gamma}(p) + (1+\epsilon)\hat{\mathrm{s}}(p)$ where $\epsilon = O(\hat{\mathrm{s}}(p)^{\frac{1}{2}})$. On the other hand let $\bar{\gamma}$ be a length minimizing geodesic joining p to H. Let $\bar{q}$ be the point of intersection to H_{Γ_i}. Then we have

$$\mathrm{s}(p) = dist(p,\bar{q}) + \mathrm{s}(\bar{q}) \geq \mathrm{s}_{\Gamma_i}(p) + \hat{\mathrm{s}}(\bar{q}) + O(\hat{\mathrm{s}}(\bar{q})^{\frac{3}{2}}) \geq \mathrm{s}_{\Gamma_i}(p) + (1-\epsilon)\hat{\mathrm{s}}(q),$$

where $\epsilon = O(\hat{\mathrm{s}}(q)^{\frac{1}{2}})$. Thus for every point p in γ we have

$$(1-\epsilon)\hat{\mathrm{s}}_0 + \mathrm{s}_{\Gamma_i}(p) \leq \mathrm{s}(p) \leq \mathrm{s}_{\Gamma_i}(p) + (1+\epsilon)\hat{\mathrm{s}}_0,$$

where we have made $\hat{\mathrm{s}}_0 = \hat{\mathrm{s}}(q)$ to simplify the notation. This establishes the relation between s and s_{Γ_i}. We obtain now the desired relation between s_{Γ_i} and $\hat{\mathrm{s}}$. We will keep the notation as before. Precisely, γ will be length minimizing geodesic segment to H_{Γ_i} and q and q_1 will be its initial and final points. From Proposition 16, we know that $|\nabla\hat{\mathrm{s}}| \leq 1$, therefore, for any point p between q and q_1 we have

$$\hat{\mathrm{s}}(q_1) - \hat{\mathrm{s}}(p) \leq \mathrm{s}_{\Gamma_i}(q_1) - \mathrm{s}_{\Gamma_i}(p),$$
$$\hat{\mathrm{s}}(p) - \hat{\mathrm{s}}(q) \leq \mathrm{s}_{\Gamma_i}(p).$$

Using this we have

$$(1+\epsilon)\hat{\mathrm{s}}_0 \geq \hat{\mathrm{s}}(q) \geq \hat{\mathrm{s}}(p) - \mathrm{s}_{\Gamma_i}(p) \geq \hat{\mathrm{s}}(q_1) - \mathrm{s}_{\Gamma_i}(q_1) \geq \hat{\mathrm{s}}(q_1) - \mathrm{s}(q_1) + (1-\epsilon)\hat{\mathrm{s}}_0.$$

Now if $\mathrm{s}(q_1) \geq \bar{L}$ and $\bar{L} = \bar{L}(\Gamma_i)$ is big enough we have $\hat{\mathrm{s}}(q_1) - \mathrm{s}(q_1) \geq -\epsilon\hat{\mathrm{s}}_0$. As a result, we have the relation

$$(1+\epsilon)\hat{\mathrm{s}}_0 \geq \hat{\mathrm{s}}(p) - \mathrm{s}_{\Gamma}(p) \geq (1-2\epsilon)\hat{\mathrm{s}}_0. \tag{68}$$

We have now all the elements to proceed with the proof of the proposition. Consider the set of the initial points on H_{Γ_i} of the geodesics in $\mathcal{F}_{\Gamma_i}$ whose lengths are greater than $\bar{L}(\Gamma_i)$. Denote such set by Ω_{Γ_i}. We will show now that as $\Gamma_i \downarrow 0$, and therefore as H_{Γ_i} approaches H, the area of Ω_{Γ_i} with respect to the area element induced from g tends to the total area of the horizon H.

Consider the argument in the exponential function of (67) with the upper limit of integration equal to $\bar{L}$. Using the relation (68), we obtain

$$\begin{aligned}\int_0^{\bar{L}} \frac{\mathcal{M}_0}{N^2(\mathrm{s}_{\Gamma_i}+2m)^2}d\mathrm{s}_{\Gamma_i} &= \int_0^{\bar{L}} \frac{\mathcal{M}_0(\hat{\mathrm{s}}+2m)}{\hat{\mathrm{s}}^2(\mathrm{s}_{\Gamma_i}+2m)^2}d\mathrm{s}_{\Gamma_i}\\ &\leq \int_0^{\bar{L}} \frac{\mathcal{M}_0(\mathrm{s}_{\Gamma_i}+2m+(1+\epsilon)\hat{\mathrm{s}}_0)}{(\mathrm{s}_{\Gamma_i}+(1-2\epsilon)\hat{\mathrm{s}}_0)(\mathrm{s}_{\Gamma_i}+2m)^2}d\mathrm{s}_{\Gamma_i}.\end{aligned}$$

This last integral can be further split into

$$\int_0^{\bar{L}} \frac{\mathcal{M}_0}{(\mathrm{s}_{\Gamma_i}+(1-2\epsilon)\hat{\mathrm{s}}_0)(\mathrm{s}_{\Gamma_i}+2m)}d\mathrm{s}_{\Gamma_i} + R(\hat{\mathrm{s}}_0),$$

where $R(\hat{\mathsf{s}}_0)$ is an expression which is easily seen to tend to zero as $\hat{\mathsf{s}}_0$ tends to zero. We integrate now Eq. (67) in dA. After integrating in dA, the left-hand side tends to 4π for a suitable divergent sequence of $\bar{L}$'s. The right-hand side is easily integrated to be (discard the term $R(\hat{\mathsf{s}}_0)$)

$$\int_{\Omega_{\Gamma_i}} \frac{\hat{\mathsf{s}}_0}{(\hat{\mathsf{s}}_0 + 2m)(2m)^2} \Big(\frac{2m}{(1-2\epsilon)\hat{\mathsf{s}}_0}\Big)^{\frac{2\mathcal{M}_0}{2m-(1-2\epsilon)\hat{\mathsf{s}}_0}} dA_g,$$

where $dA_g = N^2 dA_0 = \frac{\hat{\mathsf{s}}_0}{\hat{\mathsf{s}}_0+2m} dA$ is the element of area induced on H_{Γ_i} from the metric g. As a result we get the inequality

$$4\pi \leq \frac{\limsup A(\Omega_{\Gamma_i})}{4m^2} \limsup \hat{\mathsf{s}}_0^{\frac{2\mathcal{M}_0 - 2m + (1-2\epsilon)\hat{\mathsf{s}}_0}{2m-(1-2\epsilon)\hat{\mathsf{s}}_0}}. \tag{69}$$

Now, from the proof of Proposition 4 it is seen that $|\mathcal{M}_0 - m| \leq c_1 \hat{\mathsf{s}}_0^{\frac{1}{2}}$, where c_1 is a positive constant. Thus we get

$$\hat{\mathsf{s}}_0^{\frac{2\mathcal{M}_0 - 2m + (1-2\epsilon)\hat{\mathsf{s}}_0}{2m-(1-2\epsilon)\hat{\mathsf{s}}_0}} \leq \hat{\mathsf{s}}_0^{c_2 \hat{\mathsf{s}}_0^{\frac{1}{2}}} \to 1 \text{ as } \hat{\mathsf{s}}_0 \to 0,$$

where c_2 is a positive constant. Therefore, we get from this and Eq. (69)

$$16\pi m^2 \leq \limsup A(\Omega_{\Gamma_i}) \leq A = 16\pi m^2,$$

where A is the area of the horizon. Thus $\limsup A(\Omega_{\Gamma_i}) = A$. This was the crucial estimate. From it, it will follow that for any $L < \infty$ fixed, there is a subsequence Γ_{i_j} such that the area of the set of initial points in $H_{\Gamma_{i_j}}$ of the geodesics in $\mathcal{F}_{\Gamma_{i_j}}$ whose length is less or equal than L, tends actually to zero. This would finish the proof of the proposition. We do that now. For every j, denote by $\Omega_{L,\Gamma_{i_j}}$ such set. For every q in $\Omega_{L,\Gamma_{i_j}}$ let γ_q be the corresponding geodesic in $\mathcal{F}_{\Gamma_{i_j}}$ whose total length is less than or equal to L. Denote by $U_{L,\Gamma_{i_j}}$ the union $U = \cup_{q\in\Omega_{L,\Gamma_{i_j}}} \{\gamma_q\}$. Now, recalling that $dV' = dA$, integrating equation (67), and following the same treatment at the horizon as before gives

$$Vol_{\mathsf{g}}(U_{L,\Gamma_{i_j}}) \leq c(L) A_g(\Omega_{L,\Gamma_{i_j}}).$$

Note that in this equation, the volume is found with g, while the area is found with g. As $A(\Omega_{i_j}) \to 0$, the proposition follows. □

The proposition before has the following quite important Corollary.

Corollary 5: Let $(\Sigma, \mathsf{g}, \ln N)$ *be an asymptotically flat static solution with regular and connected horizon. Then*

1. $\mathsf{s} = \hat{\mathsf{s}}$ *and therefore* s *is smooth.*
2. $|\nabla\hat{\mathsf{s}}|^2 = 1$.
3. *The integral curves of* $\nabla\hat{\mathsf{s}}$ *are geodesics minimizing the length between any two of its points.*
4. *The set of integral curves of* $\nabla\hat{\mathsf{s}}$ *form an integrable congruence of geodesics.*

Proof: Let $p \in \Sigma\backslash H$. Let $\{\Gamma_i\}$ such that $\Gamma_i \downarrow 0$. Following Proposition 20 there is a sequence $\{\gamma_i\}$ of length minimizing geodesics to H_{Γ_i} with initial point q_i (at H_{Γ_i}), $l(\gamma_i) \to \infty$ and $\gamma_i(s(p)) \to p$. Let p_i be either the end point of γ_i or, if $l(\gamma_i) = \infty$, a point on γ_i such that $s(p_i) \to \infty$. We have

$$\hat{\mathsf{s}}(p_i) - \hat{\mathsf{s}}(q_i) = \int_{\bar{s}(q_i)=0}^{\bar{s}(p_i)} < \nabla\hat{\mathsf{s}}, \gamma' > d\bar{s} = \bar{s}(p_i) - \bar{s}(q_i) - \int_{\bar{s}(q_i)}^{\bar{s}(p_i)} (1- < \nabla\hat{\mathsf{s}}, \gamma' >) d\bar{s}. \tag{70}$$

where $\bar{\mathsf{s}}$ is the arc-length. But by Proposition 19, we have $\lim \delta(p_i) = \mathsf{s}(p_i) - \hat{\mathsf{s}}(p_i) = 0$ and thus we have $\lim \bar{\mathsf{s}}(p_i) - \hat{\mathsf{s}}(p_i) = 0$ (note that $\lim |\mathsf{s}(p_i) - \bar{\mathsf{s}}(p_i)| = 0$). By Proposition 16, we have $(1- < \nabla\hat{\mathsf{s}}, \gamma' >) \geq 0$, thus from Eq. (70) we get

$$0 \leq \lim \int (1- < \nabla\hat{\mathsf{s}}, \gamma' >) d\bar{\mathsf{s}} = 0.$$

This shows $|\nabla \mathsf{s}|(p)=1$. Moreover, we have

$$\hat{\mathsf{s}}(p)=\lim \hat{\mathsf{s}}(p_i)-\hat{\mathsf{s}}(q_i)=\lim\ \bar{\mathsf{s}}(p_i)-\bar{\mathsf{s}}(q_i)-\int_{\bar{\mathsf{s}}(q_i)}^{\bar{\mathsf{s}}(p_i)}(1-<\nabla\hat{\mathsf{s}},\gamma'>)d\bar{\mathsf{s}}=\lim\bar{\mathsf{s}}(p_i)=\mathsf{s}(p).$$

Because p is an arbitrary point we have thus proved *items 1 and 2* of the proposition.

To prove the third *item* we proceed like this. Let γ be an integral curve of $\nabla\hat{\mathsf{s}}$ with initial point p and final point q. Suppose that γ does not minimize the distance between p and q, namely, that there is another curve $\tilde{\gamma}$ joining p and q and having smaller length. Then

$$\mathsf{s}(q)=\mathsf{s}(p)+(\mathsf{s}(q)-\mathsf{s}(p))=\mathsf{s}(p)+l(\gamma)<\mathsf{s}(p)+l(\tilde{\gamma})\leq\mathsf{s}(q),$$

which is a contradiction.

Item 4 of the proposition follows directly from the fact that the congruence is orthogonal to the level set of any regular value of s. □

3. The uniqueness of the Schwarzschild solutions

Theorem 3: *Let* $(\Sigma, \mathsf{g}, \ln N)$ *be an asymptotically flat static solution with regular and connected horizon. Then the solutions is a Schwarzschild solution of positive mass.*

Proof: By Corollary 5 the set of integral curves of $\nabla\hat{\mathsf{s}}$ is an integrable congruence of geodesics. Recalling that $|\nabla\hat{\mathsf{s}}|=1$ and $\Delta\hat{\mathsf{s}}=\theta$, where θ is the mean curvature of the congruence. Using these facts in Eq. (34) we get that

$$\mathcal{M}_{a=2m}=(\frac{\theta(\mathsf{s}+2m)^2}{2}-(\mathsf{s}+2m))N^2=m,$$

over any geodesic of the congruence. The conclusion that the solution is the Schwarzschild solution follows from Proposition 2 and the remark after it. □

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