A classification of static vacuum black-holes

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Abstract

The celebrated uniqueness's theorem of the Schwarzschild solution by Israel, Robinson and Bunting/Masood-ul-Alam, asserts that the only asymptotically flat static solution of the vacuum Einstein equations with compact but non-necessarily connected horizon is Schwarzschild. Here we extend this result by proving a classification theorem for all (metrically complete) solutions of the static vacuum Einstein equations with compact but non-necessarily connected horizon without making any further assumption on the topology or the asymptotic. It is shown that any such solution is either: (i) a Boost, (ii) a Schwarzschild black hole, or (iii) is of Myers/Korotkin-Nicolai type, that is, it has the same topology and Kasner asymptotic as the Myers/Korotkin-Nicolai black holes. In a broad sense, the theorem classifies all the static vacuum black holes in 3+1-dimensions.

Contents

1 Introduction

The vacuum static solutions of the Einstein equations have played since early days a fundamental role in the study of Einstein's theory and the classification theorems have been at the centre of the work. A fundamental result in this respect is the uniqueness theorem of the Schwarzschild solution asserting that the Schwarzschild black holes are the only asymptotically flat vacuum static solutions with compact but non-necessarily connected horizon (Israel [11], Robinson et al [30], Bunting/Masood-ul-Alam [4]; for a nice review on the history of this theorem see [5]). In this article we prove a classification theorem that extends Schwarzschild's uniqueness theorem to static solutions with no specified topology or asymptotic.

To state the classification theorem we need to introduce first a setup.

Formally, a (vacuum) static data set $(\Sigma; g, N)$ consists of a three manifold Σ, a function N positive in the interior of Σ (called the lapse) and a Riemannian metric g on Σ satisfying the vacuum static equations,

$$N Ric = \nabla\nabla N, \tag{1.0.1}$$

$$\Delta N = 0 \tag{1.0.2}$$

A static data set $(\Sigma; g, N)$ gives rise to a vacuum static spacetime,

$$\mathbf{\Sigma} = \mathbb{R} \times \Sigma, \quad \mathbf{g} = N^2 dt^2 + g, \tag{1.0.3}$$

where ∂_t is the static Killing field. A static black hole data set is a static data $(\Sigma; g, N)$ such that $\partial\Sigma = \{N = 0\} \neq \emptyset$ is compact and $(\Sigma; g)$ is metrically complete. In this definition no special asymptotic or global topological structure is assumed. The boundary of Σ is non-necesarily connected and is called the horizon. The goal of this article is to classify static black hole data sets.

The paradigmatic examples of static black holes data sets are the Schwarzschild black holes $(\Sigma_S; g_S, N_S)$ given by,

$$\Sigma_S = \mathbb{R}^3 \setminus B(0, 2m), \quad g_S = \frac{1}{1 - 2m/r} dr^2 + r^2 d\Omega^2 \quad \text{and} \quad N_S = \sqrt{1 - 2m/r} \tag{1.0.4}$$

where $m > 0$ is the mass and $B(0, 2m)$ is the open ball of radius $2m$[(1)]. The Schwarzschild black holes are of course asymptotically flat.

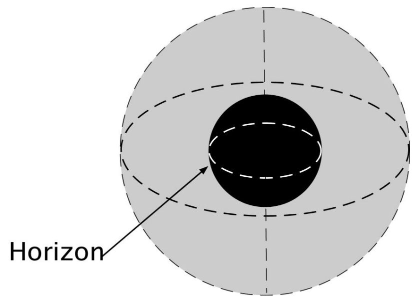

Figure 1: A Schwarzschild black hole. The grey region is Σ and is diffeomorphic to $\mathbb{R}^3$ minus the open (black) ball $B(0, 2m)$. The solution is spherically symmetric and thus axisymmetric.

In this setup the uniqueness theorem of Schwarzschild can be stated as follows.

Theorem 1.0.1 (Israel-Robinson et al-Bunting/Masood-ul-Alam)**.** *The only asymptot-*

[(1)] The spacetime (1.0.3) corresponding to (1.0.4) is just the region of exterior communication of a Schwarzschild black hole of mass m. The horizon is the boundary $\partial\Sigma_S = \{N = 0\}$. Restricted to $r \geq R(t) > 2m$, the Schwarzschild space models the gravitational field of any isolated but spherically symmetric physical body of radius $R(t)$. The object itself may be transiting a dynamical process (for instance in a star), but the spacetime outside remains spherically symmetric and thus Schwarzschild by Birkhoff's theorem. If the radius $R(t)$ goes below the threshold of $2m$, no equilibrium is possible, the body undergoes a complete gravitational collapse and a Schwarzschild black hole remains.

ically flat static black hole data sets are the Schwarzschild black holes.

The classification theorem that we prove claims that, besides Schwarzschild, there are only two more families of black hole static data sets: the *Boosts* and the data sets of *Myers/Korotkin-Nicolai type*, (from now on M/KN type). Both are, of course, non asymptotically flat.

The Boosts data sets $(\Sigma_B; g_B, N_B)$ are given by,

$$\Sigma_B = [0, \infty) \times \mathbb{T}^2, \quad g_B = dx^2 + h, \quad N_B = x \tag{1.0.5}$$

where $(\mathbb{T}^2; h)$ is any flat two-torus and x is the coordinate in the factor $[0, \infty)$. The origin of these data is simple as the spacetime (1.0.3) associated to the universal cover of any Boost is the Rindle wedge of the Minkowski spacetime,

$$\mathbf{\Sigma} = \{(x, y, z, t) : |t| \leq x\}, \quad \mathbf{g} = -dt^2 + dx^2 + dy^2 + dz^2 \tag{1.0.6}$$

and the static Killing field is the boost generator $x\partial_t$, (hence the name of the family).

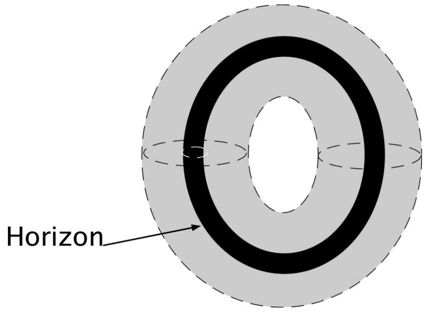

Figure 2: A Boost black hole. The grey region is Σ and is diffeomorphic to a solid torus minus an open (black) solid torus.

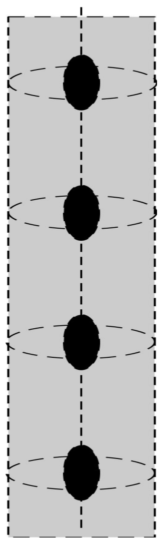

Figure 3: A 'universal M/KN data'. The grey region is Σ and is diffeomorphic to $\mathbb{R}^3$ minus an infinite number of (black) open balls. The solution is axisymmetric.

The data sets of Myers/Korotkin-Nicolai type are defined as any static black hole data set whose topology is that of a solid three-torus minus a finite number of balls and whose asymptotic is Kasner. Black holes with such properties were found by Myers in [21]. They were rediscovered and further investigated by Korotkin and Nicolai in [16], [15]. The idea of their construction is the following: use first Weyl's method to find a 'periodic' solution by superposing along an axis an infinite number of Schwarzschild solutions separated by a same distance L. This gives the 'universal cover solution'. Taking suitable covers one obtains M/KN solutions with any number of holes(2). The details of the M/KN data $(\Sigma_{MKN}; g_{MKN}, N_{MKN})$ won't be relevant for us but for the sake of completeness the main features of the 'universal data' can be summarized as follows (see [21], [16]). The metric and the lapse have the form,

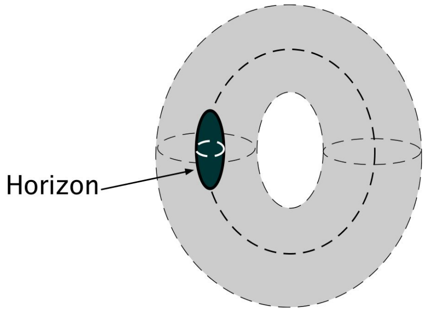

Figure 4: A M/KN data with one hole. The grey region is Σ and is diffeomorphic to a solid torus minus an open (black) ball. The solution is axisymmetric.

$$g_{MKN} = e^{-\omega}(e^{2k}(dx^2 + d\rho^2) + \rho^2 d\phi^2), \qquad N_{MKN} = e^{\omega/2}, \tag{1.0.7}$$

where (x, ρ) are Weyl coordinates ($\rho > 0$ is the radial coordinate) and $\phi \in [0, 2\pi)$ is the angular coordinate. The function ω is defined through the convergent series,

$$\omega(x,\rho) = \omega_0(x,\rho) + \sum_{n=1}^{\infty}[\omega_0(x+nL,\rho) + \omega_0(x-nL,\rho) + \frac{4M}{nL}] \tag{1.0.8}$$

where $\omega_0(x, \rho)$ is,

$$\omega_0 = \ln \mathcal{E}_0, \qquad \mathcal{E}_0(x,\rho) = \frac{\sqrt{(x-M)^2+\rho^2} + \sqrt{(x+M)^2+\rho^2} - 2M}{\sqrt{(x-M)^2+\rho^2} + \sqrt{(x+M)^2+\rho^2} + 2M} \tag{1.0.9}$$

and the function $k(x, \rho)$ is found by quadratures through the equations,

$$k_\rho = \frac{\rho}{4}(\omega_\rho^2 - \omega_x^2), \qquad k_x = \frac{\rho}{2}\omega_x\omega_\rho, \tag{1.0.10}$$

The metric g, the lapse N and the function k are invariant under the translations

(2) As the Schwarzschild solutions are axisymmetric, they can be superposed along an axis by Weyl's method. When superpossing a finite number of holes, angle deficiencies appear on the axis between them and the solution resulting is non-smooth. This defficiency can be understood from the fact that a repulsive force must keep the holes in equilibrium. However when infinitely many of them are superposed along the axis, say at a distance L from each other, no extra force is needed and the angle defficiency is no longer present. This gives a 'periodic' solution that can be quotiented to obtain M/KN solutions with any number of holes.

$x \to x + L$, hence periodic. The asymptotic of the solution is Kasner and has the form,

$$g_{MKN} \approx c_1 \rho^{\alpha^2/2-\alpha}(dx^2 + d\rho^2) + c_2 \rho^{2-\alpha} d\phi^2, \qquad N_{MKN} \approx c_3 \rho^{\alpha/2} \tag{1.0.11}$$

where $\alpha = 4M/L$ and $0 \leq \alpha < 2$.

Other solutions also with more than one hole can be constructed by superposing periodically Schwarzschild holes of different masses but separated at right distances to prevent struts. In this article the Myers/Korotkin-Nicolai data sets are defined as the axisymmetric static black hole data sets that can be obtained using the Myers/Korotkin-Nicolai method. The characteristics of them won't be relevant to us (for a discussion see [15]).

The goal of this article is therefore to prove the following classification theorem.

Theorem 1.0.2 (The classification Theorem)**.** *Any static black hole data set is either,*

(I) *a Schwarzschild black hole, or,*

(II) *a Boost, or,*

(III) *is of Myers/Korotkin-Nicolai type.*

It could be that the M/KN data sets are the only black hole static data sets of Myers/Korotkin-Nicolai type. We leave this as an open problem.

Vacuum static solutions with local symmetries have been investigated since early days by Schwarzschild [], Levi-Civita [19], [18], Kasner [14], [13], Weyl [34] and many others, (for a review see [12] and references therein), and there is a good understanding of them. Without symmetries the problem is vast more complex. The fundamental uniqueness theorem of Schwarzschild mentioned earlier is perhaps the first

However in 1967 Israel proved that Schwarzschild is unique among the asymptotically flat black hole static solutions for which the lapse N can be chosen as a global harmonic coordinate. Although the problem has no stated symmetry it does have definite boundary conditions, namely that $\partial\Sigma$ is a horizon and that the solution is asymptotically flat. These conditions are enough to grant spherical symmetry and therefore uniqueness. A relevant aspect of this work is that it makes it evident the importance of the geometry of the level sets of the lapse in the global geometry of the solution. This will play a role later when we discuss Kasner asymptotic in Section 7.2. The technical conditions impossed by Israel were removed by Müller, Robinson and Seifert in [10] ('73). Later in 1977 Robinson found a simple proof based on a remarkable integral formula [30] (the proof required also previous work by Künzle [**?**]). This proved that the only asymptotically flat solution with a connected compact horizon is Schwarzschild. The uniqueness of Schwarzschild with multiple horizons was settled by Bunting/Masood-ul-Alam in 1986 [4], where it was apparent the important role of the positive mass theorem. Recently other proofs of the Israel-Robinson theorem were given by the author in [24] and by Agostiniani and Mazzieri in [1]. In [24] techniques in comparison geometry were used and in [1] monotonic quantities along the level sets of the lapse were introduced. Some of the arguments in this article will follow similar ideas though technically distinct.

In [3] ('00) M. Anderson carried out a general investigation of static solutions from a general modern point of view. In particular he proved a fundamental decay estimate for the curvature and the gradient of the logarithm of the lapse, allowing him to prove the first uniqueness theorem of the Minkowski solution (as a static solution) without assuming any type of assymptotic but just geodesic completeness. Anderson's estimate holds too in any dimension, [29]. This was proved by importing some techniques in comparison geometry á la Backry-Émery that were introduced by J. Case in [6] in a context somehow related to that of static solutions of the Einstein equations. It turns out that these techniques in comparison geometry a la Bakry-Emery play a fundamental role in the first part of the paper as we will explain below. The global study of the lapse function that we do is based largely upon these ideas.

1.0.1 The idea of the proof and the structure of the article

We pass now to explain the structure of the proof. This should help also the reader as a guide to understand the main arguments behind the proofs.

Let $(\Sigma; g, N)$ be a static black hole data set. The structure of the proof is indeed simple and consists essentially of proving the following three parts,

1. Σ has only one end.
2. The horizons are *weakly outermost* (see Definition 2.1.3).
3. The end is asymptotically flat or asymptotically Kasner.

Once this is achieved the proof of the classification theorem is carried out using know results. Assume 1-3 hold. First if the data is asymptotically flat then it is Schwarzschild by the uniqueness theorem. If the data is asymptotically Kasner, then one deduces that the data is either a Boost or is of M/KN type as follows. As the horizons are weakly outermost then it follows from results of Galloway [], [], that either the data is a Boost or every horizon is a totally geodesic sphere. On the other hand if the horizons are spheres and the asymptotic is Kasner it follows from another result of Galloway [] that Σ is diffeomorphic to an open three torus minus a finite number of open three-balls. Finally this topology and Kasner asymptotic imply, by definition, that the data is of M/KN type.

From this point of view above the difficult roots in proving steps 1-3. Their proofs are entagled through the different sections, so it seems more appropriate to discuss these sections now, making comments when necessary. The basic argument behind the proof will be clear afterwards.

The Section 2.1 is dedicated to give the precise definitions and statements of the article. The classification theorem is stated as Theorem 2.1.6. The section includes a carefull discussion of the Kasner spaces, and the proof of their uniqueness statement, which will be used throughout Section 7 when we discuss asymptotic. At the end of this section we give a definition of Kasner asymptotic addapted to the needs of the proofs. We will comment on it.

In Section **??** we introduce background material, including the notation, terminology and conventions. Several of these notions have to do with 'scaling'. Scaling plays a fundamental role in the study of ends in Section **??**. In turn, the use of scaling techniques is possible due to the scale invariance of Anderson's decay estimates for the curvature and for the gradient of the logarithm of the lapse, Theorem **??**. We also include some elementary material on Cheeger-Gromov-Fukaya theory of convergence and collapse of Riemannian manifold under curvature bounds. This will be necessary to study the asymptotic of static data sets again through scaling.

The body of the article begins in Section 4 where we discuss the geometry of static metrics transformed by powers of the lapse, namely metrics of the form $N^{-2\epsilon}g$ where

g is the static metric of the data set and ϵ is just a coeficient. The main reasons to study these conformal metrics are the following. First, we will use the properties of the family of metrics $\overline{g} = N^{-2\epsilon}g$ with $\epsilon > 0$ to prove that the manifold of a static black hole data set has only one end, Theorem 4.4.2. This accomplishes step 1. Second, the proof of step 2, that the horizons are weakly outermost, requires using that the metric $\mathfrak{g} = N^2 g$ is complete away from the boundary(3), Proposition 4.4.3. The proof of that completeness is done through a careful understanding of the family of metrics $\overline{g} = N^{-2\epsilon}g$ for ϵ in a certain range, Theorem 4.3.1. Third, in Section 7 we will rely exclusively in the conformal metric $\mathfrak{g}$ to study the asymptotic of data sets. Again it is necessary to grant that $\mathfrak{g}$ is complete at infinity.

Making $f = -\ln N$ the static equations can be cast in the form,

$$Ric_f^1 = 0, \qquad \Delta_f f = 0 \tag{1.0.12}$$

where for any α the α-Bakry-Émery Ricci tensor Ric_f^α is defined as,

$$Ric_f^\alpha := Ric + \nabla\nabla f - \alpha \nabla f \nabla f, \tag{1.0.13}$$

whereas the f-Laplacian of a function ϕ is defined as,

$$\Delta_f \phi := \Delta\phi - \langle \nabla f, \nabla \phi \rangle \tag{1.0.14}$$

These equations are enough to deduce, via a f-Bochner formula (equation **??**), that

$$\Delta |\nabla f|^2 \geq 2|\nabla f|^4 \tag{1.0.15}$$

Then, the short Lemma 4.2.3 shows that if $Ric_f^\alpha \geq 0$, $\alpha > 0$, and for $\phi \geq 0$ we have,

$$\Delta\phi \geq c\phi^2 \tag{1.0.16}$$

then

$$\phi(p) \leq \frac{\eta}{d^2(p, \partial\Sigma)} \tag{1.0.17}$$

In particular $|\nabla N/N|^2 \leq \eta / d^2(p, \partial\Sigma)$, thus proving one of Anderson's estimates.

This is the way Anderson's estimates

$$|Ric|(p) \leq \frac{\eta}{d^2(p, \partial\Sigma)}, \qquad |\nabla \ln N|^2(p) \leq \frac{\eta}{d^2(p, \partial\Sigma)}, \tag{1.0.18}$$

were re-proved in [] (avoiding thus using some specific issues of dimension three) and that can be easily generalised to higher dimensions.

As we show in Proposition **??** the structure of these equations is preserved under conformal transformations by powers of the lapse, namely if $\overline{g} = N^{-2\epsilon}g$, then

$$\overline{Ric}_f^\alpha = 0, \qquad \overline{\Delta}_f f = 0 \tag{1.0.19}$$

where $\alpha = (1 - 2\epsilon - \epsilon^2)/(1+\epsilon)^2$ and $f = -(1+\epsilon)\ln N$. When $-1-\sqrt{2} < \epsilon < -1+\sqrt{2}$ then $\alpha > 0$. Elaborated on the ideas above it is proved in Theorem 4.3.1 that if $(\Sigma; g)$ is metrically complete (and $\partial\Sigma$ compact and $N|_\Sigma > 0$) then for any $\epsilon \in (-1-\sqrt{2}, -1+\sqrt{2})$

(3)Namely $(\Sigma_\delta; \overline{g})$ is metrically complete where Σ_δ is Σ with a collar around the boundary removed. Note that the metric $\mathfrak{g}$ is singular at $\partial\Sigma$, so to speak about completeness we need to remove a collar around $\partial\Sigma$.

the space $(\Sigma; N^{-2\epsilon}g)$ is complete.

Spaces with $Ric_f^\alpha \geq 0$ with $\alpha > 0$ but f arbitrary, have been studied in recent years under the context of comparison geometry.

Acknowledgment

2 Main statements and definitions

2.1 Static data sets and the main Theorem

Definition 2.1.1. *A static (vacuum) data set $(\Sigma; g, N)$ consists of a smooth three-manifold Σ, possibly with boundary, a smooth Riemannian metric g, and a smooth function N, such that,*

(i) $(\Sigma; g)$ *is metrically complete,*

(ii) N *is strictly positive in the interior* $\Sigma^\circ (= \Sigma \setminus \partial\Sigma)$ *of* Σ,

(iii) (g, N) *satisfy the vacuum static Einstein equations,*

$$N Ric = \nabla\nabla N, \qquad \Delta N = 0 \tag{2.1.1}$$

The definition is quite general. Observe in particular that Σ and $\partial\Sigma$ could be compact or non-compact. To give an example, a data set $(\Sigma; g, N)$ can be simply the data inherited on any region with smooth boundary of the Schwarzschild data. This flexibility in the definition of static data set allows us to write statements with great generality.

A horizon is defined as usual.

Definition 2.1.2. *Let $(\Sigma; g, N)$ be a static vacuum data set. A horizon is a connected component of $\partial\Sigma$ where N is identically zero.*

Note that the Definition 2.1.1 doesn't require $\partial\Sigma$ to be a horizon, though the data sets that we classify in this article are those with $\partial\Sigma$ consisting of a finite set of compact horizons (Σ is a posteriori non compact). It is known that the norm $|\nabla N|$ is constant on any horizon and different from zero. It is called the surface gravity.

It is convenient to give a name to those spaces that are the final object of study of this article. Naturally we will call them *static black hole* data sets.

Definition 2.1.3. *A static data set $(\Sigma; g, N)$ with $\partial\Sigma = \{N = 0\}$ and $\partial\Sigma$ compact, is called a static black hole data set.*

The following definition, taken from [9], recalls the notion of *weakly outermost* horizon.

Definition 2.1.4 (Galloway, [9])**.** *Let $(\Sigma; g, N)$ be a static black hole data set. Then, a horizon H is said weakly outermost if there are no embedded surfaces S homologous to H having negative outward mean curvature.*

The next is the definition of data set of Korotkin-Nicolai type that we use.

Definition 2.1.5. *A static data set $(\Sigma; g, N)$ is of Korotkin-Nicolai type if*

1. *$\partial\Sigma$ consist of $h \geq 1$ weakly outermost (topologically) spherical horizons,*

2. Σ is diffeomorphic to a solid three-torus minus h-open three-balls,

3. the asymptotic is Kasner.

We aim to prove

Theorem 2.1.6 (The classification Theorem)**.** *Let $(\Sigma; g, N)$ be a static black hole data set. Then, the associated space-time is either,*

(I) *covered by the Boost, or,*

(II) *is the Schwarzschild solution, or,*

(III) *is of Korotkin-Nicolai type.*

We do not know if the only solutions of type (III) are the Korotkin-Nicolai solutions. We state this as an open problem.

Problem 2.1.7. *Prove or disprove that the only static solutions of type* (III) *are the Korotkin-Nicolai solutions.*

2.2 The Kasner solutions

2.2.1 Explicit form and parameters

The Kasner data, denoted by $\mathbb{K}$, are $\mathbb{R}^2$-symmetric solutions explicitly given by

$$g = dx^2 + x^{2\alpha}dy^2 + x^{2\beta}dz^2, \tag{2.2.1}$$

$$N = x^\gamma \tag{2.2.2}$$

with (x, y, z) varying in the manifold $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$, and where (α, β, γ) satisfy

$$\alpha + \beta + \gamma = 1 \quad \text{and} \quad \alpha^2 + \beta^2 + \gamma^2 = 1 \tag{2.2.3}$$

but are otherwise arbitrary. The solutions corresponding to two different triples (α, β, γ) and $(\alpha', \beta', \gamma')$ are equivalent (i.e. isometric) iff $\alpha = \beta'$, $\beta = \alpha'$ and $\gamma = \gamma'$.

The metrics (2.2.1)-(2.2.2) are flat only when $(\alpha, \beta, \gamma) = (1, 0, 0), (0, 1, 0)$ or $(0, 0, 1)$. We will give to them the following names,

$$A: \ (\alpha, \beta, \gamma) = (1, 0, 0), \tag{2.2.4}$$

$$C: \ (\alpha, \beta, \gamma) = (0, 1, 0), \tag{2.2.5}$$

$$B: \ (\alpha, \beta, \gamma) = (0, 0, 1) \tag{2.2.6}$$

The solution B is the Boost.

$\mathbb{Z}$-actions, $\mathbb{Z} \times \mathbb{K} \to \mathbb{K}$, are given by fixing a (non-zero) vector field X, combination of ∂_y and ∂_z, and letting $n \times p \to p + nX$. The quotients are S^1-symmetric static solutions. Similarly, $\mathbb{Z}^2$ quotients give $\mathrm{S}^1 \times \mathrm{S}^1$-symmetric static solutions. $\mathbb{Z}^2$-quotient of the Kasner space will also be called Kasner spaces.

2.2.2 The harmonic presentation

The Kasner spaces in the harmonic presentation are

$$\mathfrak{g} = dx^2 + x^{2a}dy^2 + x^{2b}dz^2, \tag{2.2.7}$$

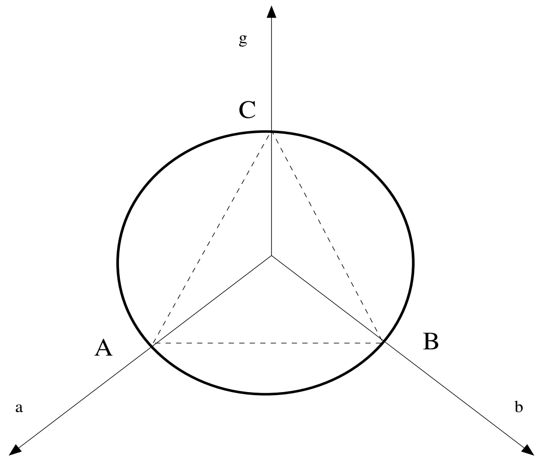

Figure 5: The circle that defines the range of parameters α, β, γ.

$$U = c \ln x \tag{2.2.8}$$

where a, b and c satisfy

$$2c^2 + (a - \frac{1}{2})^2 + (b - \frac{1}{2})^2 = \frac{1}{2} \quad \text{and} \quad a + b = 1 \tag{2.2.9}$$

Thus, the circle (2.2.3), (see Figure 5), is seen now as an ellipse in the plane $a + b = 1$, (see Figure 6). The flat solutions A, B and C are,

$$A: \ (a, b, c) = (1, 0, 0), \tag{2.2.10}$$
$$C: \ (a, b, c) = (0, 1, 0), \tag{2.2.11}$$
$$B: \ (a, b, c) = (1/2, 1/2, 1/2) \tag{2.2.12}$$

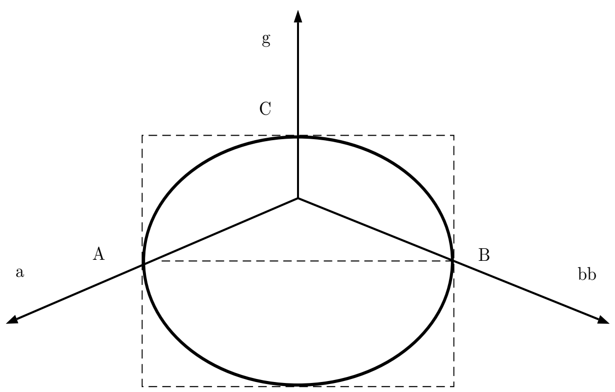

Figure 6: The ellipse that defines the range of the parameters a, b and c.

The Kasner solutions (2.2.7)-(2.2.8) are scale invariant. Namely, for any $\lambda > 0$, $(\mathbb{R}^+ \times \mathbb{R}^2; \lambda^2 \mathfrak{g})$ represents the same Kasner space as $(\mathbb{R}^+ \times \mathbb{R}^2; \mathfrak{g})$ does. This can be seen by making the change

$$\mathtt{x} = \lambda x, \quad \mathtt{y} = \lambda^{1-a} y, \quad \mathtt{z} = \lambda^{1-b} z \tag{2.2.13}$$

that transforms (2.2.7)-(2.2.8) into

$$\mathfrak{g} = d\mathtt{x}^2 + \mathtt{x}^{2a} d\mathtt{y}^2 + \mathtt{x}^{2b} d\mathtt{z}^2, \tag{2.2.14}$$

$$U = c \ln \mathtt{x} - c \ln \lambda \tag{2.2.15}$$

Another way to say this is that $(1-2c)t\partial_t + x\partial_x + (1-a)y\partial_y + (1-b)z\partial_z$ is a homothetic Killing of the space-time. The scale invariance can of course be seen also in the original space $(\mathbb{R}^+ \times \mathbb{R}^2; g, N)$. Note that in general, the isometry that exists between $(\mathbb{R}^+ \times \mathbb{R}^2; \mathfrak{g})$ and $(\mathbb{R}^+ \times \mathbb{R}^2; \lambda^2\mathfrak{g})$ does not pass to the quotient by a $\mathbb{Z} \times \mathbb{Z}$-action.

2.2.3 Uniqueness

The Kasner data are the only data with a free $\mathbb{R} \times \mathbb{R}$-symmetry other than Minkowski. This is due to [?]. We give now a proof of this fact in a way that becomes useful when we study the Kasner asymptotic later in Section 7.2.

The proof is as follows. We work in the harmonic presentation $(\Sigma; \mathfrak{g}, U)$, therefore geometric tensors are defined with respecto $\mathfrak{g}$. If the data set $(\Sigma; \mathfrak{g}, U)$ has a free $\mathbb{R}^2$-symmetry, and is not the Minkowski solution, then U can be taken as a harmonic coordinate with range in an interval I. Then, on $\mathbb{R}^2 \times I$ we can write

$$\mathfrak{g} = \lambda^2 dU^2 + h \tag{2.2.16}$$

where $\lambda = \lambda(U)$, and where $h = h(U)$ is a family of flat metrics on $\mathbb{R}^2$. Without loss of generality assume that $U = 0$ at the left end of I. Let (z_1, z_2) be a (flat) coordinate system on $\mathbb{R}^2 \times \{0\}$. In the coordinate system (z_1, z_2, U) the static equation $Ric_{\mathfrak{g}} = 2\nabla U \nabla U$ reduces to

$$\partial_U h_{AB} = 2\lambda \Theta_{AB}, \tag{2.2.17}$$

$$\partial_U \Theta_{AB} = \lambda(-\theta\Theta_{AB} + 2\Theta_{AC}\Theta^C_{\ B}), \tag{2.2.18}$$

$$\Theta_{AB}\Theta^{AB} - \theta^2 = -\frac{2}{\lambda^2}, \tag{2.2.19}$$

where Θ is the second fundamental form of the leaves $\mathbb{R}^2 \times \{U\}$ and $\theta = \Theta_A^{\ A}$ is the mean curvature. The static equation $\Delta_{\mathfrak{g}} U = 0$ reduces to

$$\partial_U \left(\frac{\sqrt{|h|}}{\lambda} \right) = 0 \tag{2.2.20}$$

where $|h|$ is the determinant of h_{AB}. Hence

$$\Gamma\sqrt{|h|} = \lambda \tag{2.2.21}$$

for a constant $\Gamma > 0$. This can be inserted in (2.2.17)-(2.2.18) to get the autonomous system of ODE

$$\partial_U h_{AB} = 2\Gamma\sqrt{|h|}\Theta_{AB}, \tag{2.2.22}$$

$$\partial_U \Theta_{AB} = \Gamma\sqrt{|h|}(-\theta\Theta_{AB} + 2\Theta_{AC}\Theta^C_{\ B}), \tag{2.2.23}$$

The equation (2.2.19) transforms into

$$\Theta_{AB}\Theta^{AB} - \theta^2 = -\frac{2}{\Gamma^2|h|}, \tag{2.2.24}$$

and (it is direct to see) that it holds for all U provided it holds for $U = 0$ and (2.2.22) and (2.2.23) hold for all U. The (2.2.24) is thus only a "constraint" equation. Therefore the system (2.2.17)-(7.2.64) is solved by giving $h_{AB}(0), \Theta_{AB}(0)$ and $\Gamma > 0$ satisfying (2.2.24), then running (2.2.22)-(2.2.23) and finally obtaining λ from (2.2.21).

To solve (2.2.22)-(2.2.23) first change variables from U to s, where $ds = \Gamma\sqrt{|h|}dU$. The system (2.2.22)-(2.2.23) now reads

$$\partial_S h_{AB} = 2\Theta_{AB}, \tag{2.2.25}$$

$$\partial_S \Theta_{AB} = -\theta\Theta_{AB} + 2\Theta_{AC}\Theta^{C}_{\ B}, \tag{2.2.26}$$

Use these equations to check that

$$\partial_s \theta = -\theta^2, \tag{2.2.27}$$

$$\partial_s \Theta_{12} = (\Theta_{11}h^{11} + \Theta_{22}h^{22} - 2\Theta_{12}h^{12})\Theta_{12} \tag{2.2.28}$$

Thus, θ has its own evolution equation which gives $\theta(S) = 1/(S + 1/\theta(0))$. Moreover if we choose (z_1, z_2) on $\{U = 0\}$ to diagonalise $h(0)$ and $\Theta(0)$ simultaneously (i.e. $h_{11}(0) = 1, h_{22} = 1, h_{12}(0) = 0$ and $\Theta_{12}(0) = 0$), then (2.2.28) shows that $\Theta_{12} = 0$ and $h_{12} = 0$ for all s and therefore that the evolutions for h_{11} and h_{22} decouple to independent ODEs. With this information it is straightforward to see that the solutions to (2.2.27)-(2.2.28), which at the initial times satisfy also (2.2.24) are only the Kasner solutions.

We will use all the previous discussion later in Section 7.2.

2.2.4 Definition of Kasner asymptotic

3 Background material

1 Manifolds. Manifolds will always be smooth (C^∞). Riemannian metrics as well as tensors will also be smooth.

2 Distance. If g is a Riemannian metric on a manifold Σ, then

$$d_g(p,q) = \inf\big\{L_g(\gamma_{pq}) : \gamma_{pq} \text{ smooth curve joining } p \text{ to } q\big\}. \tag{3.0.1}$$

is a metric, where L_g is the notation we will use for length. When it is clear from the context we will remove the sub-index g and write simply d and L.

- If C is a set and p a point then $d_g(C,p) = \inf\{d_g(q,p) : q \in C\}$. Very often we take $C = \partial\Sigma$.

- If C is a set and $r > 0$, then, define the open ball of "centre" C and radius r as,

$$B_g(C,r) = \{p \in \Sigma : d_g(C,p) < r\} \tag{3.0.2}$$

- $(\Sigma; g)$ is *metrically complete* if the metric space $(\Sigma; d)$ is complete.

3 Scaling. Very often we will work with scaled metrics. To avoid a cumbersome notation we will use often the subindex r (the scale) on scaled metrics, tensors and other geometric objects. Precisely, let $r > 0$, then for the scaled metric g/r^2 we use the notation g_r, namely,

$$g_r := \frac{1}{r^2} g \tag{3.0.3}$$

Similarly, $d_r(p,q) = d_{g_r}(p,q)$, $\langle X, Y\rangle_r = \langle X, Y\rangle_{g_r}$, $|X|_r = |X|_{g_r}$, and for curvatures and related tensors too, for instance if R is the scalar curvature of g, then R_r is the scalar curvature of g_r.

This notation will be used very often and is important to keep track of it.

4 Annuli. Let $(\Sigma; g)$ be a metrically complete and non-compact Riemannian manifold with non-empty boundary $\partial\Sigma$.

- Let $0 < a < b$, then we define the open annulus $\mathcal{A}_g(a,b)$ as

$$\mathcal{A}_g(a,b) = \{p \in \Sigma : a < d_g(p, \partial\Sigma) < b\} \tag{3.0.4}$$

We write just $\mathcal{A}(a,b)$ when the Riemannian metric g is clear from the context.

- When working with scaled metrics g_r, we will alternate often between the following notations

$$\mathcal{A}_r(a,b), \quad \mathcal{A}_{g_r}(a,b), \quad \mathcal{A}_g(ra, rb), \tag{3.0.5}$$

(to denote the same set), depending on what is more simple to write or to read. For instance we could write $\mathcal{A}_{2^j}(1,2)$ instead of $\mathcal{A}_{g_{2^j}}(1,2)$ or $\mathcal{A}_g(2^j, 2^{1+j})$.

- If C is a connected set included in $\mathcal{A}_g(a,b)$, then we define,

$$\mathcal{A}^c_g(C; a, b) \tag{3.0.6}$$

to denote the connected component of $\mathcal{A}_g(a,b)$ containing C. The set C could be for instance a point p in which case we write $\mathcal{A}^c_g(p; a, b)$.

5 Partitions cuts and end cuts. Let $(\Sigma; g)$ be a metrically complete and non-compact Riemannian manifold with non-empty and compact boundary $\partial\Sigma$.

- To understand the asymptotic geometry of data sets, we will study the geometry of scaled annuli. Sometimes however it will be more convenient and transparent to use certain sub-manifolds instead of annuli. For this purpose we define partitions. A set of compact manifolds with non-empty boundary

$$\{\mathcal{P}^m_{j,j+1},\ j = j_0, j_0+1, \ldots,\ m = 1, 2, \ldots, m_j\}, \tag{3.0.7}$$

($j_0 \geq 0$), is a *partition* if,

(a) $\mathcal{P}^m_{j,j+1} \subset \mathcal{A}(2^{1+2j}, 2^{4+2j})$ for every j and m,

(b) $\partial\mathcal{P}^m_{j,j+1} \subset (\mathcal{A}(2^{1+2j}, 2^{2+2j}) \cup \mathcal{A}(2^{3+2j}, 2^{4+2j}))$ for every j and m.

(c) The union $\cup_{j,m}\mathcal{P}^m_{j,j+1}$ covers $\Sigma \setminus B(\partial\Sigma, 2^{2+2j_0})$.

Figure **??** shows skematically a partition. The existence of partitions is easy and done (succinctly) as follows. Let $j_0 \geq 0$ and let $j \geq j_0$. Let $f : \Sigma \to [0, \infty)$ be a (any) smooth function such that $f \equiv 1$ on $\{p : d(p, \partial\Sigma) \leq 2^{1+2j}\}$ and $f \equiv 0$ on

$\{p : d(p, \partial\Sigma) \geq 2^{2+2j}\}$, [(4)]. Let x be any regular value of f in $(0,1)$. For each j let $\mathcal{Q}_j$ be the union of the connected components of $\Sigma \setminus \{f = x\}$ containing at least a component of $\partial\Sigma$. Then the manifolds $\mathcal{P}^m_{j,j+1}$, $m = 1, \ldots, m_j$, are defined as the connected components of $\mathcal{Q}_{j+1} \setminus \mathcal{Q}_j$.

- We let $\partial^-\mathcal{P}^m_{j,j+1}$ be the union of the connected components of $\partial\mathcal{P}^m_{j,j+1}$ contained in $\mathcal{A}(2^{1+2j}, 2^{2+2j})$. Similarly, we let $\partial^+\mathcal{P}^m_{j,j+1}$ be the union of the connected components of $\partial\mathcal{P}^m_{j,j+1}$ contained in $\mathcal{A}(2^{3+2j}, 2^{4+2j})$.

- We let $\{\mathcal{S}_{jk}, k = 1, \ldots, k_j\}$ be the set of connected components of the manifolds $\partial^-\mathcal{P}^m_{j,j+1}$ for $m = 1, \ldots, m_j$. The set of surfaces

$$\{\mathcal{S}_{jk}, j \geq j_0, \ldots, k = 1, \ldots, k_j\} \tag{3.0.8}$$

is called a *partition cut*.

- Suppose now that Σ has only one end. Let $\{\mathcal{S}_{jk}, j \geq j_0 \ldots, k = 1, \ldots, k_j\}$ be a partition cut. For each j one can always remove if necessary manifolds from $\{\mathcal{S}_{jk}, k = 1, \ldots, k_j\}$ and consider a subset $\{\mathcal{S}_{jk_l}, l = 1, \ldots, l_j\}$ such that: if we remove all the surfaces $\mathcal{S}_{jk_l}$, $l = 1, \ldots, l_j$, from Σ, then every connected component of $\partial\Sigma$ belongs to a bounded component of the resulting manifold, whereas if we remove all but one of the surfaces $\mathcal{S}_{jk_l}$, then at least one connected componet of $\partial\Sigma$ belongs to an unbounded component of the resulting manifold. The set of surfaces

$$\{\mathcal{S}_{jk_l}, j \geq j_0, \ldots, l = 1, \ldots, l_j\} \tag{3.0.9}$$

is called an *end cut*.

- If an end cut $\{\mathcal{S}_{jk_l}, j \geq j_0, l = 1, \ldots, l_j\}$ has $l_j = 1$ for each $j \geq j_0$ then we say that the end is a *simple end cut* and write simply $\{\mathcal{S}_j\}$.

- If $\{\mathcal{S}_j\}$ is a simple end cut and $j_0 \leq j < j'$ we let $\mathcal{U}_{j,j'}$ be the compact manifold enclosed by $\mathcal{S}_j$ and $\mathcal{S}_{j'}$.

We begin stating a simple fact (easily proved arguing by contradiction). We will use this below. Let γ be a ray and fix an integer $k \geq 2$. Then, given $\epsilon > 0$ there are $\delta > 0$ and $r_0 > 0$ such that for any $p \in \gamma$ with $r = r(p) \geq r_0$, such that the annulus $(\mathcal{A}^c_r(p; 1/2, 2); g_r)$ is δ-close in the GH-distance to the segment $[1/2, 2]$, then there is a neighbourhood $\mathcal{B}$ of $\mathcal{A}^c_r(p; 1/2, 2)$ and a finite cover $\tilde{\mathcal{B}}$ such that $(\tilde{\mathcal{B}}; \tilde{g}_r)$ is ϵ-close in C^k to a T^2-symmetric flat space $([1/2, 2] \times T^2; g_F)$. Furthermore, there is $\theta_0 > 0$ small, such that if ϵ is chosen small enough and $(\mathcal{A}^c_r(p; 1/2, 2); g_r)$ is δ-close in the GH-distance to the segment $[1/2, 2]$ and the mean curvature θ_F of the torus $\{1\} \times T^2$ in $([1/2, 2] \times T^2; g_F)$ is less or equal than θ_0, then $(\mathcal{A}^c_{r'}(p'; 1/2, 2); g_r)$ is $2\delta/3$-close in the GH-distance to the segment $[1/2, 2]$ where p' is the point in γ such that $r' = r(p') = 2r$.

4 Conformal transformations by powers of the lapse

In this section we study conformal transformations of static metrics by powers of the lapse from from a point of view á la Backry-Émery. Subsection 4.1 explains the structure of the conformal equations (Proposition ??). Subsection 4.2 proves generalised

(4) Consider a partition of unity $\{\chi_i\}$ subordinate to a cover $\{\mathcal{B}_i\}$ where the neighbourhoods $\mathcal{B}_i$ are small enough that if $\mathcal{B}_i \cap \{p : d(p, \partial\Sigma) \leq 2^{1+2j}\} \neq \emptyset$ then $\mathcal{B}_i \cap \{p : d(p, \partial\Sigma) \geq 2^{2+2j}\} = \emptyset$. Then define $f = \sum_{i \in I} \chi_i$, where $i \in I$ iff $\mathcal{B}_i \cap \{p : d(p, \partial\Sigma_i) \leq 2^{1+2j}\} \neq \emptyset$.

Anderson's decay estimates for the conformally related data (Lemma **??**). Section **??** shows metric completeness of the manifolds $(\Sigma; \overline{g} = N^{-2\epsilon} g)$ (provided $\partial\Sigma$ is compact and $N|_{\Sigma} > 0$), Theorem **??**. In Section **??** a few important remarks are pointed out on the conformal data $(\Sigma; N^{-2\epsilon} g)$ of a static data $(\Sigma; g)$, (Proposition **??**).

In section **??** we make a few important remaks on the conformal data $(\Sigma; \overline{g} = N^{-2\epsilon} g)$

4.1 Conformal metrics, the Bakry-Émery Ricci tensor and the static equations

Given a Riemannian metric g, function f and constant α, the α-Bakry-Émery Ricci tensor Ric_f^α is defined as (see [33]; note that [33] uses the notation $1/N$ instead of α),

$$Ric_f^\alpha := Ric + \nabla\nabla f - \alpha \nabla f \nabla f, \tag{4.1.1}$$

where the tensors Ric and ∇ on the right hand side are with respect to g. The f-Laplacian Δ_f acting on a function ϕ is defined as

$$\Delta_f \phi := \Delta\phi - \langle \nabla f, \nabla\phi \rangle \tag{4.1.2}$$

where again Δ on the right hand side are with respect to g and $\langle\ ,\ \rangle = g(\ ,\ )$. Now observe that letting $f := -\ln N$, the static Einstein equations (2.1.1) read

$$Ric = -\nabla\nabla f + \nabla f \nabla f, \qquad \Delta f - \langle \nabla f, \nabla f \rangle = 0 \tag{4.1.3}$$

In the notation above, this is nothing else than to say that

$$Ric_f^\alpha = 0, \qquad \Delta_f f = 0 \tag{4.1.4}$$

with $\alpha = 1$ and $f = -\ln N$. It is an important fact that the structure of these equations is preserved along a one parameter family of conformal transformations. The following calculation explains this fact.

Proposition 4.1.1. *Let $(\Sigma; g, N)$ be a static data set. Fixed ϵ define*

$$\overline{g} = N^{-2\epsilon} g. \tag{4.1.5}$$

Then,

$$\overline{Ric}_f^\alpha = 0, \qquad \overline{\Delta}_f f = 0 \tag{4.1.6}$$

where $\alpha = (1 - 2\epsilon - \epsilon^2)/(1+\epsilon)^2$ and $f = -(1+\epsilon)\ln N$.

We used the notation $\overline{Ric}$ for $Ric_{\overline{g}}$ and $\overline{\Delta}$ for $\Delta_{\overline{g}}$.

Note that when $\epsilon = -1$, we obtain $\alpha = +\infty$, $f = 0$ and $\overline{Ric}_f^\alpha = \overline{Ric} - 2\nabla \ln N \nabla \ln N$. In particular we recover $\overline{Ric} = 2\nabla \ln N \nabla \ln N$.

Proof. We prove first $\overline{\Delta}_f f = 0$. Recall from standard formulae that if $\overline{g} = e^{2\psi} g$ then for every ϕ we have

$$e^{-2\psi}\Delta\phi = \overline{\Delta}\phi - \langle \nabla\phi, \nabla\psi \rangle_{\overline{g}} \tag{4.1.7}$$

Making $\phi = \ln N$ and $e^\psi = N^{-\epsilon}$, the left hand side of (4.1.7) is equal to $-|\nabla \ln N|^2_{\overline{g}}$ because $\Delta \ln N = -|\nabla \ln N|^2_g$. Thus (4.1.7) is $\overline{\Delta} \ln N - \langle \nabla \ln N, -(1+\epsilon)\nabla \ln N \rangle_{\overline{g}} = 0$ as wished.

Let us prove now $\overline{Ric}^{\alpha}_{f} = 0$. Recall first that if $\overline{g} = e^{2\psi} g$ then

$$\overline{Ric} = Ric - (\nabla\nabla\psi - \nabla\psi\nabla\psi) - (\Delta\psi + |\nabla\psi|^2)g \tag{4.1.8}$$

Choosing $\psi = -\epsilon \ln N$ and replacing Ric by (4.1.3) then gives

$$\overline{Ric} = (1+\epsilon)\nabla\nabla \ln N + (1+\epsilon^2)\nabla \ln N \nabla \ln N - (\epsilon + \epsilon^2)|\nabla \ln N|^2 g \tag{4.1.9}$$

Use now the usual general formula

$$\overline{\nabla}_i V_j = \nabla_i V_j - \big[V_j \nabla_i \psi + V_i \nabla_j \psi - (V^k \nabla_k \psi) g_{ij}\big] \tag{4.1.10}$$

with $V^j = \nabla_j \ln N$ and with $\psi = -\epsilon \ln N$, to obtain

$$\nabla\nabla \ln N = \overline{\nabla}\nabla \ln N - \epsilon\big[2\nabla \ln N \nabla \ln N - |\nabla \ln N|^2 g\big] \tag{4.1.11}$$

Plugging (4.1.11) in (4.1.9) gives

$$\overline{Ric} = (1+\epsilon)\overline{\nabla}\nabla \ln N + (1 - 2\epsilon - \epsilon^2)\nabla \ln N \nabla \ln N \tag{4.1.12}$$

which is $\overline{Ric}^{\alpha}_{f} = 0$ as claimed. □

4.2 Conformal metrics and Anderson's curvature decay

In [2] Anderson proved the following fundamental quadratic curvature decay for static data sets.

Lemma 4.2.1 (Anderson, [2])**.** *There is a constant $\eta > 0$ such that for any static data set $(\Sigma; g, N)$ we have,*

$$|Ric|(p) \leq \frac{\eta}{d^2(p, \partial\Sigma)}, \qquad |\nabla \ln N|^2(p) \leq \frac{\eta}{d^2(p, \partial\Sigma)}, \tag{4.2.1}$$

for any $p \in \Sigma^\circ$.

This decay estimate is linked to a similar one for the metric $\mathfrak{g} = N^2 g$ that we state below. It was proved also by Anderson in [2]. We require $N > 0$ everywhere and not only on Σ°, to guarantee that $\mathfrak{g}$ is regular on $\partial\Sigma$. Note that imposing $N > 0$ on Σ, does not make $(\Sigma; \mathfrak{g} = N^2 g)$ automatically metrically complete. Indeed if Σ is non-compact then N could tend to zero over a divergent sequence of points and this may cause the metric incompleteness of the space $(\Sigma; \mathfrak{g})$.

Lemma 4.2.2 (Anderson [2])**.** *There is a constant $\eta > 0$ such that, for any static data set $(\Sigma; g, N)$ with $N > 0$ and for which $(\Sigma; \mathfrak{g} = N^2 g)$ is metrically complete, we have*

$$|Ric_{\mathfrak{g}}|_{\mathfrak{g}}(p) \leq \frac{\eta}{d^2_{\mathfrak{g}}(p, \partial\Sigma)}, \qquad |\nabla \ln N|^2_{\mathfrak{g}}(p) \leq \frac{\eta}{d^2_{\mathfrak{g}}(p, \partial\Sigma)} \tag{4.2.2}$$

for any $p \in \Sigma^\circ$.

The estimates (4.2.1) and (4.2.2) are particular intances of a whole family of estimates for the conformal metrics $\overline{g} = N^{-2\epsilon} g$, with ϵ ranging in the interval $(-1 - \sqrt{2}, -1 + \sqrt{2})$ which is the interval where the polynomial $1 - 2\epsilon - \epsilon^2$ is positive. We prove the estimates below using the results in Section 4.1. As a byproduct we provide concise proofs of Lemmas 4.2.1 and 4.2.2. This will be the goal of this section.

We start with a lemma that to our knowledge is essentially due to J. Case [6] (though similar techniques are well known too at least in the theory of minimal surfaces). This lemma was first presented in [23], but due to its importance we prove it again here.

Lemma 4.2.3. *Let (Σ, g) be a metrically complete Riemannian three-manifold with $Ric^{\alpha}_{f} \geq 0$ for some function f and constant $\alpha > 0$. Let ϕ be a non-negative function such that*

$$\Delta_f \phi \geq c\phi^2 \tag{4.2.3}$$

for some constant $c > 0$. Then, for any $p \in \Sigma^{\circ}$ we have

$$\phi(p) \leq \frac{\eta}{d^2(p, \partial\Sigma)} \tag{4.2.4}$$

where $\eta = (36 + 4/\alpha)/c$.

Observe that the lemma applies too to manifolds with $Ric \geq 0$ as this corresponds to the case $Ric^{\alpha}_{f=0} \geq 0$ for any $\alpha > 0$.

Proof. For any function χ the following general formula holds

$$\Delta_f(\chi\phi) = \phi(\Delta_f \chi) + 2\langle \nabla\chi, \nabla\phi\rangle + \chi\Delta_f\phi \tag{4.2.5}$$

Thus, if $\chi \geq 0$ and if q is a local maximum of $\chi\phi$ on Σ°, we have

$$0 \geq \Big[\Delta_f(\chi\phi)\Big]\Big|_q \geq \Big[\phi\Delta_f\chi - 2\frac{|\nabla\chi|^2}{\chi}\phi + c\chi\phi^2\Big]\Big|_q \tag{4.2.6}$$

where to obtain the second inequality we used (4.2.3). Let $r_p = d(p, \partial\Sigma)$. On $B(p, r_p)$ let the function $\chi(x)$ be $\chi(x) = (r_p^2 - r(x)^2)^2$. To simplify notation make $r = r(x) = d(x, p)$. Let q be a point in the closure of $B(p, r_p)$ where the maximum of $\chi\phi$ is achieved. If $\phi(q) = 0$, then $\phi = 0$ and (4.2.4) holds for any $\eta > 0$. So let us assume that $\phi(q) > 0$. In particular p belongs to the interior of $B(p, r_p)$. By (4.2.6) we have

$$cr_p^4\phi(p) \leq c(\chi\phi)(q) \leq \Big[2\frac{|\nabla\chi|^2}{\chi} - \Delta_f\chi\Big]\Big|_q \tag{4.2.7}$$

$$= \Big[4(r_p^2 - r^2)r\Delta_f r + 4r_p^2 + 20r^2\Big]\Big|_q \tag{4.2.8}$$

But if $Ric^{\alpha}_{f} \geq 0$ then $\Delta_f r \leq (3 + 1/\alpha)/r$, (see [33] Theorem A.1; On non-smooth points of r this equations holds in the barrier sense[(5)]). Using this in (4.2.7) and after a simple computation we deduce,

$$\phi(p) \leq \frac{(4(3 + 1/\alpha) + 24)}{cr_p^2}, \tag{4.2.9}$$

which is (4.2.4). □

Let us see now an application of the previous Lemma. Let $(\Sigma; g, N)$ be a static data with $N > 0$. Let ϵ be a number in $(-1 - \sqrt{2}, -1 + \sqrt{2})$ and assume that the space $(\Sigma;\ \overline{g} = N^{-2\epsilon}g)$ is metrically complete. We claim that there is $\eta(\epsilon) > 0$, such that for all

[(5)]This is an important property as it allows us to make analysis as if r were a smooth function, see [22].

$p \in \Sigma^\circ$ we have

$$|\nabla \ln N|^2_{\overline{g}}(p) \leq \frac{\eta(\epsilon)}{d^2_{\overline{g}}(p, \partial\Sigma)} \tag{4.2.10}$$

Let us prove the claim. Assume first $\epsilon \neq -1$. From Lemma 4.2.3 we know that $\overline{Ric}^\alpha_f = 0$ where $f = -(1+\epsilon)\ln N$ and where $\alpha = (1-2\epsilon-\epsilon^2)/(1+\epsilon)^2$. The factor $(1-2\epsilon-\epsilon^2)$ is greater than zero by the assumption on the range of ϵ. Now use the general formula (see [6])

$$\frac{1}{2}\overline{\Delta}_f|\nabla\phi|^2_{\overline{g}} = |\overline{\nabla}\nabla\phi|^2_{\overline{g}} + \langle\nabla\phi, \nabla(\overline{\Delta}_f\phi)\rangle_{\overline{g}} + \overline{Ric}^\alpha_f(\nabla\phi,\nabla\phi) + \alpha\langle\nabla f, \nabla\phi\rangle^2_{\overline{g}} \tag{4.2.11}$$

with $\phi = \ln N$, together with $\overline{Ric}^\alpha_f = 0$, to obtain

$$\overline{\Delta}_f|\nabla \ln N|^2_{\overline{g}} \geq 2(1-2\epsilon-\epsilon^2)|\nabla\ln N|^4_{\overline{g}} \tag{4.2.12}$$

and thus (4.2.10) from Lemma 4.2.3. When $\epsilon = -1$ then $\overline{Ric}^\alpha_{f=0} \geq 0$ for any $\alpha > 0$ and

$$\overline{\Delta}_{f=0}|\nabla\ln N|^2_{\overline{g}} \geq 4|\nabla \ln N|^4_{\overline{g}} \tag{4.2.13}$$

The claim again follows from Lemma 4.2.3.

Note that Lemma 4.2.3 provides the following explicit expression for $\eta(\epsilon)$,

$$\eta(\epsilon) = \frac{1}{2(1-2\epsilon-\epsilon^2)}\left[36 + \frac{4(1+\epsilon)^2}{(1-2\epsilon-\epsilon^2)}\right] \tag{4.2.14}$$

What we just showed is a part of the *generalised Anderson's quadratic curvature decay* mentioned earlier, that we now state and prove.

Lemma 4.2.4. *Let ϵ be a number in the interval $(-1-\sqrt{2}, -1+\sqrt{2})$. Then there is $\eta(\epsilon)$ such that for any static data set $(\Sigma; g, N)$ with $N > 0$ and for which $(\Sigma; \overline{g} = N^{-2\epsilon}g)$ is metrically complete, we have,*

$$|\overline{Ric}|_{\overline{g}}(p) \leq \frac{\eta(\epsilon)}{d^2_{\overline{g}}(p,\partial\Sigma)}, \qquad |\nabla\ln N|^2_{\overline{g}}(p) \leq \frac{\eta(\epsilon)}{d^2_{\overline{g}}(p,\partial\Sigma)}, \tag{4.2.15}$$

for any $p \in \Sigma^\circ$.

Proof. We have already shown the second estimate of (4.2.15). If $\partial\Sigma = \emptyset$ then N is constant and $\overline{g}$ is flat. So let us assume that $\partial\Sigma \neq \emptyset$. Let $p \in \Sigma^\circ$. By scaling we can assume without loss of generality that $N(p) = 1$ and $\overline{d}_p = d_{\overline{g}}(p,\partial\Sigma) = 1$. In this setup, we need to prove that

$$|\overline{Ric}|_{\overline{g}}(p) \leq c_0(\epsilon), \tag{4.2.16}$$

for c_0 independent of the data.

The second estimate of (4.2.15) yields,

$$|\nabla\ln N|_{\overline{g}}(x) \leq c_1, \tag{4.2.17}$$

for all $x \in B_{\overline{g}}(p, 1/2)$ and where $c_1 = c_1(\epsilon)$ is independent of the data. Therefore, as,

$$\overline{Ric} = (1+\epsilon)\overline{\nabla}\nabla\ln N + (1-2\epsilon-\epsilon^2)\nabla\ln N\nabla\ln N, \tag{4.2.18}$$

then to prove (4.2.16) it is enough to prove

$$|\overline{\nabla}\nabla \ln N|_{\overline{g}}(p) \leq c_0'(\epsilon) \tag{4.2.19}$$

for a $c_0'(\epsilon)$ independent of the data.

Let $\gamma(s)$ be a geodesic segment joining p to x. Then we can write,

$$\big|\ln \frac{N(x)}{N(p)}\big| = \big|\int \nabla_{\gamma'} \ln N ds\big| \leq \int |\nabla \ln N|_{\overline{g}} ds \leq c_1/2 \tag{4.2.20}$$

where we used (4.2.17). Because $N(p) = 1$, this inequality gives,

$$0 < c_2 \leq N(x) \leq c_3 < \infty \tag{4.2.21}$$

for all $x \in B_{\overline{g}}(p, 1/2)$ and where $c_2 = c_2(\epsilon)$ and $c_3 = c_3(\epsilon)$.

Let $\mathfrak{g} = N^{2+2\epsilon}\overline{g} = N^2 g$. If $\epsilon \geq -1$ let $r_0 = c_2^{1+\epsilon}$, whereas if $\epsilon < -1$ let $r_0 = c_3^{1+\epsilon}$. Then, clearly $B_{\mathfrak{g}}(p, r_0) \subset B_{\overline{g}}(p, 1/2)$. Moreover (4.2.17) and (4.2.21) show that for all $x \in B_{\mathfrak{g}}(p, r_0)$ we have,

$$|\nabla \ln N|_{\mathfrak{g}}(x) \leq c_4(\epsilon), \tag{4.2.22}$$

As $Ric_{\mathfrak{g}} = 2\nabla \ln N \nabla \ln N$, we deduce that

$$|Ric_{\mathfrak{g}}|_{\mathfrak{g}}(x) \leq c_5(\epsilon) \tag{4.2.23}$$

for all $x \in B_{\mathfrak{g}}(p, r_0)$. In dimension three the Ricci tensor determines the Riemann tensor, so,

$$|Rm_{\mathfrak{g}}|_{\mathfrak{g}}(x) \leq c_6(\epsilon) \tag{4.2.24}$$

Hence, by standard arguments, there is $r_1(\epsilon) \leq r_0$ such that the exponential map $exp : B^{\mathcal{T}}_{\mathfrak{g}}(p, r_1) \to \Sigma$, is a diffeomorphism into the image, ($B^{\mathcal{T}}_{\mathfrak{g}}(p, r_1)$ is a ball in $\mathcal{T}_p\Sigma$). Let $\tilde{\mathfrak{g}}$ be the lift of $\mathfrak{g}$ to $B^{\mathcal{T}}_{\mathfrak{g}}(p, r_1)$ by exp^{-1}. We still have the bound (4.2.24) for $\tilde{\mathfrak{g}}$ and as the injectivity radius $inj_{\mathfrak{g}}(p)$ is bounded from below by r_1, then the *harmonic radius* $i_h(p)$, which controls the geometry in C^2 (see [22]), is bounded from below by $r_2(\epsilon) \leq r_1$. As $\Delta_{\tilde{\mathfrak{g}}} \ln N = 0$, then standard elliptic estimates give

$$|\nabla^{\tilde{\mathfrak{g}}}\nabla \ln N|_{\tilde{\mathfrak{g}}}(p) \leq c_7(\epsilon), \tag{4.2.25}$$

where $\nabla^{\tilde{\mathfrak{g}}}$ is the covariant derivative of $\tilde{\mathfrak{g}}$. Finally, (4.2.21), (4.2.22), (4.2.25) and the general formula,

$$\overline{\nabla}\nabla \ln N = \nabla^{\mathfrak{g}}\nabla \ln N - (1+\epsilon)\big[2\nabla \ln N \nabla \ln N - |\nabla \ln N|^2_{\mathfrak{g}}\mathfrak{g}\big] \tag{4.2.26}$$

provide the required bound (4.2.19). This completes the proof. □

It is easy to check using elliptic estimates that the proof of the Lemma (4.2.4) leads also to the estimates

$$|\overline{\nabla}^{(k)}\overline{Ric}|_{\overline{g}}(p) \leq \frac{\eta_k(\epsilon)}{d^{2+k}_{\overline{g}}(p, \partial\Sigma)}, \qquad |\overline{\nabla}^{(k)}\nabla \ln N|^2_{\overline{g}}(p) \leq \frac{\eta_k(\epsilon)}{d^{2+2k}_{\overline{g}}(p, \partial\Sigma)} \tag{4.2.27}$$

for every $k \geq 1$, where $\overline{\nabla}^{(k)}$ is $\overline{\nabla}$ applied k-times and where the positive constants $\eta(\epsilon)$, $\eta_1(\epsilon)$, $\eta_2(\epsilon)$, $\eta_3(\epsilon), \ldots$ are independent of the data set.

4.3 Conformal metrics and metric completeness

In this section we aim to prove that metric completeness of data sets (with $N > 0$ and $\partial\Sigma$ compact) imply the metric completeness of the conformal spaces $(\Sigma; \overline{g} = N^{-2\epsilon} g)$ for any ϵ in the range $(-1-\sqrt{2}, -1+\sqrt{2})$. Note that until now, when it was necessary we have benn including the completeness of the metrics $\overline{g}$ as a hypothesis.

Theorem 4.3.1. *Let ϵ be a number in the interval $(-1-\sqrt{2}, -1+\sqrt{2})$. Let $(\Sigma; g, N)$ be a static data set with $N > 0$ and $\partial\Sigma$ compact. Then $(\Sigma; \overline{g} = N^{-2\epsilon} g)$ is metrically complete.*

We start proving a corollary to Lemma 4.2.4 that estimates N.

Corollary 4.3.2. (to Lemma 4.2.4) *Let ϵ be a number in the interval $(-1-\sqrt{2}, -1+\sqrt{2})$. Let $(\Sigma; g, N)$ be a static data set with $N > 0$ and $\partial\Sigma$ compact, and for which $(\Sigma, \overline{g} = N^{-2\epsilon} g)$ is metrically complete. Then, there is $c > 0$ (depending on the data) such that*

$$\frac{1}{c(1 + d_{\overline{g}}(p, \partial\Sigma))^{\sqrt{\eta}}} \leq N(p) \leq c(1 + d_{\overline{g}}(p, \partial\Sigma))^{\sqrt{\eta}} \tag{4.3.1}$$

for any $p \in \Sigma^\circ$, where $\eta = \eta(\epsilon)$ is the coefficient in the decay estimate (4.2.15) of Lemma 4.2.4.

Proof. Let $p \in \Sigma$ such that $\overline{d}_p := d_{\overline{g}}(p, \partial\Sigma) \geq 1$. Let $\gamma(\overline{s})$ be a $\overline{g}$-geodesic segment joining $\partial\Sigma$ to p and realising the $\overline{g}$-distance between them (in particular $N(\gamma(\overline{d}_p)) = N(p)$). Then we can write

$$\left| \ln \frac{N(\gamma(\overline{d}_p))}{N(\gamma(1))} \right| = \left| \int_1^{\overline{d}_p} \nabla_{\gamma'} \ln N d\overline{s} \right| \leq \int_1^{\overline{d}_p} \left| \nabla \ln N \right| d\overline{s} \leq \sqrt{\eta(\epsilon)} \ln \overline{d}_p \tag{4.3.2}$$

where to obtain the last inequality we have used (4.2.10). Therefore,

$$N(p) \leq N(\gamma(1)) \overline{d}_p^{\sqrt{\eta}} \quad \text{and} \quad N(p) \geq N(\gamma(1)) / \overline{d}_p^{\sqrt{\eta}} \tag{4.3.3}$$

Thus,

$$\overline{m} \overline{d}_p^{\sqrt{\eta}} \geq N(p) \geq \underline{m} / d_p^{\sqrt{\eta}} \tag{4.3.4}$$

where $\overline{m} = \max\{N(q) : d_{\overline{g}}(q, \partial\Sigma) = 1\}$ and $\underline{m} = \min\{N(q) : d_{\overline{g}}(q, \partial\Sigma)\}$. This clearly implies (4.3.1). Obtaining (4.3.1) for all $p \in \Sigma^\circ$, namely even for those with $\overline{d}_p \leq 1$, is direct due to the compactness of $\partial\Sigma$. □

Proposition 4.3.3. *Let ϵ be a number in the interval $(-1-\sqrt{2}, -1+\sqrt{2})$. Let $(\Sigma; g, N)$ be a static data set with $N > 0$ and for which $(\Sigma, \overline{g} = N^{-2\epsilon} g)$ is metrically complete. Then, for any ζ such that $|\zeta| \leq 1/(2\sqrt{\eta})$, the space $(\Sigma; N^{2\zeta} \overline{g})$ is metrically complete, where $\eta = \eta(\epsilon)$ is the coefficient in (4.2.15).*

Proof. Let us assume that Σ is non-compact otherwise there is nothing to prove. Let $\hat{g} = N^{2\zeta} \overline{g}$. To prove that $(\Sigma; \hat{g})$ is complete, we need to show that the following holds: for any sequence of points p_i whose $\overline{g}$-distance to $\partial\Sigma$ diverges, then the $\hat{g}$-distance to $\partial\Sigma$ also diverges. Equivalently, we need to prove that for any sequence of curves α_i starting at $\partial\Sigma$ and ending at p_i we have

$$\int_0^{\overline{s}_i} N^{\zeta}(\alpha_i(\overline{s})) d\overline{s} \longrightarrow \infty \tag{4.3.5}$$

where $\overline{s}$ is the $\overline{g}$-arc length of α_i counting from $\partial\Sigma$.

From (4.3.1) we get,

$$N^{\zeta}(p) \geq \frac{c^{-|\zeta|}}{(1+d_{\overline{g}}(p,\partial\Sigma))^{|\zeta|\sqrt{\eta}}} \tag{4.3.6}$$

for all p. But, $d_{\overline{g}}(\alpha_i(\overline{s}),\partial\Sigma) \leq \overline{s}$ and $|\zeta| \leq 1/(2\sqrt{\eta})$, so we deduce,

$$N^{\zeta}(\alpha_i(\overline{s})) \geq \frac{c^{-|\zeta|}}{(1+\overline{s})^{1/2}} \tag{4.3.7}$$

Thus,

$$\int_0^{\overline{s}_i} N^{\zeta}(\alpha_i(\overline{s}))d\overline{s} \geq \int_0^{\overline{s}_i} \frac{c^{-|\zeta|}}{(1+\overline{s})^{1/2}} d\overline{s} \longrightarrow \infty \tag{4.3.8}$$

as $\overline{s}_i \to \infty$ as wished. □

We prove now Theorem 4.3.1.

Proof of Theorem 4.3.1. Let $\epsilon \in (-1-\sqrt{2}, -1+\sqrt{2})$. Assume $\epsilon \neq 0$ otherwise there is nothing to prove. Let $n > 0$ be an integer such that for any $i = 0, 1, \ldots, n-1$,

$$\left|\frac{\epsilon}{n}\right| \leq \frac{1}{2\sqrt{\eta(i\epsilon/n)}} \tag{4.3.9}$$

where η is the coefficient in (4.2.15). According to Proposition 4.3.3, the condition (4.3.9) says that if $\overline{g}_i = N^{-2(i\epsilon/n)}g$ is complete then so is $\overline{g}_{i+1} = N^{-2\epsilon/n}\overline{g}_i = N^{-2(i+1)\epsilon/n}g$ for any $i = 0, 1, \ldots, n-1$. Therefore, as g is complete, then so are $\overline{g}_1$, $\overline{g}_2$, $\overline{g}_3$, until $\overline{g}_n = N^{-2\epsilon}g$ as wished. □

4.4 Applications.

4.4.1 Conformal transformations of black hole metrics

Let $(\Sigma; g, N)$ be a static black hole data set. We denote by Σ_δ the manifold resulting after removing from Σ the g-tubular neighbourhood of $\partial\Sigma$ and radius δ, i.e. $\Sigma_\delta = \Sigma \setminus B(\partial\Sigma, \delta)$. Let δ_0 be small enough that $\partial\Sigma_\delta$ is always smooth and isotopic to $\partial\Sigma$ for any $\delta \leq \delta_0$.

Given $\epsilon > 0$ let $\overline{g} = N^{-2\epsilon}g$. Let $\delta > 0$ such that $\delta < \delta_0$. The second fundamental form $\overline{\Theta}$ of $\partial\Sigma_\delta$, (with respect to $\overline{g}$ and with respect to the inward normal to Σ_δ), is

$$\overline{\Theta} = N^{\epsilon}\Theta - \epsilon\frac{\nabla_n N}{N^{1-\epsilon}}g \tag{4.4.1}$$

where Θ is the second fundamental form of $\partial\Sigma_\delta$ with respect to g and n is the inward g-unit normal. If we let $\delta \to 0$, the function $\nabla_n N|_{\partial\Sigma_\delta}$ converges (on each connected component) to a positive constant (the surface gravity) while $N|_{\partial\Sigma_\delta}$ converges to zero. Hence if δ is small enough, the second term on the right hand side of (4.4.1) dominates over the first, and the boundary $\partial\Sigma_\delta$ is strictly convex with respect to $\overline{g}$.

Combining this discussion with Theorem 4.3.1 we deduce the following Proposition that was proved for the first time in [27] and that will be used fundamentally in the next section.

Proposition 4.4.1. *Let $(\Sigma; g, N)$ be a static black hole data set. Then, for every $0 < \epsilon < -1 + \sqrt{2}$ there is $0 < \delta < \delta_0$ such that $(\Sigma_\delta; \overline{g} = N^{-2\epsilon} g)$ is metrically complete and $\partial\Sigma_\delta$ is strictly convex (with respect to $\overline{g}$ and with respect to the inward normal).*

The Riemannian spaces $(\Sigma_\delta; \overline{g})$ have a metric, as discussed earlier, that we will denote by $d^\delta_{\overline{g}}$. The strict convexity of the boundaries as well as the metric completeness of the spaces $(\Sigma_\delta; \overline{g})$ imply two basic, albeit important, geometric facts:

(i) The distance $d^\delta_{\overline{g}}(p, q)$ between two points in Σ_δ is always realised by the length of a geodesic segment joining p to q, and disjoint from $\partial\Sigma_\delta$ except, possibly, at the end-points p and q.

(ii) Given a curve I embedded in Σ_δ and with end-points p and q, there is always a geodesic segment minimising length in the class of curves embedded in Σ_δ, isotopic to I and having the same end-points. The minimising segment is disjoint from $\partial\Sigma_\delta$ except, possibly, at the end points p and q.

These properties allow us to make analysis as if the manifold Σ_δ were in practice boundaryless, and thus to import a series of results from *comparison geometry*, as developed for instance in [33], without worrying about the existence of the boundary.

4.4.2 The structure of infinity

The following proposition shows that static black hole data sets have only one end and moreover admit simple end cuts.

Proposition 4.4.2. *Let $(\Sigma; g, N)$ be a static black hole data set. Then Σ has only one end. Moreover $(\Sigma; g)$ admits a simple end cut.*

Proof. We work with the manifolds $(\Sigma_\delta, \overline{g} = N^{-2\epsilon} g)$ from Proposition 4.4.1, with $0 < \epsilon < -1 + \sqrt{2}$ and $\delta = \delta(\epsilon) \leq \delta_0$. We argue first in a fixed $(\Sigma_\delta; \overline{g})$ and then let $\epsilon \to 0$. If $i_\Sigma > 1$, i.e. if Σ has at least two ends, then Σ_δ has also at least two ends. Hence Σ_δ, (which has convex boundary) contains a line diverging through two of them. The presence of a line is relevant because, even having $\partial\Sigma_\delta \neq \emptyset$, the geometry of $(\Sigma_\delta; \overline{g}, N)$ is such (recall the discussion in Section 4.4.1) that the *Splitting Theorem* as proved in [33] applies [(6)]. More precisely, repeating line by line the proof of Theorem 6.1 in [33], one concludes that (see comments below after (a), (b) and (c)),

(a) there is a smooth Busemann function b^+_ϵ, (b^+ in the notation of [33]), with $|\nabla b^+_\epsilon|_{\overline{g}} = 1$ and whose level sets are totally geodesic,

(b) the Ricci tensor is zero in the normal direction to the level sets, that is

$$\overline{Ric}(\nabla b^+_\epsilon, -) = 0, \tag{4.4.2}$$

(c) N is constant in the normal directions to the level sets, that is $\langle \nabla b^+_\epsilon, \nabla N \rangle_{\overline{g}} = 0$.

[(6)] Theorem 6.1 in [33] is stated for spaces with $Ric^0_f \geq 0$ and f bounded. The boundedness of f is required to have a Laplacian comparison for distance functions (§ [33] Theorem 1.1). No such condition on f (hence on N, because $f = -(1+\epsilon)\ln N$) is required in our case, as we have $\overline{Ric}^0_f = \alpha \nabla f \nabla f$ with $\alpha > 0$ and a Laplacian comparison holds without further assumptions (§ [33], Theorem A.1).

The item (a) is what is proved in Theorem 6.1 of [33] and requires no comment. The items (b) and (c) follow instead from formula (6.11) in [33] after recalling that in our case we have $\overline{Ric}^0_f = \alpha\nabla f\nabla f$, with $f = -(1+\epsilon)\ln N$ and $\alpha > 0$.

Of course (a) implies that $\overline{g}$ locally splits. Namely, defining a coordinate x by $x = b^+$, one can locally write $\overline{g} = dx^2 + \overline{h}$, where $\overline{h}$ is the metric inherited from $\overline{g}$ on the level sets of x, that (under a natural identification) does not depend on x.

The conclusions (a), (b) and (c) imply a contradiction as follows. Fix a point p in $\Sigma^\circ_{\delta_0}$ and take a sequence $\epsilon_i \to 0$. Then, in a small but fixed neighbourhood $\mathcal{U}$ of p, the sequence $b^+_{\epsilon_i}$ sub-converges to a limit function b^+_0, with the same properties (a), (b), (c) as each $b^+_{\epsilon_i}$ but now on $(\mathcal{U}; g, N)$, [(7)]. Hence $(\mathcal{U}; g)$ also splits. We claim that the Gaussian curvature κ of the level sets of b^+_0 in $\mathcal{U}$ is zero. Indeed, as: (i) the level sets of b^+_0 are totally geodesic by (a), (ii) $Ric(\nabla b^+_0, \nabla b^+_0) = 0$ by (b), and (iii) the scalar curvature R of g is zero by the static equations, then the Gauss-Codazzy equations yield $\kappa = 0$. As $(\mathcal{U}; g)$ is flat then the static solution is flat everywhere by analyticity. The only flat solution with compact boundary is the Boost. As Boots have only one end we reach a contradiction. Hence $i_\Sigma = 1$.

Let us prove now that $(\Sigma; g)$ admits simple cuts. Let $\{\mathcal{S}_{jk}, j = 0, 1, 2, \ldots, k = 1, \ldots, k_j\}$ be an end cut. Suppose that $k_j > 1$ for some $j \geq 0$. If we cut Σ along $\mathcal{S}_{j1}$ we obtain a connected manifold, say Σ', with two new boundary components, say $\mathcal{S}'_1$ and $\mathcal{S}'_2$, both of which are copies of $\mathcal{S}_{j1}$ (if cutting Σ along $\mathcal{S}_{j1}$ results in two connected components then $k_j = 1$ because of how simple cuts are constructed). Consider another copy of Σ', denoted by Σ'' and denote the corresponding new boundary components as $\mathcal{S}''_1$ and $\mathcal{S}''_2$. By gluing $\mathcal{S}'_1$ to $\mathcal{S}''_2$ and $\mathcal{S}'_2$ to $\mathcal{S}''_1$ we obtain a static solution (a double cover of the original) with two ends, and one can proceed as earlier to obtain a contradiction. □

4.4.3 Horizons's types and properties

The following Proposition, about the structure of horizons, makes use of Sections 5 and 4.4.2 and a combination of results due to Galloway [?], [?].

Proposition 4.4.3. *Let $(\Sigma; g, N)$ be a static black hole data set. Then, either*

(i) $(\Sigma; g, N)$ *is a Boost and therefore $\partial\Sigma$ is a totally geodesic flat torus, or,*

(ii) *every component of $\partial\Sigma$ is a totally geodesic, weakly outermost, minimal sphere.*

Proof. The idea is to prove that every component H of $\partial\Sigma$ is a weakly outermost. Then, it is direct from Theorem 1.1 and 1.2 in [9] that either H is a sphere or is a torus and if it is a torus then the whole space is a Boost. So let us prove that every component is weakly outermost.

Let $\{H_1, \ldots, H_h\}$, $h \geq 1$, be the set of horizons, i.e. the connected components of $\partial\Sigma$. Assume that there is an embedded orientable surface $\mathcal{S}$, homologous to one of the H's, (say H_1), and with outer-mean curvature $\theta_{\mathcal{S}}$ strictly negative. For reference below define the negative constant c as

$$c = \sup\left\{\frac{\theta_{\mathcal{S}}(q)}{N(q)} : q \in \mathcal{S}\right\} \tag{4.4.3}$$

Let $\{\mathcal{S}_j, j = j_0, j_1, \ldots\}$ be a simple end cut of $(\Sigma; g)$ (Proposition 4.4.2). For each j, let $\Omega(\partial\Sigma, \mathcal{S}_j)$ be the closure of the connected component of $\Sigma \setminus \mathcal{S}_j$ containing $\partial\Sigma$. Let

(7) The existence of the limit is easy to see because $|\nabla b^+_\epsilon|_{\overline{g}} = 1$ and the level sets of b^+_ϵ are totally geodesic, (for every ϵ). At every point the level set is just defined by geodesics perpendicular to ∇b^+_ϵ

$\mathcal{U}$ be the closed region enclosed by H_1 and $\mathcal{S}$ and assume that j_0 is large enough that $\mathcal{S}_j \cap \mathcal{U} = \emptyset$ for all $j \geq j_0$. For every $j \geq j_0$ let $\mathcal{M}_j$ be the closed region enclosed by $\mathcal{S}, H_2, \ldots, H_h$ and $\mathcal{S}_j$, that is $\mathcal{M}_j = \Omega(\partial\Sigma, \mathcal{S}_j) \setminus \mathcal{U}^\circ$. Finally let

$$\hat{\mathcal{M}}_j = \mathcal{M}_j \setminus (H_2 \cup \ldots \cup H_h) \tag{4.4.4}$$

and note that now $\partial\hat{\mathcal{M}}_j = \mathcal{S} \cup \mathcal{S}_j$. On $\hat{\mathcal{M}}_j$ consider the optical metric $\overline{g} = N^{-2}g$. The Riemannian space $(\hat{\mathcal{M}}_j; \overline{g})$ is metrically complete, (roughly speaking the horizons $H_i, i \geq 2$ have been blown to infinity).

Now, for every $j \geq j_0$ let γ_j be the $\overline{g}$-geodesic segment inside $\hat{\mathcal{M}}_j$, realising the $\overline{g}$-distance between $\mathcal{S}$ and $\mathcal{S}_j$. The segments γ_j are perpendicular to $\mathcal{S}$. Also, as they are length-minimising the $\overline{g}$-expansion $\overline{\theta}$ of the congruence of $\overline{g}$-geodesics emanating perpendicularly from $\mathcal{S}$, remains finite all along γ_j. Let $s \in [0, s_j]$ be the g-arc-length of γ_j measured from $\mathcal{S}$. Note that s is not the arc-length with respect to $\overline{g}$, that would be natural. We are going to use this parameterisation of γ_j below. Observe that $s_j \to \infty$ as $j \to \infty$.

Along $\gamma_j(s)$ let

$$F(s) = \overline{\theta}(\gamma_j(s)) + \frac{2}{N^2(\gamma_j(s))}\frac{dN(\gamma_j(s))}{ds} \tag{4.4.5}$$

Then, as shown by Galloway [8] (see also [20]), the function F satisfies the following differential inequality

$$\frac{dF}{ds} \leq -\frac{N}{2}F^2 \tag{4.4.6}$$

Now, a simple computation shows that $F(0) = \theta(0)/N(0) \leq c < 0$. But from (4.4.5) it is easily deduced that if

$$\int_0^{s_j} N(\gamma_j(s))ds > -\frac{2}{c} \tag{4.4.7}$$

then there is $s^* \in (0, s_j)$ such that $F(s^*) = -\infty$, thus $\overline{\theta}(s^*) = -\infty$ and the γ_j would not be $\overline{g}$-length minimising. Thus, a contradiction is reached if we prove that $\int_0^{s_j} N(\gamma_j(s))ds \to \infty$. But his follows from the completness of the metric $\mathfrak{g} = N^2 g$ from Theorem 4.3.1.

□

4.4.4 The ball-covering property and a Harnak-type of estimates for the Lapse

Let $(\Sigma; g, N)$ be a static data set. In [3], Anderson observed that as the four-metric $N^2dt^2 + g$ is Ricci-flat, then Liu's ball-covering property holds. Namely, for any $b > a > \delta > 0$ there is n and r_0 such that for any $r \geq r_0$ the annulus $\mathcal{A}(ra, rb)$ can be covered by at most n balls of g-radius $r\delta$ centred in the same annulus (equivalently $\mathcal{A}_r(a, b)$ can be covered by at most n balls of g_r-radius δ centred in the same annulus). Hence any two points p and q in a connected component of $\mathcal{A}_r(a, b)$ can be joined through a chain, say α_{pq}, of at most $n+2$ radial geodesic segments of the balls of radius δ covering $\mathcal{A}_r(a, b)$. On the other hand Anderson's estimate implies that the g_r-gradient $|\nabla \ln N|_r$ is uniformly bounded (i.e. independent on r) on $\mathcal{A}_r(a-\delta, a+\delta)$ and therefore uniformly bounded over any curve α_{pq}. Integrating $|\nabla \ln N|_r$ along the curves α_{pq} and using the bound we arrive at a relevant Harnak estimate controling uniformly (i.e. independly of r) the quotients $N(p)/N(q)$. The estimate is due to Anderson and is summarised in the next Proposition (for further details see, [27]).

Proposition 4.4.4. (Anderson, [3]) *Let $(\Sigma; g, N)$ be a static data set and let $0 < a < b$. Then,*

1. *There is r_0 and $\eta > 0$, such that for any $r > r_0$ and for any set Z inluded in a connected component of $\mathcal{A}_r(a,b)$ we have,*

$$\max\{N(p) : p \in Z\} \leq \eta \min\{N(p) : p \in Z\} \tag{4.4.8}$$

2. *Furthermore, if $r_i \to \infty$ and if Z_i is a sequence of sets such that for each i the set Z_i is included in a connected component $\mathcal{A}^c_{r_i}(a,b)$ of $\mathcal{A}_{r_i}(a,b)$ and we have,*

$$\max\{|\nabla \ln N|_{r_i}(p) : p \in \mathcal{A}^c_{r_i}(a/2, 2b)\} \to 0 \tag{4.4.9}$$

then,

$$\frac{\max\{N(p) : p \in Z_i\}}{\min\{N(p) : p \in Z_i\}} \to 1. \tag{4.4.10}$$

as $i \to \infty$.

Let $(\Sigma; g, U)$ be a static data set in the harmonic presentation (assume $N > 0$). We have shown that $(\Sigma; \mathfrak{g})$ is metrically complete and that $|\nabla U|^2_{\mathfrak{g}}$ decays quadratically. But as $Ric_{\mathfrak{g}} \geq 0$ Liu's ball covering property also holds on $(\Sigma; \mathfrak{g})$. Repeating then Anderson's argument we arrive at the following Harnak estimate but in the harmonic presentation.

Proposition 4.4.5. *Let $(\Sigma; \mathfrak{g}, U)$ be a static data set and let $0 < a < b$. Then,*

1. *There is $r_0 > 0$ and $\eta > 0$, such that for any $r > r_0$ and set Z included in a connected component of $\mathcal{A}_r(a,b)$ we have,*

$$\max\{U(q) : q \in Z\} \leq \eta + \min\{U(q) : q \in Z\}, \tag{4.4.11}$$

2. *Furthermore, if $r_i \to \infty$ and if Z_i is a sequence of sets such that for each i the set Z_i is included in a connected compoent $\mathcal{A}^c_{r_i}(a,b)$ of $\mathcal{A}_{r_i}(a,b)$ and we have,*

$$\max\{|\nabla U|_{r_i}(q) : q \in \mathcal{A}^c_{r_i}(a/2, 2b)\} \to 0 \tag{4.4.12}$$

then,

$$\max\{U(q) : q \in Z_i\} - \min\{U(q) : q \in Z_i\} \to 0 \tag{4.4.13}$$

as $i \to \infty$.

Both propositions will be used later.

4.4.5 The asymptotic of isolated systems.

Theorem 4.3.1 shows that if $N > 0$ and $\partial\Sigma$ is compact then $(\Sigma; \mathfrak{g} = N^2 g)$ is metrically complete. On the other hand it was proved in [25], [26], that if Σ is diffeomorphic to $\mathbb{R}^3$ minus a ball and $\mathfrak{g}$ is complete then the space $(\Sigma; g, N)$ is asymptotically flat. Combining these two results we obtain that: if Σ minus a compact set K is diffeomoprhic to $\mathbb{R}^3$ minus a closed ball then the data set $(\Sigma; g, N)$ is asymptotically flat. Asymptotic flatness is thus characterised only by the asymptotic topology of Σ.

This fact has physically interesting consequences. Following physical intuition define a *static isolated system* as a static space-time $(\mathbb{R} \times \Sigma; -N^2 dt^2 + g)$, ($\partial\Sigma = \emptyset$ and $(\Sigma; g)$

metrically complete), for which there is a set $K \subset \Sigma$ such that $\Sigma \setminus K$ is diffeomoprhic to $\mathbb{R}^3$ minus a closed ball and such that the region $\mathbb{R} \times (\Sigma \setminus K)$ is vacuum (i.e. matter lies only in $\mathbb{R} \times K$). The most obvious example of static isolated system one can think of is that of body like a planet or a star. Then, using what we explained in the previous paragraph, static isolated systems are always asymptotically flat. This conclusion was reached in [27] but requiring as part of the definition of static isolated system that the space-time is null geodesically complete at infinity. What we are showing here is that this condition is indeed unnecessary and the completeness of the hypersurface $(\Sigma; g)$ is sufficient.

5 Global properties of the lapse

We aim to prove that the lapse N of any black hole data set is bounded away from zero at infinity, namely that there is $c > 0$ such that for any divergent sequence p_n we have $\lim N(p_n) \geq c$.

Theorem 5.0.1. *Let $(\Sigma; g, N)$ be a static black hole data set. Then, N is bounded away from zero at infinity.*

The proof of this theorem will follow after some propositions that we state and prove below.

Proposition 5.0.2. *Let $(\Sigma_\delta; \overline{g})$ be a space as in Proposition 4.4.1, with $0 < \epsilon < 1/4$. Let p and q be two different points in Σ_δ and let $\gamma : [0, L] \to \Sigma_\delta$ be a $\overline{g}$-geodesic (parameterised with the arc-length $\overline{s}$) starting at p and ending at q and minimising the $\overline{g}$-length in its own isotopy class. Then, for any $0 < s < t < L$ we have*

$$-\sqrt{50\left[\frac{(t-s)}{s} + \frac{(t-s)}{L-t}\right]} \leq \ln\left[\frac{N(\gamma(t))}{N(\gamma(s))}\right] \leq \sqrt{50\left[\frac{(t-s)}{s} + \frac{(t-s)}{L-t}\right]} \tag{5.0.1}$$

Note that in this statement, s, $t-s$ and $L-t$ are, respectively, the $\overline{g}$-distances along γ between the pairs of points $(p, \gamma(s))$, $(\gamma(s), \gamma(t))$ and $(\gamma(t), q)$.

Proof. Let f and α be as in Proposition 4.1.1. Let γ, s and t be as in the hypothesis. Let $\theta(\overline{s})$ be the expansion along γ of the congruence of geodesics emanating from p, where $\overline{s}$ is the arc-length. From (4.1.6) we can write

$$\overline{Ric}_f^{\alpha/2} = \overline{Ric} + \overline{\nabla}\overline{\nabla} f - \frac{\alpha}{2}\overline{\nabla} f \overline{\nabla} f = \frac{\alpha}{2}\overline{\nabla} f \overline{\nabla} f \tag{5.0.2}$$

where $0 < \alpha$ because $0 < \epsilon < 1/4 < -1 + \sqrt{2}$. Let $\theta_f = \theta - f'$ where $f' = df(\gamma(\overline{s}))/d\overline{s}$. As shown in [33], (5.0.2) implies that,

$$\theta_f' \leq -\frac{1}{2/\alpha + 3}\theta_f^2 - \frac{\alpha}{2}(f')^2 = -a^2\theta_f^2 - b^2\left(\frac{N'}{N}\right)^2 \tag{5.0.3}$$

where $' = d/d\overline{s}$ and

$$a^2 = \frac{1}{2/\alpha + \epsilon}, \quad \text{and} \quad b^2 = \frac{(1+\epsilon)^2\alpha}{2} \tag{5.0.4}$$

From the differential inequality $\theta_f' \leq -a^2\theta_f^2$ we deduce,

$$\theta_f(s) \leq \frac{1}{a^2 s} \tag{5.0.5}$$

and also we deduce

$$\theta_f(t) \geq -\frac{1}{a^2(L-t)} \tag{5.0.6}$$

because if $\theta_f(s) < -\frac{1}{L-t}$ then there exists r, with $t < r < L$, for which $\theta_f(r) = -\infty$, and therefore $\theta(r) = -\infty$, contradicting that γ is length minimising within its isotopy class.

Hence, we can use (5.0.5) and (5.0.6) and $\theta'_f \leq -b^2(N'/N)^2$ to deduce

$$\left| \ln \frac{N(t)}{N(s)} \right|^2 = \left| \int_s^t \frac{N'}{N} d\bar{s} \right|^2 \leq (t-s) \int_s^t \left(\frac{N'}{N} \right)^2 d\bar{s} \tag{5.0.7}$$

$$\leq (t-s) \frac{1}{b^2} (\theta_f(s) - \theta_f(t)) \leq \frac{(t-s)}{a^2 b^2} \left(\frac{1}{s} + \frac{1}{L-t} \right) \tag{5.0.8}$$

which gives (5.0.1) if one observes that $1/a^2b^2 \leq 50$, after a short computation involving (5.0.4), the form of α from Proposition 4.1.1, and the fact that $\epsilon < 1/4$. □

Proposition 5.0.3. *Let $(\Sigma; g, N)$ be a static black hole data set. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two disjoint, connected, compact, boundaryless and orientable surfaces, embedded in Σ°. Let $W : \mathbb{R} \to \Sigma^\circ$ be a smooth embedding, intersecting $\mathcal{S}_1$ and $\mathcal{S}_2$ only once and transversely and with $W(t)$ diverging as $t \to \pm\infty$. Then, there is $p_1 \in \mathcal{S}_1$ and $p_2 \in \mathcal{S}_2$ such that $N(p_1) = N(p_2)$.*

Proof. We work in a manifold $(\Sigma_\delta; \bar{g})$ as in Proposition 4.4.1 and with $0 < \epsilon < 1/4$. Assume thus that δ is small enough that $(W \cup \mathcal{S}_1 \cup \mathcal{S}_2) \subset \Sigma^\circ_\delta$. Orient W in the direction of increasing t. Orient also $\mathcal{S}_1$ and $\mathcal{S}_2$ in such a way that the intersection number between $\mathcal{S}$ and W, and between $\mathcal{S}_2$ and W, are both equal to one. All intersection numbers below are defined with respect to these orientations.

Redefine the parameter t if necessary to have $W(-1) \in \mathcal{S}_1$ and $W(1) \in \mathcal{S}_2$. Then, for every natural number $m \geq 1$ let $\gamma_m(\bar{s})$ be a $\bar{g}$-geodesic minimising the $\bar{g}$-length among all the curves embedded in Σ°_δ, with end points $W(-1-m)$ and $W(1+m)$ and having non-zero intersection number with $\mathcal{S}_1$ and $\mathcal{S}_2$, [(8)]. We denoted by $\bar{s}$ the $\bar{g}$-arc length starting from $W(-1-m)$. The $\bar{g}$-length of γ_m is denoted by L_m.

We want to prove that there are points $p^1_m := \gamma_m(\bar{s}^1_m) \in \mathcal{S}_1$ and $p^2_m := \gamma_m(\bar{s}^2_m) \in \mathcal{S}_2$, (for some $\bar{s}^1_m$ and $\bar{s}^2_m$), with $|\bar{s}^2_m - \bar{s}^1_m|$ uniformly bounded above. Once this is done the proof is finished as follows. As the initial and final points $W(-1-m)$ and $W(1+m)$ get further and further away from $\mathcal{S}_1$ and $\mathcal{S}_2$, then we have $\bar{s}^1_m \to \infty$, $\bar{s}^2_m \to \infty$, $L_m - \bar{s}^2_m \to \infty$, and $L_m - \bar{s}^1_m \to \infty$. Therefore we can rely in Proposition 5.0.2 used with $\gamma = \gamma_m$, $\gamma(s) = p^1_m$, and $\gamma(t) = p^2_m$, to conclude that

$$\lim_{m\to\infty} |N(p^1_m) - N(p^2_m)| = 0 \tag{5.0.9}$$

Hence, if p_1 is an accumulation point of $\{p^1_m\}$ and p_2 an accumulation point of $\{p^2_m\}$ we will have $N(p_1) = N(p_2)$ as desired.

[(8)] The existence of such geodesic is as follows. Let $\mathcal{C}$ be the family of all curves joining $W(-1-m)$ and $W(1+m)$ and having non-zero intersection number with $\mathcal{S}_1$ and $\mathcal{S}_2$. As the intersection number is an isotopy-invariant, the family $\mathcal{C}$ is a union of isotopy classes. In each class consider a representative minimising length inside the class (recall the discussion in Section 4.4.1). Let C_i be a sequence of such representatives and (asymptotically) minimising length in the family $\mathcal{C}$. Such sequence has a convergent subsequence, to, say, C_∞. As for $i \geq i_0$ with i_0 big enough, C_i is isotopic to C_∞ we conclude that $C_\infty \in \mathcal{C}$ as wished.

Consider now the set of embedded curves $X : [-1, 1] \rightarrow \Sigma^{\circ}$, starting at $\mathcal{S}_1$ and transversely to it, ending at $\mathcal{S}_2$ and transversely to it, and not intersecting $\mathcal{S}_1$ and $\mathcal{S}_2$ except of course at the initial and final points. There are at most four classes of curves X, distinguished according to the direction to which the vectors $X'(-1)$ and $X'(1)$ point. For each non-empty class fix a representative, so there are at most four of them, and let B be a common upper bound of their lengths.

Without loss of generality assume that each γ_m, as defined earlier, intersects $\mathcal{S}_1$ and $\mathcal{S}_2$ transversely[(9)]. Let also $\{\gamma_m(\overline{s}^1_{1m}), \ldots, \gamma_m(\overline{s}^1_{l_1 m})\}$ and $\{\gamma_m(\overline{s}^2_{1m}), \ldots, \gamma_m(\overline{s}^2_{l_2 m})\}$ be the points of intersection of γ_m with $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively. For each m choose any two $\overline{s}^1_{i_1 m}$ and $\overline{s}^2_{i_2 m}$ consecutive, namely that the open interval

$$(\min\{\overline{s}^1_{i_1 m}, \overline{s}^2_{i_2 m}\}, \max\{\overline{s}^1_{i_1 m}, \overline{s}^2_{i_2 m}\}) \tag{5.0.10}$$

does not contain any of the elements $\{\overline{s}^1_{1m}, \ldots, \overline{s}^1_{l_1 m}; \overline{s}^2_{1m}, \ldots, \overline{s}^2_{l_2 m}\}$. Without loss of generality we assume that $\overline{s}^1_{i_1 m} < \overline{s}^2_{i_2 m}$ for all m.

To simplify notation let $\overline{s}^1_m := \overline{s}^1_{i_1 m}$ and $\overline{s}^2_m := \overline{s}^2_{i_2 m}$. The curves $X_m(\overline{s}) := \gamma_m(\overline{s})$, $\overline{s} \in [\overline{s}^1_m, \overline{s}^2_m]$, can be thought (after reparameterisation) as belonging to one of the four classes of curves X described above. For every m let then $\hat{X}_m$ be the representative, chosen earlier, of the class to which X_m belongs.

We compare now the length of γ_m with the length of a competitor curve, that we denote by $\hat{\gamma}_m$, and that is constructed out of $\hat{X}_m$ and γ_m itself. The construction of $\hat{\gamma}_m$ is better described in words. Starting from $\gamma_m(0)$ we move forward through γ_m, reach $\mathcal{S}_1$ at $\gamma_m(\overline{s}^1_m)$, and cross it slightly. From there we move through a curve very close to $\mathcal{S}_1$ and of length less than $2\text{diam}(\mathcal{S}_1)$ until reaching a point in $\hat{X}_m$. Then we move through $\hat{X}_m$ until a point right before $\mathcal{S}_2$. Finally we move through a curve very close to $\mathcal{S}_2$ and of length less than $2\text{diam}(\mathcal{S}_2)$ until reaching a point in γ_m right before $\gamma_m(\overline{s}^2_m)$, from which we move through γ_m until reaching $\gamma_m(L_m)$. Clearly γ_m has the same intersections numbers with $\mathcal{S}_1$ and $\mathcal{S}_2$ as γ_m has, hence non-zero. Thus, by the definition of γ_m we have,

$$L(\gamma_m) \leq L(\hat{\gamma}_m) \tag{5.0.11}$$

But we have

$$L(\gamma_m) = \overline{s}^1_m + (\overline{s}^2_m - \overline{s}^1_m) + (L_m - \overline{s}^2_m) \tag{5.0.12}$$

and (if the construction of $\hat{\gamma}_m$ is fine enough)

$$L(\hat{\gamma}_m) \leq \overline{s}^1_m + 2\text{diam}(\mathcal{S}_1) + L(\hat{X}_m) + 2\text{diam}(\mathcal{S}_2) + (L_m - \overline{s}^2_m) \tag{5.0.13}$$

Hence, as $L(\hat{X}_m) \leq B$ we conclude that

$$\overline{s}^2_m - \overline{s}^1_m \leq B + 2\text{diam}(\mathcal{S}_1) + 2\text{diam}(\mathcal{S}_2) \tag{5.0.14}$$

That is, $|\overline{s}^2_m - \overline{s}^1_m|$ is uniformly bounded as wished. □

Let us introduce the setup required for the next Proposition 5.0.4 and for the proof of Theorem 5.0.1. Choose $\Sigma_i, i = 1, \ldots, i_{\Sigma} \geq 1$ a set of non-compact and *connected* regions of Σ°, with compact (and smooth) boundaries, each containing only one end, and the union covering Σ except for a *connected* set of compact closure, (i.e. $\Sigma \setminus (\cup \Sigma^{\circ}_i)$ is compact and connected). For each end Σ_i we consider an end cut $\{\mathcal{S}_{ijk}, j \geq 0, k = 1, \ldots, k_{ij}\}$.

(9) Otherwise use suitable small deformations

The surfaces $\mathcal{S}_{ijk}$ are considered only to serve as a 'reference'. Their geometry plays no role. The condition that the union of the ends Σ_i covers Σ except for a connected set of compact closure will be technically relevant in the proof below. It ensures that given any two $\mathcal{S}_{ijk}$ and $\mathcal{S}_{i'j'k'}$ with either: $i \neq i'$ (j, k, j', k' any), or $i = i'$, $j = j'$ (k, k' any), one can always find an inmersed curve $W : \mathbb{R} \to \Sigma$ intersecting $\mathcal{S}_{ijk}$ and $\mathcal{S}_{i'j'k'}$ only once and such that $W(t)$ diverges as $t \to \pm\infty$. This fact follows directly from the definition of end cut.

Proposition 5.0.4. *(setup above) Let $(\Sigma; g, N)$ be a static black hole data set. Then,*

1. *If $i_\Sigma > 1$, then for any $\mathcal{S}_{ijk}$ and $\mathcal{S}_{i'j'k'}$, with $i \neq i'$, there are points $p \in \mathcal{S}_{ijk}$ and $p' \in \mathcal{S}_{i'j'k'}$ such that $N(p) = N(p')$.*

2. *If $i_\Sigma = 1$, then for every j with $k_{1j} > 1$ and $1 \leq k \neq k' \leq k_{1j}$, there are points $p \in \mathcal{S}_{1jk}$ and $p' \in \mathcal{S}_{1jk'}$ such that $N(p) = N(p')$.*

Proof. If $i_\Sigma > 1$ then we can easily construct an embedding $W : \mathbb{R} \to \Sigma^\circ$ intersecting the manifolds $\mathcal{S}_{ijk}$ and $\mathcal{S}_{i'j'k'}$ only once and with $W(t) \to \infty$ as $t \to \pm\infty$. The existence of $p \in \mathcal{S}_{ijk}$ and $p' \in \mathcal{S}_{i'j'k'}$ for which $N(p) = N(p')$ then follows from Proposition 5.0.3. The case $i_\Sigma = 1$ is treated in exactly the same way. $\square$

We are ready to prove Theorem 5.0.1.

Proof of Theorem 5.0.1. We use the same stup as in Proposition 5.0.4. Also we let $\mathcal{S}_{ij} := \cup_{k=1}^{k=k_{ij}} \mathcal{S}_{ijk}$ and given $j' > j$, $\mathcal{U}_{i;jj'}$ denotes the closed region enclosed by $\mathcal{S}_{ij}$ and $\mathcal{S}_{ij'}$. Also, given a closed set C, we let $\min\{N; C\} := \min\{N(x) : x \in C\}$ and similarly for $\max\{N; C\}$.

We want to show that N is bounded from below away from zero at every one of the ends Σ_i. We distinguish two cases: $i_\Sigma > 1$ and $i_\Sigma = 1$.

Case $i_\Sigma > 1$. Without loss of generality we prove this only for Σ_1. Let us fix a surface $\mathcal{S}_{2j_0k_0}$ in Σ_2. By Proposition 5.0.4 we know that at every $\mathcal{S}_{1jk}$ we have

$$0 < \min\{N; \mathcal{S}_{2j_0k_0}\} \leq \max\{N; \mathcal{S}_{1jk}\} \tag{5.0.15}$$

On the other hand the Harnak estimate (4.4.8) in Proposition 4.4.4 gives us

$$\max\{N; \mathcal{S}_{1jk}\} \leq \eta' \min\{N; \mathcal{S}_{1jk}\} \tag{5.0.16}$$

where η' is independent of j and k. Combined with (5.0.15) this gives us the bound

$$0 < \eta'' < \min\{N; \mathcal{S}_{1jk}\} \tag{5.0.17}$$

where η'' is independent of j and k. Now, recall that the manifolds $\mathcal{U}_{1;j,j+1}, j = 0, 1, \ldots$ cover Σ_1 up to a set of compact closure and that for each j, $\partial\mathcal{U}_{1;j,j+1}$ is the union of the surfaces $\mathcal{S}_{1jk}; k = 1, \ldots, k_{1j}$ and $\mathcal{S}_{1,j+1,k}; k = 1, \ldots, k_{1,j+1}$. Therefore by (5.0.17) and the maximum principle we deduce,

$$0 < \eta'' < \min\{N; \partial\mathcal{U}_{1;j,j+1}\} \leq \min\{N; \mathcal{U}_{1;j,j+1}\} \tag{5.0.18}$$

from which the lower bound for N away from zero over Σ_1 follows.

Case $i_\Sigma = 1$. We observe first that, as in this case Σ_1 is the only end and as $N = 0$ on $\partial\Sigma$, then N cannot go uniformly to zero at infinity (this would violate the maximum principle). We prove now that, if there is a diverging sequence p_l such that $N(p_l) \to 0$, then N must go to zero uniformly at infinity. The proof will then be finished.

As $i_\Sigma = 1$ we will remove the index $i = 1$ everywhere from now on. For every l let j_l be such that $p_l \in \mathcal{U}_{j_l,j_l+1}$ and let $\mathcal{U}^c_{j_l,j_l+1}$ be the connected component of $\mathcal{U}_{j_l,j_l+1}$ containing p_l. By the maximum principle we have

$$\min\{N; \partial\mathcal{U}^c_{j_l,j_l+1}\} \leq \min\{N; \mathcal{U}^c_{j_l,j_l+1}\} \leq N(p_l) \tag{5.0.19}$$

Therefore we can extract a sequence of connected components of $\partial\mathcal{U}^c_{j_l,j_l+1}$, denoted by $\mathcal{S}_{j^l k_l}$ (j^l is either j_l or $j_l + 1$), such that

$$\min\{N; \mathcal{S}_{j^l k_l}\} \to 0 \tag{5.0.20}$$

From this and (5.0.16) we obtain

$$\max\{N; \mathcal{S}_{j^l k_l}\} \to 0 \tag{5.0.21}$$

Then, by Proposition 5.0.4 we have

$$\min\{N; \mathcal{S}_{j^l k}\} \leq \max\{N; \mathcal{S}_{j^l k_l}\} \tag{5.0.22}$$

(note the difference in the subindexes k and k_l) for all $k = 1, \ldots, k_{j^l}$ (it could be of course $k_{j^l} = 1$). Using (5.0.16) in the left hand side of (5.0.22) and using (5.0.21) we get

$$\max\{N; \mathcal{S}_{j^l}\} \to 0 \tag{5.0.23}$$

By the maximum principle again we deduce for any $l' > l$ the inequality

$$\max\{N; \mathcal{U}_{j^l j^{l'}}\} \leq \max\{\max\{N; \mathcal{S}_{j^l}\}; \max\{N; \mathcal{S}_{j^{l'}}\}\} \tag{5.0.24}$$

Taking the limit $l' \to \infty$ we deduce that the supremum of N over the unbounded connected component of $\Sigma \setminus \mathcal{S}_{j^l}$ is less or equal than the maximum of N over $\mathcal{S}_{j^l}$. Hence N must tend uniformly to zero at infinity because of (5.0.23). $\square$

6 Free S^1- symmetric solutions

6.1 The reduced data and the reduced equations

Let $(\Sigma; g, N)$ be a static data set invariant under a free S^1-action. The action induces a foliation of Σ by S^1-invariant circles. Let $(\Sigma; \mathfrak{g}, U)$ be the harmonic presentation. We will quotient the data $(\Sigma; \mathfrak{g}, U)$ by the Killing field and study the reduced system.

The complete list of reduced variables and other necessary notation, is the following.

- As usual let $\mathfrak{g} = N^2 g$,
- let ξ be the Killing field generating the S^1-action.
- let $\Lambda = |\xi|_\mathfrak{g}$ be the $\mathfrak{g}$-norm of ξ,
- let $\Omega = \epsilon^\mathfrak{g}_{abc}\xi^a\nabla^b\xi^c$ be the $\mathfrak{g}$-twist of ξ ($\epsilon^\mathfrak{g}$ is the $\mathfrak{g}$-volume form and ∇ any cov. der.),
- let $U = \ln N$,

- let $V = \ln \Lambda$,
- let S be the quotient manifold of Σ by the S^1-action,
- let q be the quotient two-metric of $\mathfrak{g}$,
- let κ be the Gaussian curvature of q.

With all this at hand the following is the definition of a reduced static data set.

Definition 6.1.1. *A data set $(S; q, U, V)$ arising from reducing a S^1-invariant static data set is a reduced static data set.*

The next proposition presents the reduced equations of a reduced data set(10). The equations involve only q, U and V, therefore the tensor Ric and the operators, Δ, ∇ and $\langle\ ,\ \rangle$ are with respect to q.

Proposition 6.1.2. *The reduced static equations of a reduced data set $(S; q, U, V)$ are,*

$$Ric = \nabla\nabla V + \nabla V \nabla V + \frac{1}{2}\Omega^2 e^{-4V} q + 2\nabla U \nabla U, \tag{6.1.1}$$

$$\Delta V + \langle \nabla V, \nabla V \rangle = \frac{1}{2}\Omega^2 e^{-4V}, \tag{6.1.2}$$

$$\Delta U + \langle \nabla U, \nabla V \rangle = 0. \tag{6.1.3}$$

where Ω (introduced earlier) is constant. Moreover Ω is zero iff ξ is hypersurface orthogonal inside Σ.

Before passing to the proof let us make some comments on the reduced equations.

- When $\Omega = 0$ the system (6.1.1)-(6.1.2) is locally equivalent to the Weyl equations around any point where $\nabla\Lambda \neq 0$. We won't use this information however in the rest of the article.

- The solutions to (6.1.1)-(6.1.3) are invariant under the simultaneous transformations

$$q \to \lambda^2 q, \quad V \to V + \frac{1}{2}\ln\nu, \quad U \to U + \mu, \quad \Omega \to \frac{\nu}{\lambda}\Omega \tag{6.1.4}$$

for any $\lambda > 0, \nu > 0$ and μ. Namely, if we replace (q, V, U) and Ω in (6.1.1)-(6.1.3) for $(\lambda^2 q, V + \frac{1}{2}\ln\nu, U + \mu)$ and $\nu\Omega/\lambda$ respectively, then the equations are still verified. We will call them simply "scalings" and denote them by (λ, ν, μ).

- Given a solution to (6.1.1)-(6.1.2), the metric $\mathfrak{g}$ can be recovered using the expression

$$\mathfrak{g} = h_{ab}dx^a dx^b + \Lambda^2(d\varphi + \theta_i dx^i)^2 \tag{6.1.5}$$

where (x^1, x^2) are coordinates on S and where the one form θ is found by solving

$$d(\theta_i dx^i) = \frac{\Omega}{\Lambda^3}\sqrt{|q|}dx^1 \wedge dx^2 \tag{6.1.6}$$

where $|q|$ is the determinant of q_{ij} and where $\partial_\varphi = \xi$ is the original Killing field. As ξ is the generator of a S^1-action, the range of φ is $[0, 2\pi)$. Without this information the range of φ is undetermined. This is related to the fact that, locally, the reduction procedure requires only that ξ is a non-zero Killing field. If the orbits of ξ do not close

(10) We haven't found a reference for these equations though most likely they are given somewhere

up in parametric time 2π, still the reduced equations (6.1.1)-(6.1.3) hold, and to recover $\mathfrak{g}$ using (6.1.5) and (6.1.6) the right range of φ needs to be provided.

This indeterminacy gives rise to two globally inequivalent ways to scale data $(\Sigma; \mathfrak{g}, U; \xi)$ giving rise to the same reduced variables and equations. We assume that $\xi \neq 0$ and has closed orbits. The first is the scaling,

$$\mathfrak{g} \to \lambda^2 \mathfrak{g}, \qquad \xi \to \frac{\sqrt{\nu}}{\lambda}\xi \tag{6.1.7}$$

the second is (recall $\mathfrak{g} = q_{ij}dx^i dx^j + \Lambda^2(d\varphi + \theta_i dx^i)^2$),

$$\mathfrak{g} \to \lambda^2 q_{ij}dx^i dx^j + \nu\Lambda^2(d\varphi + \frac{\lambda}{\nu^{1/2}}\theta_i dx^i)^2, \quad \xi \to \xi \tag{6.1.8}$$

In either case, the reduced variables (q, U, V) scale in the same way (6.1.4). The two new three-metrics are locally isometric but the new length of the orbits of the killing field ξ do not necessarily coincide. The length of the orbits is scaled by λ in the first case, and by $\sqrt{\nu}$ in the second case.

- As in dimension two we have $Ric = \kappa q$, then (6.1.1)-(6.1.2) imply that the Gaussian curvatures acquires the expression

$$\kappa = \frac{3}{4}\Omega^2 e^{-4V} + |\nabla U|^2. \tag{6.1.9}$$

In particular κ is non-negative. This will be an important property when analysing the geometry of the reduced data.

The proof of of Proposition 6.1.2 is just computational and relies on formulae in [7]. We include it for the sake of completeness, but it can be skipped otherwise.

Proof of Proposition 6.1.2. We use calculations from [7], but the notation is different. Precisely we use the following notation: $\mathcal{N}$ is the quotient of the spacetime manifold $\mathcal{M}$ by the S^1-action, ω_a is the twist one form of the Killing field ξ in the spacetime and λ its norm. Naturally, we have the commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\quad i_S \quad} & \mathcal{N} \\ \pi \uparrow & & \uparrow \pi \\ \Sigma & \xrightarrow{\quad i_\Sigma \quad} & \mathcal{M} \end{array}$$

where the π's are the projections into the quotient spaces and the inclusions i_Σ and i_S are totally geodesic, namely the second fundamental form K of Σ in $\mathcal{M}$ and the second fundamental form χ of S in $\mathcal{N}$, are both zero. Let $\boldsymbol{n}$ be the normal to S in $\mathcal{N}$.

Equation (45) from [7] implies $\boldsymbol{n}(\lambda) = 0$ and $i_S^*\omega_a = 0$. Using this information inside (18) of [7] we obtain,

$$\tilde{\boldsymbol{\nabla}}_a \tilde{\boldsymbol{\nabla}}^a \lambda = \frac{\omega(\boldsymbol{n})^2}{2\lambda^3} \tag{6.1.10}$$

where $\tilde{\boldsymbol{\nabla}}_a$ is the covariant derivative of the quotient metric on $\mathcal{N}$. We compute

$$\tilde{\boldsymbol{\nabla}}_a \tilde{\boldsymbol{\nabla}}^a \lambda = -\boldsymbol{n}^a \boldsymbol{n}^b \tilde{\boldsymbol{\nabla}}_a \tilde{\boldsymbol{\nabla}}_b \lambda + \Delta\lambda = \langle \frac{\nabla N}{N}, \nabla\lambda \rangle + \Delta\lambda \tag{6.1.11}$$

where now Δ and $\langle\ ,\ \rangle$ are defined with respect to the quotient two-metric over S that we denote by h. Thus

$$\Delta\lambda + \langle \frac{\nabla N}{N}, \nabla\lambda\rangle = \frac{\omega(\boldsymbol{n})^2}{2\lambda^3} \tag{6.1.12}$$

On the other hand as N is harmonic in (Σ, g) we have

$$\Delta N + \langle \nabla N, \frac{\nabla\lambda}{\lambda}\rangle = 0 \tag{6.1.13}$$

where the operators are again with respect to h. Finally, the equations (26) and (30) in [7] give

$$\kappa_h = \frac{\Delta\lambda}{\lambda} + \frac{1}{4}\frac{\omega(\boldsymbol{n})^2}{\lambda^4} \tag{6.1.14}$$

where κ_h is the gaussian curvature of h. Now, $q = N^2 h$, hence

$$N^2\kappa = \kappa_h - \Delta \ln N = \hat{\kappa} - \frac{\Delta N}{N} + \frac{|\nabla N|^2}{N^2} \tag{6.1.15}$$

where again Δ and $|\ \ |$ are with respect to h. Combining (6.3.14), (6.1.13) and (6.1.15) we obtain

$$\kappa = \frac{3}{4}\frac{\omega(\boldsymbol{n})^2}{N^2\lambda^4} + \frac{|\nabla N|^2}{N^4} \tag{6.1.16}$$

Now, the spacetime expression

$$\partial_t^a \boldsymbol{\epsilon}_{abcd}\xi^b \boldsymbol{\nabla}^c \xi^d = N\omega(\boldsymbol{n}) \tag{6.1.17}$$

is well known to be constant where $\boldsymbol{\nabla}$ is the spacetime covariant derivative and $\boldsymbol{\epsilon}$ the spacetime volume form (see [32] Theorem 7.1.1). On the other hand

$$\Omega = N\epsilon_{abc}\xi^a\nabla^b\xi^c = \partial_t^a \boldsymbol{\epsilon}_{abcd}\xi^b \boldsymbol{\nabla}^c \xi^d \tag{6.1.18}$$

where ϵ^g_{abc} is the g-volume form. Expressing (6.1.12), (6.1.13), (6.1.16) and (6.1.18) in terms of U, V, and expressing the Laplacians and norms in terms of q we obtain (6.1.2)-(6.1.3). To obtain (6.1.1) use

$$\kappa_h h_{ab} = \frac{\nabla_a\nabla_b\lambda}{\lambda} + \frac{\omega(\boldsymbol{n})^2}{2\lambda^4}h_{ab} + \frac{\nabla_a\nabla_b N}{N} \tag{6.1.19}$$

taken from eqs. (20) and (25) in [7], and re-express it in terms of q_{ab} and its covariant derivative. □

6.2 Example: the reduced Kasner

The most simple examples of reduced static data sets come from reducing the Kasner solutions through suitable Killing fields. Below we describe the reduced Kasner in detail.

Recall that the Kasner data sets (in the harmonic representation) are

$$\mathfrak{g} = dx^2 + x^{2a}dy^2 + x^{2b}dz^2, \qquad U = U_1 + c\ln x \tag{6.2.1}$$

where c, a and b satisfy $c^2 + (a - \frac{1}{2})^2 = \frac{1}{4}$ and $a + b = 1$. If we reduce these metrics through the Killing field $\xi = \lambda\partial_z$ we obtain the reduced data (q, U, V),

$$q = dx^2 + x^{2a} d\varphi^2, \tag{6.2.2}$$

$$U = U_1 + c\ln x, \tag{6.2.3}$$

$$V = V_1 + b\ln x \tag{6.2.4}$$

where of course

$$c^2 + (a - \frac{1}{2})^2 = \frac{1}{4}, \qquad a + b = 1. \tag{6.2.5}$$

and also

$$\Omega = 0 \tag{6.2.6}$$

Above we made $V_1 = \ln\lambda$, (note that $V_1 = V(1)$ and that $U_1 = U(1)$). If we make this solution periodic along φ and vary a, (hence b and c) and λ we obtain all the possible reduced solutions with $\Omega = 0$ and with a S^1-symmetry (in φ).

More general than this we can quotient the Kasner solutions by the Killing field

$$\xi = \lambda(\cos\omega\ \partial_y + \sin\omega\ \partial_z) \tag{6.2.7}$$

for any $\lambda > 0$ and $\omega \in [0, 2\pi)$, (fixed). A direct calculation shows that the reduced data set (q, U, V) is

$$q = dx^2 + \left[\frac{x^2}{x^{2a}\cos^2\omega + x^{2b}\sin^2\omega}\right] d\varphi^2, \tag{6.2.8}$$

$$U = U_1 + c\ln x, \tag{6.2.9}$$

$$V = V_1 + \frac{1}{2}\ln(x^{2a}\cos^2\omega + x^{2b}\sin^2\omega), \tag{6.2.10}$$

where of course

$$c^2 + (a - \frac{1}{2})^2 = \frac{1}{4}, \qquad a + b = 1. \tag{6.2.11}$$

and furthermore

$$\Omega^2 = 4e^{4V_1}(a - b)^2\cos^2\omega\sin^2\omega \tag{6.2.12}$$

Above we made $e^{V_1} = \lambda$, (note that $V_1 = V(1)$ and that $U_1 = U(1)$). If we make this solution periodic along φ and vary a, (hence b and c) and λ and ω, we obtain all the possible reduced solutions with a $\Omega \neq 0$ and with a S^1-symmetry (in φ).

A simple computation shows that as long as $\Omega \neq 0$ the norm Λ of the Killing field ξ grows at least as fast as the square root of the distance to the boundary of the data set. More precisely we have

$$\Lambda^2 \geq \eta|\Omega|x \tag{6.2.13}$$

where η does not depend on the data set. As we will see later this is indeed a general property for the asymptotic of any reduced data set.

6.3 A subclass of the reduced Kasner: the cigars

When either $(a, b) = (1, 0)$ or $(a, b) = (0, 1)$ and $\omega \notin \{0, \pi/2, \pi, 3\pi/2\}$ we obtain an important class of solutions that we will call the *cigars* (motivated by their shape, see Figure 7). Their metrics are complete in $\mathbb{R}^2$. After a convenient change of variables, the cigars are given by

$$U = U_0, \quad V = V_0 + \frac{1}{2}\ln(1+r^2) \quad \text{and} \quad q = 4\Omega^{-2}e^{4V_0}\big(dr^2 + \frac{r^2}{1+r^2}d\varphi^2\big) \qquad (6.3.1)$$

where U_0 and V_0 are arbitrary constants and where r is the radial coordinate from the origin and φ is the polar angle ranging in $[0, 2\pi)$, (note that $V_0 = V(r=0)$). The asymptotic metric is $q = 4\Omega^{-2}e^{4V_0}(dr^2 + d\varphi^2)$, hence cylindrical of section equal to $4\pi\Omega^{-1}e^{2V_0}$.

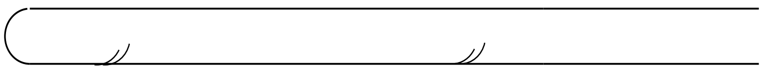

Figure 7: Representation of the cigar.

As U is constant, then the lapse N is also constant and the original static solution, (from where the data (6.3.1) is coming from), is flat. Let us explain now which quotient of $\mathbb{R}^3$ gives rise to the cigars. For any positive δ we let T_δ be the translation in $\mathbb{R}^3$ of magnitude δ along the z-axis and for any φ we let R_φ be the rotation in $\mathbb{R}^3$ of angle φ around the z-axis. Consider the isometric $\mathbb{R}$-action I on $\mathbb{R}^3$ given by

$$I : (t) \times (x, y, z) \longrightarrow T_{te^{V_0}}\big(R_{t\Omega(e^{-V_0})/2}(x, y, z)\big) \qquad (6.3.2)$$

Now, we quotient $\mathbb{R}^3$ as follows: two points (x, y, z) and (x', y', z') are identified iff $(x', y', z') = I(2\pi n, (x, y, z))$ for some $n \in \mathbb{Z}$. The quotient is free S^1-symmetric where the action is by restricting I to $[0, 2\pi)$. A straight forward calculation shows that the quotient data (q, U, V) is the cigar solution.

6.3.1 The cigars's uniqueness

The cigars (6.3.1) are the only complete non-compact boundaryless solutions to (6.1.1)-(6.1.3) with $\Omega \neq 0$. To see this observe that any complete non-compact solution must have U constant because U satisfies

$$|\nabla U|(p) \leq \frac{\eta}{d(p, \partial S)} \qquad (6.3.3)$$

and if S is complete and non-compact then $d(p, \partial S) = \infty$ and U is constant (this decay is direct from Anderson's estimate; We will make another proof of it in Proposition 6.4.1). Thus, as before, the original static $(\Sigma; g, N)$ solution is flat (and a S^1-bundle). It is not difficult to see that the only possibility must be a quotient of $\mathbb{R}^3$ as described above. However in Proposition 6.3.2 we give an alternative proof whose technique will be useful later when we present the cigar as the singularity model. Before and for the sake of completeness we prove that the only complete (reduced) data set with $\Omega = 0$ is M or a quotient thereof.

Proposition 6.3.1. *The only complete boundaryless (reduced) static data with $\Omega = 0$ is M or a quotient thereof.*

Proof. As $U = U_0$ and $\Omega = 0$ then $\nabla\nabla\Lambda = 0$ (eq. (6.1.1)). This implies that Λ is linear along geodesics. Thus, as the space is complete and $\Lambda > 0$ then Λ must be constant and q flat. The result follows. □

Proposition 6.3.2. *The only complete boundaryless (reduced) static data with $\Omega \neq 0$ are the cigars.*

Proof. The estimate (6.3.3) shows that U must be constant, i.e. $U = U_0$. Hence, making $\overline{\Lambda} = \sqrt{2/\Omega}\ \Lambda$ we have

$$\nabla\nabla\overline{\Lambda} = \frac{1}{\overline{\Lambda}^3} q, \qquad \kappa = \frac{3}{\overline{\Lambda}^4} \tag{6.3.4}$$

The first is an equation of Killing type and can be integrated easily along geodesics. If $\gamma(s)$ is a geodesic parametrised by arc-length then we have $\overline{\Lambda}'' = \overline{\Lambda}^{-4}$ (make $\overline{\Lambda}(\gamma(s)) = \overline{\Lambda}(s)$) which has the solutions

$$\overline{\Lambda}^2(s) = \frac{1}{(\overline{\Lambda}'^2_0 + 1/\overline{\Lambda}^2_0)} \big(1 + (\overline{\Lambda}_0 \overline{\Lambda}'_0 + (\overline{\Lambda}'^2_0 + 1/\overline{\Lambda}^2_0) s)^2\big) \tag{6.3.5}$$

where $\overline{\Lambda}_0 = \overline{\Lambda}(0)$ and $\overline{\Lambda}'_0 = \overline{\Lambda}'(0)$. We have thus the bound

$$\overline{\Lambda}^2(s) \geq \frac{1}{(|\nabla\overline{\Lambda}_0|^2 + 1/\overline{\Lambda}^2_0)} \tag{6.3.6}$$

where $|\nabla\overline{\Lambda}_0| = |\nabla\overline{\Lambda}|(0)$. This lower bound is achieved only at $s = \overline{\Lambda}_0|\nabla\overline{\Lambda}_0|/(|\nabla\overline{\Lambda}^2_0 + 1/\overline{\Lambda}^2_0)$ on the geodesic that points in the direction of least Λ'_0, i.e. when it is equal to $-|\nabla\overline{\Lambda}_0|$. Therefore at the point p where the minimum is achieved we have $\nabla\overline{\Lambda}(p) = 0$. Hence, along any geodesic $\gamma(s)$ emanating from p, (i.e. $\gamma(0) = p$), we have

$$\overline{\Lambda}^2 = \overline{\Lambda}^2_0 \left(1 + \frac{s^2}{\overline{\Lambda}^4_0}\right) \tag{6.3.7}$$

Thus, near p we can write

$$q = ds^2 + \ell^2 d\varphi^2 \tag{6.3.8}$$

with $\ell = \ell(s)$ satisfying

$$\ell'' = -\kappa\ell = -\frac{3}{\overline{\Lambda}^4}\ell \tag{6.3.9}$$

and with $\ell(0) = 0$ and $\ell'(0) = 1$. The solution is

$$\ell^2 = \frac{s^2}{\big(1 + s^2/\overline{\Lambda}^4_0\big)} \tag{6.3.10}$$

recovering (6.3.1) at least near p. It is simple to see that this q indeed represents the metric all over S which in turn must be diffeomorphic to $\mathbb{R}^2$. □

6.3.2 The cigars as models near high-curvature points

Lemma 6.3.3. *Let $(S_i; p_i; q_i, V_i, U_i)$ be a pointed sequence of (reduced) static data sets all having the same $\Omega \neq 0$. Suppose that*

$$d_{q_i}(p_i, \partial S_i) \geq d_0 > 0 \tag{6.3.11}$$

and that either

$$\kappa_{q_i}(p_i) \to \infty, \quad \text{or} \quad |\nabla V_i|_{q_i}(p_i) \to \infty \tag{6.3.12}$$

Then, there are scalings $(\hat{\lambda}_i, \hat{\nu}_i, \hat{\mu}_i)$ *such that the scaled sequence* $(S_i; p_i; \hat{q}_i, \hat{V}_i, \hat{U}_i)$ *converges in* C^∞ *and in the pointed sense to either a flat cylinder or a cigar with the same* Ω.

To simplify notation inside the proof, we will use the notation κ_i *for* κ_{q_i} *and* $|\nabla V_i|$ *for* $|\nabla V_i|_{q_i}$, *(the index "i" is from the sequence and of course does not represent a scaling).*

Proof. The proof is divided in various cases.

Case I. Suppose that $|\nabla V_i|(p_i)$ diverges but that $\kappa_i(p_i)$ remains uniformly bounded. To start on we make scalings $(\overline{\lambda}_i, \overline{\nu}_i, \overline{\mu}_i)$ where

$$\overline{\lambda}_i = |\nabla V_i|(p_i), \quad \overline{\nu}_i = e^{-2V_i(p_i)}, \quad \overline{\mu}_i = -U_i(p_i). \tag{6.3.13}$$

Let $(\overline{q}_i, \overline{V}_i, \overline{U}_i)$ be the scaled variables. Observe that Ω scales to $\overline{\Omega}_i = \overline{\nu}_i\Omega/\overline{\lambda}_i$. We have

$$\overline{\Lambda}_i(p_i) = 1, \quad |\nabla\overline{\Lambda}_i|(p_i) = 1, \tag{6.3.14}$$

where, recall, $\overline{\Lambda}_i = e^{\overline{V}_i}$. Consider now the three-dimensional static pointed data $(\Sigma_i; o_i; \overline{\mathfrak{g}}_i, \overline{U}_i)$ whose reductions are the $(S_i; p_i; \overline{q}_i, \overline{V}_i, \overline{U}_i)$. The o_i are points in Σ_i projecting into the p_i's. Let $\overline{\xi}_i$ be the scaling of ξ_i. In this context the relations (6.3.14) are

$$|\overline{\xi}_i|(o_i) = 1, \quad |\nabla|\overline{\xi}_i||(o_i) = 1, \tag{6.3.15}$$

where the norms are with respect to $\overline{\mathfrak{g}}_i$. Moreover, $\overline{\Omega}_i = \overline{\nu}_i\Omega/\overline{\lambda}_i \to 0$ because the $\overline{\nu}_i$ are bounded and the $\overline{\lambda}_i$ tend to infinity. Let us study now the convergence of the derivatives $(\overline{\nabla}\,\overline{\xi}_i)(o_i)$ of the Killings $\overline{\xi}_i$ at the points o_i. For notational simplicity we will remove for a moment the subindexes "i" (but we keep them in mind). For the calculation we consider $\overline{\mathfrak{g}}$-orthonormal basis $\{e_1, e_2, e_3\}$ around the points o, with $e_3(o) = \overline{\xi}(o)/|\overline{\xi}|(o)$ and $(\overline{\nabla}_{e_i} e_j)(o) = 0$. Then, using the relation $\overline{\Omega} = \overline{\epsilon}_{abc}\overline{\xi}^a\overline{\nabla}^b\overline{\xi}^c$ and the Killing condition $\overline{\nabla}_a\overline{\xi}_b + \overline{\nabla}_b\overline{\xi}_a = 0$, the components of $\overline{\nabla}\,\overline{\xi}$ are computed as,

$$\langle \nabla_{e_j}\overline{\xi}, e_j\rangle = 0, \tag{6.3.16}$$

$$\langle \nabla_{e_1}\overline{\xi}, e_2\rangle = -\langle \nabla_{e_2}\overline{\xi}, e_1\rangle = \frac{\overline{\Omega}}{|\overline{\xi}|}, \tag{6.3.17}$$

$$\langle \nabla_{e_3}\overline{\xi}, e_j\rangle = -\langle \nabla_{e_j}\overline{\xi}, e_3\rangle = -\nabla_{e_j}|\overline{\xi}|. \tag{6.3.18}$$

If furthermore $e_1(o)$ and $e_2(o)$ are chosen such that $\nabla_{e_1(o)}|\overline{\xi}| = 0$ and $\overline{\nabla}_{e_2(o)}|\overline{\xi}| = 1$ then, (restoring now the indexing "i"), the components $\langle\overline{\nabla}_{e_j}\overline{\xi}_i, e_k\rangle(o_i)$ are either zero or tend to zero as i goes to infinity except for $\langle\overline{\nabla}_{e_1}\overline{\xi}_i, e_3\rangle(o_i)$ and $\langle\overline{\nabla}_{e_3}\overline{\xi}_i, e_1\rangle(o_i)$ that are constant and equal to one and minus one respectively.

Now we observe that

$$d_{\overline{\mathfrak{g}}_i}(o_i, \partial\Sigma_i) = \overline{\lambda}_i d_{\mathfrak{g}_i}(o_i, \partial\Sigma_i) = \overline{\lambda}_i d_{q_i}(p_i, \partial S_i) \geq \overline{\lambda}_i d_0 \to \infty. \tag{6.3.19}$$

Therefore by Anderson's estimates, the curvature of the $\overline{\mathfrak{g}}_i$ over balls of centres o_i and any fixed radius tend to zero. Hence, there are neighbourhoods $\mathcal{B}_i$ of o_i and covers $\tilde{\mathcal{B}}_i$ such that the pointed sequence $(\tilde{\mathcal{B}}_i; \tilde{o}_i; \overline{\mathfrak{g}}_i)$ converges in C^∞ and in the pointed sense to the Euclidean three-space (for the cover metric we use also $\overline{\mathfrak{g}}_i$). We claim that the lift of the Killing fields $\overline{\xi}_i$ to the $\tilde{\mathcal{B}}_i$, (that we will denote too by $\overline{\xi}_i$) converge in C^∞

to the generator of a (non-trivial) rotation of $\mathbb{R}^3$. To see this recall first that for any Killing field χ it holds $\nabla_a\nabla_b\chi_c = -Rm_{bca}{}^{d}\chi_d$. Thus, at any point x we can find $\overline{\xi}_i(x)$ by integrating a second order linear ODE along a geodesic that extends from $\gamma(0) = \tilde{o}_i$ to x, given the initial data $\overline{\xi}_i(\gamma(0))$ and $\overline{\nabla}_{\gamma'(0)}\overline{\xi}_i$. As it was shown earlier that the data $\overline{\xi}_i(\tilde{o}_i)$ and $(\overline{\nabla}\overline{\xi}_i)(\tilde{o}_i)$ converges, hence so does $\overline{\xi}_i$ and the perpendicular distribution of the limit Killing field $\overline{\xi}_\infty$ is integrable because $\lim\overline{\Omega}_i = 0$. Thus, $\overline{\xi}_\infty$ generates a rotation in $\mathbb{R}^3$. As $|\overline{\xi}_\infty|(\tilde{o}_\infty) = 1$ and $|\nabla|\overline{\xi}_\infty||(\tilde{o}_\infty) = 1$ it must be that $\tilde{o}_\infty$ is at a distance one from the rotational axis. In coordinates (x, y, z) of $\mathbb{R}^3$ the limit vector field would, (for instance), $x\partial_y - y\partial_z$ and the limit point would be, (for instance), $(1, 0, 0)$.

This convergence of $\overline{\xi}_i$ to the generator of a rotation will be used in the following to extract a pair of relevant informations.

First we show that inside the surfaces S_i there are geodesic loops ℓ_i, based at the points p_i, whose $\overline{q}_i$ length tends to zero. Let us see this. For i large enough, the orbit of the Killing $\overline{\xi}_i$ inside $\tilde{\mathcal{B}}_i$, that starts at the point $\tilde{o}_i$, twists around an "axis" and come very close to close up into a circle when it approaches again the point $\tilde{o}_i$ (see Figure **??**). Hence, a small two-dimensional disc formed by short geodesic segments emanating perpendicularly to $\overline{\xi}_i(\tilde{o}_i)$ at $\tilde{o}_i$ must intersect the orbit at a nearby point $\tilde{o}'_i$. Moreover the geodesic segment joining $\tilde{o}_i$ and $\tilde{o}'_i$, projects into a geodesic loop ℓ_i on S_i based at p_i. The length of the loops ℓ_i clearly tend to zero as i goes to infinity.

Second, for $i \geq i_0$ large enough, the norm of the Killings $\overline{\xi}_i$ over the balls $B_{\overline{\mathfrak{g}}_i}(o_i, 1/2) \subset \tilde{\mathcal{B}}_i$ is bounded below by $1/4$. Hence, $\overline{\Lambda}_i$ is bounded below by $1/4$ over the balls $B_{\overline{q}_i}(p_i, 1/2)$ in S_i. More importantly the Gaussian curvature $\overline{\kappa}_i$ is bounded above by $100\overline{\Omega}_i^2$ also on $B_{\overline{q}_i}(p_i, 1/2)$.

From these two facts we conclude that the geometry near the points o_i is collapsing with bounded curvature. This implies that if we scale up $\overline{q}_i$ to have the injectivity radius at o_i equal to one, then the new scaled spaces converge in the pointed sense to a flat cylinder. The composition of this last scaling and the one we performed first is the scaling $(\hat{\lambda}_i, \hat{\nu}_i, \hat{\mu}_i)$ we were looking for.

Case II. Suppose now that both $|\nabla V_i|(p_i)$ and $\kappa_i(p_i)$ are diverging. If the quotient $\kappa_i(p_i)/|\nabla V_i|^2(p_i)$ tends to zero, then we can perform a scaling $(\overline{\lambda}_i, \overline{\nu}_i, \overline{\mu}_i)$ that leaves Ω invariant and that makes $\overline{\kappa}_i(p_i)$ bounded and $|\nabla\overline{V}_i|(p_i)$ diverging. We can then repeat the step in *Case I* with $(\overline{q}_i, \overline{V}_i, \overline{U}_i)$ instead of (q_i, V_i, U_i) to prove the Lemma in this case too.

Assume therefore that the quotient $\kappa_i(p_i)/|\nabla V_i|^2(p_i)$ remains bounded. Perform again a scaling $(\overline{\lambda}_i, \overline{\nu}_i, \overline{\mu}_i)$ that leaves Ω invariant and makes $\overline{\kappa}_i(p_i) = 1$ and therefore makes $|\nabla\overline{V}_i|(p_i)$ bounded because $\kappa_i(p_i)/|\nabla V_i|^2(p_i)$ is invariant. Note that as $d_{\overline{q}_i}(p_i, \partial S_i) \to \infty$, the estimate (2.2.16) impose that $|\nabla\overline{U}_i|$ must tend uniformly to zero over balls of centres p_i and fixed but arbitrary radius. We claim that the curvature $\overline{\kappa}_i$ remains uniformly bounded on balls of centres p_i and fixed radius. Let $L > 0$, let x be a point in $B_{\overline{q}_i}(p_i, L)$ and let $\gamma(s)$ be a length-minimising geodesic joining p_i to x. Let $\overline{\Lambda}_i(s) = \overline{\Lambda}_i(\gamma(s))$. Then, the value of $\overline{\Lambda}_i$ at x is found by solving the second order ODE

$$\overline{\Lambda}''_i = \frac{\Omega^2}{4\overline{\Lambda}^3} + (|\nabla U|^2 - 2U'^2)\overline{\Lambda} \tag{6.3.20}$$

subject to the initial data $\overline{\Lambda}_i(0) = \overline{\Lambda}_i(\gamma(0))$ and $\overline{\Lambda}'_i(0) = \nabla_{\gamma'(0)}\overline{\Lambda}_i$, and evaluating at $s = d_{\overline{q}_i}(x, p_i)$. If $\nabla\overline{U}_i$ were identically zero then the solutions would be exactly (6.3.5) and we would have the bound

$$\overline{\Lambda}_i^2(s) \geq \frac{1}{(\overline{\Lambda}'_i(0))^2 + 1/(\overline{\Lambda}_i(0))^2} \tag{6.3.21}$$

for all $s \geq 0$. In particular, if $\overline{\Lambda}_i(0)$ is bounded below by A and $|\overline{\Lambda}_i'(0)|$ is bounded above by B then $\overline{\Lambda}_i(s)$ is bounded below by $\sqrt{1/(B^2 + 1/A^2)}$. But as $|\nabla \overline{U}_i|$ tends to zero uniformly over balls or radius L, then the solutions to the ODE tend to (6.3.5) with initial data $\overline{\Lambda}_i(0)$ and $\overline{\Lambda}_i'(0)$. Now, as $\overline{\kappa}_i(p_i) = 1$ and $|\nabla \overline{V}_i|(p_i)$ is bounded, there are constants A and B such that

$$\overline{\Lambda}_i(0) \leq A, \quad \text{and} \quad |\overline{\Lambda}_i'(0)| \leq B \tag{6.3.22}$$

no matter which the geodesic γ is. Therefore if $i \geq i_0(L)$ is big enough then $\overline{\Lambda}_i(x) \leq 2\sqrt{1/(B^2 + 1/A^2)}$. Hence, $\overline{\kappa}_i \geq 3\overline{\Omega}_i(B^2 + 1/A^2)^2/32$ everywhere on $B_{\overline{q}_i}(p_i, L)$.

The bound we proved for the curvature implies that if for a certain subsequence the injectivity radius at the points p_i tends to zero then there are finite covers that converge to a cigar. But this is impossible because the cigars do not admit any non-trivial quotient. Hence the injectivity radius remains bounded away from zero and the pointed sequence $(S_i; p_i; \overline{q}_i, \overline{V}_i, \overline{U}_i)$ must sub-converge in the pointed sense to a solution with U constant. By uniqueness it is always a cigar and we are done. □

Let us make an extra observation about a construction made inside the proof. Recall that the spaces $(\tilde{\mathcal{B}}_i, \overline{\mathfrak{g}}_i)$ converge to $\mathbb{R}^3$ and the Killings $\overline{\xi}_i$ converge to the generator of a rotation. Let z_i be points where $(\nabla|\overline{\xi}_i|)(z_i) = 0$. These points one can think that lie in the "axis" of rotation. Naturally if we quotient the balls of centres z_i and radius two we obtain a two disc. This disc projects into a "cup" on S_i containing p_i (see Figure **??**). In the metric q_i, the "radius" of this cup (i.e. the maximum distance from a point to the boundary) goes to zero.

The Lemma 6.3.3 provides models for the scaled geometry near points of high curvature or high V-gradient, but it does not say how such points affect the unscaled geometry nearby. This is an important information that we will need later. In rough terms, what occurs is that at any finite distance from such a point the (unscaled) geometry becomes one dimensional, pretty much like a cigar highly scaled down. The next Lemma 6.3.4 explains the phenomenon. In few words it explains how the geometry looks like near geodesics that join points of high curvature or high V-gradient and the boundary of the surfaces S_i. This basic information will be sufficient to extract conclusions later.

The scaled geometry around points in such geodesics will be model essentially as regions of the cigar whose curvature at the origin is conventionally $\kappa_0 = 3(2\pi)^2$ and therefore whose metric is

$$q_0 = \frac{1}{(2\pi)^2}\big(dr^2 + \frac{r^2}{1+r^2}d\varphi^2\big) \tag{6.3.23}$$

where $r \geq 0$. Let us describe the models more explicitly. A pointed space $(\{0 \leq r \leq 40\}; x; q_0)$, where x be a point in this cigar with $r(x) \leq 25$, is a model of type Ci (from "cigar"). A pointed space $(\{r(x) - 10 \leq r \leq r(x) + 10\}; x; q_0)$, where x be a point with $r(x) > 25$, is a model of type Cy (from "cylinder"). The Figure **??** sketches these two types of models.

Lemma 6.3.4. *Let $(S_i; p_i; q_i, V_i, U_i)$ be a pointed sequence of (reduced) static data sets all having the same $\Omega \neq 0$ and suppose that*

$$d_{q_i}(p_i, \partial S_i) \geq d_0 > 0 \tag{6.3.24}$$

and that either

$$\kappa_{q_i}(p_i) \to \infty, \quad \text{or} \quad |\nabla V_i|_{q_i}(p_i) \to \infty. \tag{6.3.25}$$

For every i let γ_i be a geodesic segment joining p_i to ∂S_i and minimising the distance between them (if $\partial S_i = \emptyset$ let γ_i be an infinite ray). Fix a positive d_1 less than d_0.

Then, for every $k \geq 1$, $\epsilon > 0$ there exists i_0 such that for any $i \geq i_0$ and for any $x_i \in \gamma_i$ with $d_{q_i}(x_i, p_i) \leq d_1$ there exist a neighbourhood $\mathcal{B}_i$ of x_i and a scaled metric $\overline{q}_i = \overline{\lambda}_i^2 q_i$ such that $(\mathcal{B}_i; x_i; \overline{q}_i)$ is ϵ-close in C^k to either a model space Ci *or a model space* Cy.

Again to simplify notation inside the proof, we will use the notation κ_i for κ_{q_i} and $|\nabla V_i|$ for $|\nabla V_i|_{q_i}$.

Proof. Half of the work has been done essentially already in Lemma 6.3.3 because the geometry near points of high curvature or high V-gradient are model locally (at a right scale) by a space Ci or a space Cy. We say this formally as follows: given $\epsilon > 0$ and $k \geq 1$ there are $K_0 > 0$ and $i_1 > 0$ such that for any $i \geq i_1$ and $x_i \in \gamma_i$ such that $d_i(x_i, p_i) \leq d_1$ and either $\kappa_i(x_i) \geq K_0$ or $|\nabla V_i|(x_i) \geq K_0$, then the conclusions of the Lemma hold. Thus, it is left to show that the conclusions hold too for points on γ_i that do not have "high" curvature or high gradient, that is for which $\kappa_i(x_i) \leq K_0$ and $|\nabla V_i|(x_i) \leq K_0$. We prove that in what follows.

We will show that there is $i_2 \geq i_1$ such that for any $i \geq i_2$ and for any $x_i \in \gamma_i$ such that $d_i(x_i, p_i) \leq d_1$, $\kappa_i(x_i) \leq K_0$ and $|\nabla V_i|(x_i) \leq K_0$, the conclusion of the Lemma also holds and the local model is of type Cy.

Given i, let x_i be a point such that $x_i \in \gamma_i$ such that $d_i(x_i, p_i) \leq d_1$, $\kappa_i(x_i) \leq K_0$ and $|\nabla V_i|(x_i) \leq K_0$. We begin claiming that there are $r_0 < (d_0 - d_1)/2$ and $K_1 > 0$ independent of i such that $\kappa_i(x) \leq K_1$ for all $x \in B_{q_i}(x_i, r_0)$. Let r_0 be any number less than $(d_0 - d_1)/2$ and let x be a point such that $d(x, x_i) \leq r_0$. Let $\alpha_i(s)$ be a length minimising geodesic joining x_i to x ($\alpha_i(0) = x_i$). Denote $V_i(s) := V_i(\alpha_i(s))$. Let,

$$\hat{V}_i(s) = V_i(s) - V_i(0) \tag{6.3.26}$$

Then we have,

$$\hat{V}_i(0) = 0, \quad \text{and} \quad |\hat{V}_i'(0)| \leq K_0 \tag{6.3.27}$$

where the first equation is by the definition of $\hat{V}_i(0)$ and the second follows by assumption. On the other hand $\hat{V}_i(s)$ satisfies the differential equation (6.1.1), namely,

$$\hat{V}_i'' + \hat{V}_i'^2 = \big(\frac{1}{2}\Omega^2 e^{-4V_i(0)}\big) e^{-4\hat{V}_i} + (|\nabla U|^2 - 2U'^2) \tag{6.3.28}$$

where the last expression in parenthesis is evaluated of course on $\alpha_i(s)$.

Let us make two comments on this equation. First, the coefficient $\Omega^2 e^{-4V_i(0)}/2$ is less or equal than $\kappa_i(x_i)$ and thus less or equal than K_0 by assumption. Second, the sumand $(|\nabla U|^2 - 2U'^2)(s)$ is uniformly bounded, say by $K_2 > 0$, independently of s, x, x_i and i. This follows from the estimate (6.3.3) and $d_{q_i}(\alpha_i(s), \partial S_i) \geq (d_1 - d_0)/2$; This last inequality is due to,

$$d_{q_i}(\alpha_i(s), \partial S_i) \geq d_{q_i}(x_i, \partial S_i) - d_{q_i}(\alpha_i(s), x_i) \tag{6.3.29}$$

and the inequalities $d_{q_i}(x_i, \partial S_i) \geq (d_1 - d_0)$ and $d_{q_i}(\alpha_i(s), x_i) \leq d_{q_i}(x, x_i) \leq (d_1 - d_0)/2$.

Until now we have shown control on the ODE (6.3.28) and the initial data (6.3.27). Therefore by standard ODE analysis, it follows that one can chose r_0 small enough such that $|\hat{V}_i(s)| \leq K_1$, (i.e. preventing blow up), for a K_1 independent on s, x, x_i and i. This bound on $V_i(x)$ (we removed the hat now) and the bound on $|\nabla U|^2(x)$ gives the desired bound on $\kappa_i(x)$.

We have proved a curvature bound $\kappa_i(x) \leq K_1$ for all $x \in B_{q_i}(x_i, r_0)$. Using this bound we are going to show that the injectivity radius at x_i, namely $inj_{q_i}(x_i)$, tends to zero as i tends to infinity. Indeed, if on the contrary $inj_{q_i}(x_i) \geq r_1 > 0$ for some $r_1 > 0$, then because the curvature is bounded on $B_{q_i}(x_i, r_0)$, there is $v > 0$ and $r_2 \leq \min\{r_0, r_1\}/2$ such that the area of the ball $B_{q_i}(x_i, r_2)$ is greater or equal than v. As $B_{q_i}(x_i, r_2) \subset B_{q_i}(p_i, d_0))$ then we have

$$\frac{A_i(B_{q_i}(p_i, d_0))}{d_0^2} \geq \frac{v}{d_0^2} \tag{6.3.30}$$

On the other hand observe that by Lemma 6.3.3 the geometry near the points p_i is locally collapsing (at a right scale) to a line or to half a line. Thus, there is i_3 such that for $i \geq i_3$ there is $\delta_i \to 0$, such that the quotient

$$\frac{A_i(B_{q_i}(p_i, \delta_i))}{\delta_i^2} \tag{6.3.31}$$

is less or equal than $v/(2d_0^2)$ (in fact the quotient tends to zero). But by Bishop-Gromov the function

$$s \to \frac{A_i(B_{q_i}(p_i, s))}{s^2} \tag{6.3.32}$$

is monotonically decreasing and therefore we should have

$$\frac{v}{2d_0^2} \geq \frac{A_i(B_{q_i}(p_i, d_0))}{d_0^2} \geq \frac{v}{d_0^2} \tag{6.3.33}$$

which is impossible. Thus the injectivity radius at x_i tends to zero. Therefore the balls $B_{q_i}(x_i, r_0)$ collapse with bounded curvature and the existence of a scaling whose limit is a cylinder (Cy) is now direct. □

The Lemma 6.3.4 gives a local model for the collapsed geometry around points on the geodesics γ_i. The concatenation of the local models provide a global picture that is sumarized in the next corollary (whose proof is now direct), see Figure **??**.

Corollary 6.3.5. *Let $(S_i; p_i; q_i, V_i, U_i)$ be a pointed sequence of (reduced) static data sets all having the same $\Omega \neq 0$ and suppose that*

$$d_{q_i}(p_i, \partial S_i) \geq d_0 > 0 \tag{6.3.34}$$

and that either

$$\kappa_{q_i}(p_i) \to \infty, \quad \text{or} \quad |\nabla V_i|_{q_i}(p_i) \to \infty. \tag{6.3.35}$$

For every i let γ_i be a geodesic segment joining p_i to ∂S_i and minimising the distance between them (if $\partial S_i = \emptyset$ let γ_i be an infinite ray). Fix a positive d_1 less than d_0.

Then there is i_0 such that for any $i \geq i_0$ there is a neighbourhood $\mathcal{B}_i$ of the ball $B_{q_i}(p_i, d_1)$, diffeomorphic to a disc and metrically collapsing to a segment of length d_1 as i goes to infinity.

6.4 Decay of the fields at infinity and asymptotic topology

We know already that the gradient of U decays quadratically at infinity. In this section we show that also de gradient of V and the Gassian curvature κ decay quadratically. The proof depends on whether Ω is zero or not. The case $\Omega = 0$ is simple and relies only on the techniques a la Bakry-Émery used earlier. As a by product we re-prove the quadratic decay of the gradient of U, valid when $\Omega = 0$ or not. When $\Omega \neq 0$, the proof requires the use of Corollary 6.3.5.

6.4.1 Case $\Omega = 0$

Proposition 6.4.1. *There is a constant $\eta > 0$ such that for every (reduced) static data set we have*

$$|\nabla U|^2(p) \leq \frac{\eta}{d^2(p, \partial S)}. \tag{6.4.1}$$

Moreover when $\Omega = 0$ we have

$$|\nabla V|^2(p) \leq \frac{\eta}{d^2(p, \partial S)}, \tag{6.4.2}$$

hence also

$$\kappa(p) \leq \frac{\eta}{d^2(p, \partial S)}. \tag{6.4.3}$$

Proof. Write (6.1.1) as

$$Ric^\alpha_f = \frac{1}{2}\Omega^2 e^{-4V} q + 2\nabla U \nabla U \geq 0 \tag{6.4.4}$$

with $f = -V$, $\alpha = 1$, and recall from (6.1.3) that $\Delta_f U = 0$. Then, using (4.2.11) with $\psi = U$ we obtain

$$\Delta_f |\nabla U|^2 \geq 4|\nabla U|^4 \tag{6.4.5}$$

and hence (6.4.1) by Lemma 4.2.3.

Similarly, if $\Omega = 0$ we have $\Delta_f V = 0$ and using (4.2.11) again but with $\psi = V$ we obtain

$$\Delta_f |\nabla V|^2 \geq 2|\nabla V|^4 \tag{6.4.6}$$

and hence (6.4.2) by Lemma 4.2.3. □

The next proposition describes in simple form the asymptotic topology of data sets $(S; q, U, V)$ when $\Omega = 0$.

Proposition 6.4.2. *Let $(S; q, U, V)$ be a (reduced) static data set with $\Omega = 0$, S non-compact and ∂S compact. Then there is a set K with compact closure, such that*

$$S = K \cup \left(\cup_{i=1}^{i=n} E_i \right) \tag{6.4.7}$$

where every E_i is diffeomorphic to $[0,\infty)\times \mathrm{S}^1$.

Proof. Firs we observe that as $\kappa\geq 0$, the ball covering property holds (indeed regardless of whether $\Omega=0$ or not). Hence, $\mathcal{S}$ has a finite number of ends. In particular we can write $\mathcal{S}$ as the union of a set with compact closure and a finite number of surfaces E_i, $i=1,\ldots,i_S$, each with compact boundary and containing only one end.

It is sufficient to work with the surfaces E_i, that we denote generically as E. By Bishop-Gromov we have $\frac{A(B(\partial E,r))}{r^2}\searrow\mu$. The analysis depends on whether $\mu=0$ or $\mu>0$.

Case $\mu=0$. Let γ be a ray from ∂E and let $p_i\in\gamma$ with $r(p_i)=r_i=2^i$, for $i=0,1,2,\ldots$. If $\mu=0$, then the sequence of annuli $(\mathcal{A}^c_{r_i}(p_i,1/4,4);q_{r_i})$ collapses in volume (in area) with bounded curvature. As we have explained earlier, this type of collapse is only through thin (finite) cylinders. Thus, (outside a compact set) E is formed by an infinite concatenation of finite cylinders, (i.e. each diffeomorphic to $[0,1]\times\mathrm{S}^1$).

Case $\mu>0$. As $\kappa\geq 0$ and κ has quadratic decay, if $\mu>0$ then $(E;q)$ is asymptotic to a flat cone $(\mathcal{C};q_\mu)$ where

$$\mathcal{C}:=\mathbb{R}^2\setminus\{(0,0)\},\qquad q_\mu=dr^2+4\mu^2r^2d\varphi^2 \tag{6.4.8}$$

(r is the radius and φ is the polar angle in $\mathbb{R}^2$). It then follows that, outside a compact set of compact closure, E is diffeomorphic to $[0,\infty)\times\mathrm{S}^1$ as wished. □

6.4.2 *Case* $\Omega\neq 0$

Lemma 6.4.3. *Let $(S;q,U,V)$ be a (reduced) static data set with $\Omega\neq 0$, S non-compact and ∂S compact. Then,*

$$|\nabla U|^2(p)\leq\frac{\eta}{d^2(p,\partial S)},\qquad |\nabla V|^2(p)\leq\frac{\eta}{d^2(p,\partial S)}, \tag{6.4.9}$$

and,

$$\kappa(p)\leq\frac{\eta}{d^2(p,\partial S)} \tag{6.4.10}$$

where $\eta>0$ is independent on the data. In particular

$$\Lambda^2(p)\geq\eta'\Omega\, d(p,\partial S) \tag{6.4.11}$$

where $\eta'>0$ is also independent on the data.

Proof. The proof requires using Corollary 6.3.5. Without loss of generality assume that S is an end. Let γ be a ray from ∂S. For every $j\geq 0$ let $r_j=2^{2j}$ and let $p_j\in\gamma$ be such that $d(p_j,\partial S)=r_j$.

The first goal will be to prove that κ and $|\nabla V|^2$ decay quadratically along the union of annuli $\cup_{j\geq 0}\mathcal{A}^c_{r_j}(p_j;1/8,8)$. This union covers γ except for a finite segment of it but a priori may not cover the whole end. This follows after proving the quadratic curvature decay.

Let x_{j_i} be any sequence of points such that $x_{j_i}\in\mathcal{A}^c_{r_{j_i}}(p_{j_i};1/8,8)$ for every $i\geq 0$. Each x_{j_i} can be joined to p_{j_i} through a continuous curve α_i entirely inside the annulus $\mathcal{A}^c_{r_{j_i}}(p_{j_i};1/8,8)$. Concatenating α_i with the part of γ extending from p_{j_i} to infinity, we obtain a curve, say $\hat{\alpha}_i$, extending from x_{j_i} to infinity, and never entering the ball

$B(\partial S, 1/8)$, namely, keeping at a q_{r_j}-distance of $1/8$ from ∂S. We will use the existence of this curve below to reach a contradiction.

Suppose now that either,

$$\kappa(x_{j_i})d^2(x_{j_i}, \partial S) \to \infty, \quad \text{or} \quad |\nabla V|^2(x_{j_i})d^2(x_{j_i}, \partial S) \to \infty \tag{6.4.12}$$

We perform a sequence of scalings $(\lambda_i, \nu_i, \mu_i) = (r_{j_i}, r_{j_i}, 0)$ leading to the new fields,

$$q \to q_i = \frac{1}{r_{j_i}^2} q, \quad V \to V_i = V + \frac{1}{2}\ln r_{j_i}, \quad U \to U_i = U \tag{6.4.13}$$

With this scaling we obtain then a sequence of reduced data $(S; q_i, V_i, U_i)$ all having the same Ω (recall $\Omega \to \Omega_i = (\nu_i/\lambda_i)\Omega = \Omega$). At the same time we have $1/8 \leq d_i(x_{j_i}, \partial S) \leq 8$. Because of this, we can rewrite (6.4.12) as,

$$\kappa_i(x_{j_i}) \to \infty, \quad \text{or} \quad |\nabla V_i|_i^2(x_{j_i}) \to \infty, \tag{6.4.14}$$

(where $\kappa_i = \kappa_{q_i}$ and $|\nabla V_i| = |\nabla V_i|_{q_i}$). Taking a subsequence if necessary we can assume that $d_i(x_{j_i}, \partial S) \to d_*$ (where $d_i = d_{q_i}$).

We are clearly in the hypothesis of Corollary 6.3.5. Choosing d_1 (see the hypothesis of Corollary 6.3.5) as $d_1 = d_* + (d_* - 1/8)/2$, we conclude that there is a sequence of neighbourhoods $\mathcal{B}_i$ containing $B_i(\partial S, d_1)$ such that $(\mathcal{B}_i; q_i)$ metrically collapses to a segment of length d_1 (where $B_i = B_{q_i}$). The neighbourhood $\mathcal{B}_i$ essentially wraps around a geodesic β_i joining x_{j_i} and ∂S and minimising the distance between them, and "covering" the part of it at a distance less or equal than d_1 from x_{j_i}. Hence, for i large enough, the boundary of the $\mathcal{B}_i$ is inside the ball $B_i(\partial S, 1/8)$. Therefore for i large enough, the curve $\hat{\alpha}_i$ must enter $B_i(\partial S, 1/8)$ before going to infinity. We reach thus a contradiction.

We have then that for each j, the scaled curvature κ_{r_j} is bounded on each of the annuli $\mathcal{A}^c_{r_j}(p_j; 1/8, 8)$. Consider the areas A_{r_j} of the annuli $\mathcal{A}^c_{r_j}(p_j; 1/8, 8)$ with respect to q_{r_j}. If A_{r_j} tend to zero then the annuli $(\mathcal{A}^c_{r_j}(p_j; 1/8, 8), q_{r_j})$ collapse with bounded curvature and thus become thiner and thiner finite cylinders. The end S is then (except for a set of compact closure) a concatenation of the annuli $\mathcal{A}^c_{r_j}(p_j; 1/8, 8)$ and the quadratic curvature decay in the whole end follows as well as the quadratic decay of $|\nabla V|^2$ follows. If instead a sequence $A_{r_{j_i}}$ of the areas is bounded below away from zero then, due to the Bishop-Gromov monotonicity $A(B(\partial S, r))/r^2 \searrow$ and the curvature bound, the geometry of the annuli $(\mathcal{A}^c_{r_j}(p_j; 1/8, 8); q_{r_j})$ becomes more and more that of a flat annulus. Once a piece sufficiently close to a flat annulus forms then the whole end must be asymptotic to a flat annulus (for a detailed proof in dimension three see [26]). Again, the quadratic decay of κ and $|\nabla V|^2$ on the whole end follows. □

The following version of Proposition 6.4.2 but when $\Omega \neq 0$ is now straight forward after Proposition 6.4.3 and the proof of Proposition 6.4.2 itself.

Proposition 6.4.4. *Let $(S; q, U, V)$ be a (reduced) static data set with $\Omega \neq 0$, S non-compact and ∂S compact. Then there is a set K with compact closure, such that*

$$S = K \cup \left(\cup_{i=1}^{i=n} E_i \right) \tag{6.4.15}$$

where every E_i is diffeomorphic to $[0, \infty) \times \mathrm{S}^1$.

Taking into account the description of the asymptotic geometry of (reduced) static ends $(E;q,U,V)$, $(E\sim[0,\infty)\times S^1)$, we can easily find a simple end cut $\{\ell_j; j=1,2,\ldots\}$. Each ℓ_j is of course isotopic to ∂E and embedded in $\mathcal{A}(2^{1+2j},2^{2+2j})$. Let us be a bit more precise. Let $r_j=2^{1+2j}$ and as usual let $q_{r_j}=q/r_j^2$. If $\mu=0$ then the annuli $(\mathcal{A}_{r_j}(1,2);q_{r_j})$ metrically collapse to the segment $[1,2]$ and therefore the loops ℓ_j can be chosen to have q_{r_j}-length tending to zero. If instead $\mu>0$ then the loops can be chosen to converge to the radial circle $\{x=3/2\}$ as the annuli $(\mathcal{A}_{r_j}(1,2);q_{r_j})$ converge to the annulus $([1,2]\times S^1; dx^2+4\mu^2x^2d\varphi^2)$ as explained earlier.

Let Σ be the three-manifold whose quotient by the S^1-Killing field is E. Let $\pi:\Sigma\to E$ be the projection. The tori $S_j:=\pi^{-1}(\ell_j)$ form obviously a simple cut of $(\Sigma;\mathfrak{g})$. Let us state this in a proposition that will be recalled later.

Proposition 6.4.5. *Let $(\Sigma;\mathfrak{g},U)$ be a free S^1-symmetric data set such that the reduced state $(E;q,U,V)$ is a reduced end. Then, E and S admit simple cuts.*

The next proposition shows that U tends uniformly to a constant U_∞, on any (reduced) static end $(E;q,U,V)$. The constant U_∞ satisfies $-\infty\leq U_\infty\leq\infty$. The proposition will be used in Section 7.2.2.

Proposition 6.4.6. *Let $(E;q,U,V)$ be a reduced end. Then, $U\to U_\infty$ where the arrow signifies uniform convergence and the constant U_∞ satisfies $-\infty\leq U_\infty\leq\infty$.*

Proof. Note that the maximum principle is also applicable to U because (6.1.3) can be written as $div(e^V\nabla U)=0$. We will use this several times below.

Let $\{\ell_j, j=0,1,2,\ldots\}$ be a simple cut of E as described above. Let $r_j=2^{1+2j}$.

Assume that $\mu=0$. Then, as said, the q_{r_j}-length of the loops ℓ_j tends to zero. At the same time the norm $|\nabla U|_{r_j}$ restricted to the loops ℓ_j remains uniformly bounded. Therefore, by a simple integration along the ℓ_j it is deduced that,

$$(\max\{U(q):q\in\ell_j\}-\min\{U(q):q\in\ell_j\})\to 0 \tag{6.4.16}$$

If instead $\mu>0$ then the q_{r_j}-length of the loops ℓ_j remains uniformly bounded while the norm $|\nabla U|_{r_j}$, over the loops ℓ_j, tends to zero. So by a simple integration along the loops ℓ_j we deduce again (6.4.16).

Now suppose that for a certain sequence $p_i\in\ell_{j_i}$, $U(p_i)$ tends to a constant $-\infty\leq U_\infty\leq\infty$. Then by (6.4.16), the maximum and the minimum of U over ℓ_{j_i} also tend to U_∞. We use now the maximum principle to write for any $i<i'$

$$\max\{U(q):q\in\ell_{j_i}\cup\ell_{j_{i'}}\}\geq\max\{U(q):q\in\mathcal{L}_{j_i,j_{i'}}\}\geq \tag{6.4.17}$$
$$\geq\min\{U(q):q\in\mathcal{L}_{j_i,j_{i'}}\}\geq\min\{U(q):q\in\ell_{j_i}\cup\ell_{j_{i'}}\} \tag{6.4.18}$$

where $\mathcal{L}_{j_i,j_i'}$ is the compact region enclosed by ℓ_{j_i} and $\ell_{j_{i'}}$. Letting i' tend to infinity we deduce,

$$\max\{\max\{U(q):q\in\ell_{j_i}\},U_\infty\}\geq\max\{U(q):q\in\mathcal{L}_{j_i,\infty}\}\geq \tag{6.4.19}$$
$$\geq\min\{U(q):q\in\mathcal{L}_{j_i,\infty}\}\geq\min\{\min\{U(q):q\in\ell_{j_i}\},U_\infty\} \tag{6.4.20}$$

where $\mathcal{L}_{j_i,\infty}$ is the region enclosed by ℓ_{j_i} and infinity. As the left hand side of (6.4.19) and the right hand side of (6.4.20) tend to U_∞ then U must tend also uniformly to U_∞. □

6.5 Reduced data sets arising as collapsed limits

In this last subsection about S^1-symmetric states, it is worth to discuss the geometry of reduced data arising from scaled limit of data sets. This discussion will be recalled later in Section 7.2.3 where we prove that the asymptotic of static black hole data sets with subcubic volume growth is Kasner.

Let $(\Sigma;\mathfrak{g},U)$ be a data set, and let γ be a ray from $\partial\Sigma$. Let $p_n\in\gamma$ be a divergent sequence of points. Suppose there are neighbourhoods $\mathcal{B}_n$ of $\mathcal{A}^c_{r_n}(p_n,1/2,2)$ such that $(\mathcal{B}_n;\mathfrak{g}_{r_n})$ collapses to a two-dimensional orbifold. Having this, by a diagonal argument, one can find a subsequence of it (also indexed by n) and neighbourhoods $\mathcal{B}_n$ of $\mathcal{A}^c_{r_n}(p_n;1/2,2^{k_n})$, with $k_n\to\infty$, and collapsing to a two-dimensional orbifold $(S_\infty;q_\infty)$. As the collapse is along S^1-fibers (hence defining asymptotically a symmetry), we obtain, in the limit, a well defined reduced data $(S;q,\bar{U},V)$ where $\overline{U}$ is obtained as the limit of $U_n:=U-U(p_n)$. On smooth points the scalar curvature κ is non-negative. Orbifold points are connical with total angles an integer fraction of 2π ($2\pi/2$, $2\pi/3$, $2\pi/4$, etc) hence can be thought as having also non-negative curvature (they can be rounded off to have a smooth metric with $\kappa\geq 0$). Therefore $(S;q)$ has only a finite number of ends. Note that it has at least one end containing a limit, say $\overline{\gamma}$, of the ray γ. Let us denote that end by $S_{\overline{\gamma}}$.

We claim that every end has only a finite number of orbifold points. This is the result of a simple application of Gauss-Bonett. Indeed, let S be an end. Let ℓ_j, $j=1,2,3,\ldots$, be one-manfiolds embedded for each j in $\mathcal{A}(2^{2j},2^{2j+3})$ such that ℓ_1 and ℓ_j enclose a connected manifold Ω_{1j}. Let $\mathcal{O}$ be the set of orbifold points in S. By Gauss-Bonnet we have

$$-\int_{\ell_1}kdl-\int_{\ell_j}kdl=\int_{\Omega_{1j}\setminus\mathcal{O}}\kappa dA+\sum_{p\in\Omega_{1j}\cap\mathcal{O}}2\pi\bigg(\frac{i(p)-1}{i(p)}\bigg)\tag{6.5.1}$$

where k is the mean-curvature (or first variation of logarithm of length) on the one-manifolds ℓ_j and the angle at each orbifold point $p\in\mathcal{O}$ is $2\pi/i(p)$. As the right hand side is greater or equal than the number of orbifold points in Ω_{1j}, that is $\sharp\{\Omega_{1j}\cap\mathcal{O}\}$. Thus, if the left hand side remains bounded as $j\to\infty$ then the number of orbifold points must be finite. To see the existence of such one-manifolds ℓ_j for which the left hand side remains bounded just argue as follows. First note that the left hand side is scale invariant. Second observe that as for each j the scaled annuli $(\mathcal{A}(2^{2j},2^{2j+3});q_{2^{2j}})$ in S are scaled limits of annuli in $(\Sigma;\mathfrak{g})$, (which has quadratic curvature decay), then one can always chose a suitable subsequence j_i such that as $i\to\infty$ the annuli $(\mathcal{A}(2^{2j},2^{2j+3});q_{2^{2j}})$ either converge of collapse to a segment. The selection of the ℓ_i is then evident.

7 Volume growth and the asymptotic of ends

The asymptotic of ends is markedly divided by the volume growth. We discuss first cubic volume growth, which is the simplest and that implies AF. Then we discuss sub-cubic volume growth which implies (under certain hypothesis) AK. This last case requires an elaborated and long analysis.

7.1 Cubic volume growth and asymptotic flatness

Suppose $(\Sigma;\mathfrak{g},U)$ is a static end with cubic volume growth. Cubic volume growth, non-negative Ricci curvature and quadratic curvature decay, implies that the end is asymptotically conical, (i.e. the metric is asymptotic to a metric of the form dr^2+

$a^2r^2d\Omega^2$ in $\mathbb{R}^3$). Hence, outside an open set of compact closure, Σ is diffeomorphic to $\mathbb{R}^3$ minus a ball. It was proved in [], [] (see also []) that the data is then asymptotically flat (indeed asymptotically Schwarzschild).

7.2 Sub-cubic volume growth and Kasner asymptotic

The goal of this section will be to prove that the asymptotic of any static black hole data set with sub-cubic volume growth is Kasner. Observe that we are dealing with black hole data sets, and not just any end with sub-cubic volume growth. We do not know if just sub-cubic volume growth is enough to classify the asymptotic in general.

We aim to prove the following theorem.

Theorem 7.2.1. *Let $(\Sigma; \mathfrak{g}, U)$ be a static black hole data set with sub-cubic volume growth. Then the data is asymptotically Kasner.*

To achieve this we provide first a necessary and sufficient condition for Kasner asymptotic. This is the content of Proposition 7.2.6 for which we dedicate the whole Section 7.2.1. In seccond place, we analize the asymptotic of free S^1-symmetric static ends $(\Sigma; \mathfrak{g}, U)$ under the natural condition that $U(p) \leq U_\infty$ (recall that U_∞, the limit of U at ∞, exists by Proposition 6.4.6). We dedicate Section 7.2.2 to show Theorem 7.2.7 claiming that, for such a data, either the asymptotic is Kasner or the whole data is flat. The proof requires the results we have obtained for reduced states in section 6, as well as the development of an interesting monotonic quantity along the leaves of the level sets of U, that in turn will be used again in the proof of Theorem 7.2.1. Finally, Section 7.2.3 uses the results of the previous two sections to prove the desired Theorem 7.2.1.

7.2.1 Necessary and sufficient condition for KA.

We begin recalling the definition of the C^k-norms of a tensor with respect to a background metric. Let $(M; g)$ be a smooth Riemannian manifold. Let W be a smooth tensor of any valence. We denote by $|W|_g(x)$ the g-norm of W at $x \in M$. Given $k \geq 0$, the C^k-norm of W with respect to g is defined as

$$\|W\|^2_{C^k_g} := \sup_{x\in M}\left\{\sum_{i=0}^{i=k} |\nabla^{(i)}W|^2_g(x)\right\} \quad \text{where} \quad \nabla^{(i)}W = \underbrace{\nabla\ldots\nabla}_{\text{i-times}} W \tag{7.2.1}$$

Proposition 7.2.2. *Let $(T; h_F)$ be a flat two-torus. Let W be a smooth tensor field (of any valence), equal to zero at some point. Then for any $0 \leq j \leq k$ we have*

$$\|W\|_{C^j_{h_F}} \leq c(k)\ \mathrm{diam}^{k-j}_{h_F}(T)\ \|W\|_{C^k_{h_F}} \tag{7.2.2}$$

Proof. We will prove the inequality for functions. To prove it for tensors use the expansion $W = \sum f_I \omega_I$, where ω_I is an orthonormal and parallel basis (i.e. $\delta_{II'} =< \omega_I, \omega_{I'} >_g$ and $\nabla\omega_I = 0$), and then use the result obtained for functions.

We will work in $(\mathbb{R}^2; g_{\mathbb{R}^2})$ thought as the universal cover of $(T; h_F)$. In particular $\pi^* h_F = g_{\mathbb{R}^2}$. On a Cartesian coordinate system (x_1, x_2) we have

$$g_{\mathbb{R}^2} = dx_1^2 + dx_2^2 \tag{7.2.3}$$

and

$$\|f\|^2_{C^j_{h_F}} = \|f\|^2_{C^j_{g_{\mathbb{R}^2}}} = \sup_{x\in\mathbb{R}^2}\left\{\sum_{|I|=0}^{|I|=j}|\partial_I f|^2(x)\right\} \tag{7.2.4}$$

where for any multi-index $I=(i_1,\ldots,i_{|I|})$, $i_l\in\{1,2\}$, we denote $\partial_I=\partial_{x_{i_1}}\ldots\partial_{x_{i_{|I|}}}$.

We will need to rely on the existence of a coordinate system $(\overline{x}_1,\overline{x}_2)$ on which the metric $g_{\mathbb{R}^2}$ is written as

$$g_{\mathbb{R}^2}=d\overline{x}_1^2+\alpha(d\overline{x}_1 d\overline{x}_2+d\overline{x}_2 d\overline{x}_1)+d\overline{x}_2^2, \tag{7.2.5}$$

where α is a constant such that $|\alpha|\leq 1/2$, and where the directions $\partial_{\overline{x}_1}$ and $\partial_{\overline{x}_2}$ are periodic of period less than $6\mathrm{diam}_{h_F}(T)$, that is, any line in the direction of either ∂_{x_1} or ∂_{x_2} projects into a circle in T of length less that $6\mathrm{diam}_{h_F}(T)$. For the calculations that follow we assume that the coordinates $(\overline{x}_1,\overline{x}_2)$ are given. We will prove their existence at the end.

Observe that the norm (7.2.4), which is defined with respect to the metric (7.2.3) and the norm

$$\|f\|^2_{C^j_{\overline{g}_{\mathbb{R}^2}}}=\sup\left\{\sum_{|I|=0}^{|I|=j}|\overline{\partial}_I f|^2\right\},\qquad \overline{\partial}_I=\partial_{\overline{x}_{i_1}}\ldots\partial_{\overline{x}_{i_{|I|}}}, \tag{7.2.6}$$

which is defined with respect to the metric

$$\overline{g}_{\mathbb{R}^2}=d\overline{x}_1^2+d\overline{x}_2^2, \tag{7.2.7}$$

are equivalent, namely $c_1(j)\|f\|_{C^j_{g_{\mathbb{R}^2}}}\leq\|f\|_{C^j_{\overline{g}_{\mathbb{R}^2}}}\leq c_2(j)\|f\|_{C^j_{g_{\mathbb{R}^2}}}$. This is proved by noting that the family of metrics (7.2.5) with $|\alpha|\leq 1/2$ is compact. Thus, to prove (7.2.2) it is enough to prove

$$\|W\|_{C^j_{\overline{g}_{\mathbb{R}^2}}}\leq c(k)\ \mathrm{diam}^{k-j}_{h_F}(T)\ \|W\|_{C^k_{\overline{g}_{\mathbb{R}^2}}} \tag{7.2.8}$$

We do that in what follows.

For any function ψ which is zero at some point, say $(\overline{x}_1^0,\overline{x}_2^0)$, we have

$$\sup\big\{|\psi|\big\}\leq 12\mathrm{diam}_{h_F}(T)\sup\big\{|\partial_{\overline{x}_1}\psi|,|\partial_{\overline{x}_2}\psi|\big\} \tag{7.2.9}$$

This is seen by just writing

$$\psi(\overline{x}_1,\overline{x}_2)=\int_0^{\overline{x}_1-\overline{x}_1^0}\partial_{\overline{x}_1}\psi\Big|_{(\overline{x}_1^0+s,\overline{x}_2^0)}ds+\int_0^{\overline{x}_2-\overline{x}_2^0}\partial_{\overline{x}_2}\psi\Big|_{(\overline{x}_1,\overline{x}_2^0+s)}ds \tag{7.2.10}$$

and using that $|\overline{x}_1-\overline{x}_1^0|$ and $|\overline{x}_2-\overline{x}_2^0|$ are less or equal than $6\mathrm{diam}_{h_F}(T)$. If $\psi=f$ the ψ has a zero by hypothesis. Moreover, for any multi-index I, ($|I|\geq 1$), the function $\psi=\overline{\partial}_I f$ has also a zero. To see this just fix $\overline{x}_i$, for all $i\neq i_1$ (at any values), and observe that the function ψ as a function of $\overline{x}_{i_1}$ is the $\overline{x}_{i_1}$-derivative of a periodic function. Now, starting with $\psi=f$, use (7.2.9) repeatedly to obtain (7.2.8).

It remains to show the existence of the coordinates $(\overline{x}_1,\overline{x}_2)$. In the cartesian system (x_1,x_2), the balls $B((4\mathrm{diam}_{\mathbb{R}^2},0),\mathrm{diam}_{\mathbb{R}^2}(T))$ and $B((0,4\mathrm{diam}_{\mathbb{R}^2}),\mathrm{diam}_{\mathbb{R}^2}(T))$, possess points q_1 and q_2 projecting (in T) to the same point as the point $q_0=(0,0)$ does. Define the directions $\partial_{\overline{x}_1}$ and $\partial_{\overline{x}_1}$ as, respectively, those defined by q_0, q_1 and q_0, q_2, and finally define the origin of the coordinates $(\overline{x}_1, \overline{x}_2)$ to be $(x_1, x_2) = (0,0)$. It is direct to check that the coordinates $(\overline{x}_1, \overline{x}_2)$ thus constructed enjoy the required properties. □

Proposition 7.2.3. *Let $(T; h)$ be a Riemannian two-torus and let $p \in T$. Then there is a unique flat metric h_F, conformally related to $\star$ and equal to $\star$ at p. Moreover, for any integer $k \geq 1$, and reals $K_1 > 0$ and $K_k > 0$ there is $D(K_1) > 0$ (small enough) and $C(k, K_k) > 0$ such that if*

$$\|\kappa\|_{C^1_h} \leq K_1, \quad \|\kappa\|_{C^k_h} \leq K_k, \quad \textit{and} \quad \mathrm{diam}_h(T) \leq D \tag{7.2.11}$$

then,

$$e^{-C} h_F \leq h \leq e^C h_F \tag{7.2.12}$$

and

$$\|h\|_{C^k_{h_F}} \leq C. \tag{7.2.13}$$

Proof. We will use that there is $D(K_1)$, (small enough), such that if $\mathrm{diam}_h(T) \leq D(K_1)$ then there is a finite cover $\pi : (\tilde{T}; \tilde{h}) \to (T; h)$, (i.e. $\pi : \tilde{T} \to T$ and $\tilde{h} = \pi^* h$), such that, (i) $\mathrm{diam}_{\tilde{h}}(\tilde{T}) \leq 1$, and (ii) $inj_{\tilde{h}}(p) \geq i_0(K_1)$ for all $p \in \tilde{T}$. Because $(\tilde{T}; \tilde{h})$ is a cover of $(T; h)$ we also have (iii) $\|\tilde{\kappa}\|_{C^k_{\tilde{h}}} \leq K_k$. The claims, (i) and (ii), are well known from the standard theory of diameter-collapse with bounded curvature. In simple terms they follow easily from the fact that the exponential map $exp : \mathcal{T}_p T \to T$ restricted to a small ball in $\mathcal{T}_p T$ is an immersion and then finding an appropriate fundamental domain on $\mathcal{T}_p T$ around p that will define $\tilde{T}$. We will not discuss this further, rather we will use it from now on.

The properties (i) and (ii) imply that the geometry of $(\tilde{T}; \tilde{h})$ is controlled(11) in C^2 by K_1. Moreover if the geometry of $(\tilde{T}; \tilde{h})$ is controlled in C^2 by K_1, then the geometry of $(\tilde{T}; \tilde{h})$ is controlled in C^{k+1} by K_k. This allows us to make standard elliptic analysis in $(\tilde{T}; \tilde{h})$ as if working in a fixed manifold.

Let $\tilde{\phi}$ be the solution to

$$\Delta_{\tilde{h}} \tilde{\phi} = \tilde{\kappa}, \quad \text{with} \quad \int_{\tilde{T}} \tilde{\phi}\, dA_{\tilde{h}} = 0 \tag{7.2.14}$$

With such $\tilde{\phi}$, the conformal metric $\tilde{h}_F = e^{2\tilde{\phi}} \tilde{h}$ is flat. Multiply (7.2.14) by $\tilde{\phi}$, integrate and use Cauchy-Schwarz to obtain

$$\int_{\tilde{T}} |\nabla \tilde{\phi}|^2_{\tilde{h}}\, dA_{\tilde{h}} \leq \Big(\int_{\tilde{T}} \tilde{\kappa}^2\, dA_{\tilde{h}} \Big)^{\frac{1}{2}} \Big(\int_{\tilde{T}} \tilde{\phi}^2\, dA_{\tilde{h}} \Big)^{\frac{1}{2}} \tag{7.2.15}$$

Now, we can use the Poincaré inequality

$$\int_{\tilde{T}} \tilde{\phi}^2\, dA_{\tilde{h}} \leq I(K_1) \int_{\tilde{T}} |\nabla \tilde{\phi}|^2\, dA_{\tilde{h}} \tag{7.2.16}$$

in the right hand side of (7.2.15) to obtain an upper bound on $\|\nabla \tilde{\phi}\|_{L^2_{\tilde{h}}}$, (that $I = I(K_1)$ is justified because the geometry of $\tilde{T}$ is controlled in C^2). Such bound can be used in

(11) To be precise: A geometry is controlled in C^k by K if there is a cover of $\tilde{T}$ by $n(K)$-harmonic charts, with Lebesgue number $\delta(K)$, such that, on each chart (x_1, x_2), we have (i) $e^{K'(K)} \delta_{ij} \leq \tilde{h}_{ij} \leq e^{K'(K)} \delta_{ij}$ and (ii) $\|\tilde{h}_{ij}\|_{C^k_{\delta_{ij}}} \leq K'(K)$. See [?].

turn again in (7.2.16) to obtain $\|\tilde{\phi}\|_{L^2_{\tilde{h}}} \leq B_1(K_1)$. Using this L^2-bound together with standard elliptic estimates on (7.2.14) we obtain

$$\|\tilde{\phi}\|_{C^k_{\tilde{h}}} \leq B_2(k, K_k). \tag{7.2.17}$$

As $k \geq 1$, we deduce

$$|\tilde{\phi}| \leq B_2(k, K_k), \tag{7.2.18}$$

This implies that for a $C_1(k, K_k) > 0$ we have

$$e^{-C_1}\tilde{h} \leq \tilde{h}_F \leq e^{C_1}\tilde{h} \tag{7.2.19}$$

Moreover the covariant derivative ∂ of $\tilde{h}_F$ is related to the covariant derivative ∇ of $\tilde{h}$ by

$$\partial_A = \nabla_A + (\nabla_A \phi)h^C_B + (\nabla_B \phi)h^C_A - (\nabla^C \phi)h_{AB} \tag{7.2.20}$$

Now (7.2.17), (7.2.19) and (7.2.20) (to compute $\partial^{(j)}$) imply the bound

$$\|\tilde{\phi}\|_{C^k_{\tilde{h}_F}} \leq B_3(k, K_k). \tag{7.2.21}$$

By the uniqueness of solutions to (7.2.14), $\tilde{\phi}$ has to coincide with its average by the Deck-transformations. Hence, $\tilde{\phi}$ and $\tilde{h}_F$ descend respectively to a function $\overline{\phi}$ and a flat metric h_F. As the bound (7.2.21) is local we also have

$$\|\overline{\phi}\|_{C^k_{h_F}} \leq B_3(k, K_k). \tag{7.2.22}$$

Finally define $\phi = \overline{\phi} - \overline{\phi}(p)$. With this definition we have $h(p) = h_F(p)$. From the bound $|\overline{\phi}| \leq B_2(k, K_k)$ we obtain the bound $|\phi| \leq 2B_2(k, K_k)$ hence (7.2.12). Also from $|\phi| \leq 2B(k, K_k)$ and (7.2.22) we deduce,

$$\|\phi\|_{C^k_{h_F}} \leq B_4(k, K_k). \tag{7.2.23}$$

hence (7.2.13). □

Before passing to the next crucial propositions we make a pair of geometric observations about the Kasner solutions and introduce some terminology.

On $\mathbb{R}^+ \times \mathbb{R}^2$ consider a Kasner solution

$$\mathfrak{g} = dx^2 + x^{2a}dy^2 + x^{2b}dz^2, \tag{7.2.24}$$

$$U = c \ln x \tag{7.2.25}$$

and assume that $c \in (0, 1/2)$. Quotient the space by a $\mathbb{Z} \times \mathbb{Z}$ action to obtain a Kasner solution on $\mathbb{R}^+ \times T^2$. For every x let T_x be the two-torus $\{x\} \times T^2$. Fixed c, there are two possibilities for (a, b), (a_-, b_-) and $(a_+, b_+) = (b_-, a_-)$. In either case, and because $c \in (0, 1/2)$, we have $0 < a < 1$, $0 < b < 1$. Let

$$a_* = \max\left\{e^{2/(1-a)}, e^{2/(1-b)}\right\} \geq 4 \tag{7.2.26}$$

If $\overline{a} \geq a_*$ then

$$\operatorname{diam}_{\mathfrak{g}_{\overline{a}}}(T_{\overline{a}}) \leq \frac{1}{e^2}\operatorname{diam}_{\breve{g}}(T_1) \tag{7.2.27}$$

where, recall former notation, $\mathfrak{g}_{\overline{a}} = \mathfrak{g}/\overline{a}^2$. To see this simply note that

$$\frac{1}{\overline{a}^2}(\overline{a}^{2a}dy^2 + \overline{a}^{2b}dz^2) = (\overline{a}^{2a-2}dy^2 + \overline{a}^{2b-2}dz^2) \leq \frac{1}{e^4}(dz^2 + dy^2) \tag{7.2.28}$$

so (7.2.27) holds no matter how we quotient $\mathbb{R}^2$. Thus, the diameter of $T_{\overline{a}}$ with respect to $\mathfrak{g}_{\overline{a}}$, is at least $1/e^2$ of the diameter of T_1 with respect to $\mathfrak{g}$.

In the following propositions we will use the notation $\rho = |\nabla U|$, $\rho_r = |\nabla U|_r$ and $\lambda = 1/\rho$, $\lambda_r = 1/\rho_r$. Also, given $0 < \rho^* < 1/2$ we let

$$a^* = \max\{a_*(c) : c \in [\rho^*/4, \rho^*/4 + 3/8]\}. \tag{7.2.29}$$

The reader must keep this notation in mind.

We will also use the following definition. Let W be a tensor of any valence defined at just one point x of a flat torus $(T; h_F)$. Then the *h_F-extension* of W is the tensor field defined by translating W to all T by its isometry group.

Proposition 7.2.4. *Let $(\Sigma; \mathfrak{g}, U)$ be a static end, and let γ be a ray emanating from $\partial\Sigma$. Let $0 < \rho^* < 1/2$ and let integers $j^* \geq 0$ and $m^* \geq 1$. Then, there exist positive constants $\epsilon^*, \mu^* \leq \min\{\rho^*/2, (1/2 - \rho^*)/2\}, r^*, C^*$, such that if at a point $p \in \gamma$ with $r = r(p) \geq r^*$ we have,*

(a) $d_{GH}\big((\mathcal{A}^c_r(p; 1/2, 2); d_r), ([1/2, 2]; |\ldots|)\big) \leq \epsilon^*$, *and,*

(b) $|\rho_r(p) - \rho^*| \leq \mu^*$,

then,

(I) *there is a neighbourhood $\mathcal{U}_p$ of $\mathcal{A}^c_r(p; 1/(2a^*), 2a^*)$ foliated by level sets of U each of which is a two-torus, and,*

(II) *there is a Kasner space $\mathbb{K}$ and a smooth diffeomorphism (into the image) $\phi : \mathcal{U} \to \mathbb{K}$, preserving the toric-foliations, such that*

$$\|\phi_*\mathfrak{g}_r - \mathfrak{g}_{\mathbb{K}}\|_{C^{j^*}_{\mathfrak{g}_{\mathbb{K}}}} \leq C^* \text{diam}^{m^*}_{\mathfrak{g}_{\mathbb{K}}}\big(\phi(T_p)\big) \tag{7.2.30}$$

where T_p is the level set of U containing p.

Proof.

(I) Proceeding by contradiction, assume that for any $\epsilon^*_i = 1/i$, $\delta^*_i = 1/i$ and $r^* = i$ there is $p_i \in \gamma$ with $r_i = r(p_i) \geq r^*_i$ for which (a) and (b) hold but for which the neighbourhood $\mathcal{U}_p$ with the desired properties does not exist. But if (a) holds and p_i belongs to a ray then the space $(\mathcal{A}^c_{r_i}(p_i; 1/(2a^*), 2a^*); d_{r_i})$ necesarily metrically collapses to a segment of length $2a^* - 1/(2a^*)$. Thus there are neighbourhoods $\mathcal{B}_i$ of $\mathcal{A}^c_r(p_i; 1/(2a^*), 2a^*)$ and covers $\pi_i : \tilde{\mathcal{B}}_i \to \mathcal{B}_i$ such that $(\tilde{\mathcal{B}}_i; \tilde{g}_{r_i}, \tilde{U}_i)$ converges to a $\mathrm{S}^1 \times \mathrm{S}^1$-symmetric state. The limit state has non-constant ρ because by (b) it must be $\tilde{\rho}_{r_i}(p_i) \to \rho^*$ and $0 < \rho^* < 1/2$. Hence the limit space is a Kasner space. Therefore for i large enough the level sets of $\tilde{U}$ foliate $\tilde{\mathcal{B}}_i$ and hence $\mathcal{B}_i$. Thus the neighbourhoods $\mathcal{U}_{p_i}$ with the desired properties exist for i large enough, which is a contradiction.

It is direct to see from the argumentation above that, after choosing ϵ^* smaller if necessary and r^* bigger if necessary, ρ_r is uniformly bounded above and below away from zero; that is, for some $0 < \underline{\rho} < \overline{\rho} < 1/2$, the bound $0 < \underline{\rho} \leq \rho_r \leq \overline{\rho}$ holds on $\mathcal{U}_p$, for any $p \in \gamma$ with $r(p) \geq r^*$ for which (a) and (b) hold. In the proof of part (II) we will assume that ϵ^* and r^* were chosen accordingly. As the proof progresses the values of ϵ^* and r^* will be adjusted a few times.

Note that the estimates (4.2.27) and the uniform bound for ρ_r show that for any $i \geq 0$, $|\nabla^{(i)}\rho_r|_r$ is uniformly bounded (without the need to adjust ϵ^* or r^* for each i). Similarly for any $i \geq 0$, $|\nabla^{(i)}\lambda_r|_r$ is uniformly bounded.

It is natural then to introduce the following terminology that will be used throughout the proof of (II) below. Let $\mathcal{G}$ be a geometric quantity defined on each of the neighbourhoods $\mathcal{U}_p$ (for instance $\mathcal{G} = \lambda_r$). Then $\mathcal{G}$ is *uniformly bounded* if one can find a constant $C > 0$ such that $\mathcal{G} \leq C$ holds on $\mathcal{U}_p$, for any p with $r(p) \geq r^*$ for which (a) and (b) hold.

(II) *Thoughout this part* (II) *we will be working on the neighbourhoods* $\mathcal{U}_p$ *and at the scaled geometry, namely dealing with* $\mathfrak{g}_r$ *rather than* $\mathfrak{g}$. *However to prevent a cumbersome notation we will omit the subindex r everywhere. The reader should be aware of that.*

(II)-A. The trivialisation ϕ and the flat metric $\mathfrak{g}_F$. Given $q \in \mathcal{U}_p$ let $\zeta_q(U)$ be the integral curve of the vector field $\nabla^a U$, extending throughout $\mathcal{U}_p$ and parametrised by U. Then define $\phi : \mathcal{U}_p \to T_p \times I$ by $\phi(q) = (T_p \cap \zeta_q, U(q))$. We will be identifying $\mathcal{U}_p$ with $T_p \times I$ via the diffeomorphism ϕ.

On $T_p \times I$ the metric $\mathfrak{g}$ is written as

$$\mathfrak{g} = \lambda^2 dU^2 + h \tag{7.2.31}$$

where $\lambda = 1/\rho$ and $\star$ is the induced metric on the tori $T_U := T_p \times U$. Denote by D the intrinsic covariant derivative on the T_U's. As $T_{U(p)}$ will appear often we will use the simpler notation T_p.

Let h_F be the metric on T_p that is conformally related to $h|_{T_p}$ and that is equal to $\star$ at p (§ Proposition 7.2.3). On $T_p \times I$ define

$$\mathfrak{g}_F = dU^2 + h_F. \tag{7.2.32}$$

Around any point $q \in T_p$ we can consider coordinates (z_1, z_2) such that $h_F = dz_1^2 + dz_2^2$. On every patch (z_1, z_2, U) we have $\mathfrak{g}_F = dU^2 + dz_1^2 + dz_2^2$. For this reason the h_F-covariant derivative on the tori T_U will be denoted by ∂_A or simply ∂.

We claim that,

(i) $e^{-C_0} h_F \leq h \leq e^{C_0} h_F$, where $C_0 > 0$ is uniform,

(ii) for any $i \geq 0$ and $l \geq 0$, $|\partial_U^l \partial^{(i)} h|_{h_F}$ and $|\partial_U^l \partial^{(i)} \lambda|_{h_F}$ are uniformly bounded.

Of course these uniform bounds should be understood to hold at every point of every T_U in $\mathcal{U}_p$.

We prove first (i). We start showing that for every $i \geq 0$, $|D^i \lambda|_h$, $|D^{(i)}\Theta|_h$ and $|D^i \theta|_h$ are uniformly bounded. Let v and w be two unit vectors tangent to a T_U at one point. A normal unit vector to T_U is $n^a = \lambda \nabla^a U$. Then we compute,

$$\Theta(v, w) = \langle \nabla_v (\lambda \nabla U), w \rangle = \big(\lambda \nabla_a \nabla_b U + (\nabla_a \lambda) \nabla_b U\big) v^a w^b \tag{7.2.33}$$

By the estimates (4.2.27), $|\nabla_a U|_{\mathfrak{g}}$ and $|\nabla_a \nabla_b U|_{\mathfrak{g}}$ are uniformly bounded. Similarly, as mentioned in (I), λ and $|\nabla \lambda|_{\mathfrak{g}}$ are uniformly bounded. Hence $|\Theta|_h$ is uniformly bounded.

For the same reason ∇-derivatives of Θ are uniformly bounded, and therefore are the D-derivatives because ∇ and D differ from each other in Θ. These bounds imply the uniform bounds also for $|D^i\lambda|_h$ and $|D^i\theta|_h$.

Recall that the Gaussian curvature κ of the metric $\star$ on a slice T_U is given by

$$2\kappa = -|\Theta|^2 - \theta^2 - \frac{2}{\lambda^2}. \tag{7.2.34}$$

The previous estimates then show that for every $i \geq 0$, $|D^{(i)}\kappa|_h$ is also uniformly bounded.

So far these uniform bounds hold without the need to adjust ϵ^* or r^*, because they are due essentially to the bounds (4.2.27) and the uniform bounds for ρ. In the sequel we may need however further adjustment. Chose then ϵ^* sufficiently small such that $\mathrm{diam}_h(T_p)$ is small enough that we can use Proposition 7.2.3 on T_p to conclude first that

$$e^{-K_0}h_F \leq h\big|_{T_p} \leq e^{K_0}h_F \tag{7.2.35}$$

where $K_0 > 0$ is uniform and second that for any $i \geq 1$, $|\partial^{(i)}h|_{T_p}|_{h_F}$ is uniformly bounded.

Now we explain how (i) is a simple consequence of the boundedness of the second fundamental forms. Recall that

$$\partial_U h = 2\lambda\Theta \tag{7.2.36}$$

As λ is uniformly bounded and as $e^{-K_1}h \leq \Theta \leq e^{K_1}h$ at every T_U and for some uniform $K_1 > 0$, we deduce that $e^{-K_2}h \leq \partial_U h \leq e^{K_2}h$ for some uniform $K_2 > 0$. After integration in U we obtain $e^{-K_3}h|_{T_p} \leq h \leq e^{K_3}h|_{T_p}$ for some uniform $K_3 > 0$, which is equivalent to $e^{-C_0}h_F \leq h \leq e^{C_0}h_F$ for a uniform $C_0 > 0$ because of (7.2.35).

We turn to prove (ii). We have mentioned already that $|\nabla\lambda|_{\mathfrak{g}}$ is uniformly bounded. Thus, $|\partial_U\lambda|(= |\rho^2\langle\nabla U, \nabla\lambda\rangle|)$ is uniformly bounded and so is $|\partial\lambda|_{h_F}$ by (7.2.35). We prove then that $|\partial_U h|_{h_F}$ and $|\partial h|_{h_F}$ are uniformly bounded. The uniform bound for $|\partial_U h|_{h_F}$ follows directly from the formula (7.2.36), the uniform bound of λ and of $|\Theta|_h$, and (i). Let us turn now to prove the uniform bound for $|\partial h|_{h_F}$. We work in coordinates. We compute

$$\partial_U\partial_C h_{AB} = 2(\partial_C\lambda)\Theta_{AB} + 2\lambda\partial_C\Theta_{AB} \tag{7.2.37}$$

where we can write

$$\partial_C\Theta_{AB} = D_C\Theta_{AB} + \Gamma^M_{CA}\Theta_{MB} + \Gamma^M_{CB}\Theta_{AM} \tag{7.2.38}$$

with the Levi-Civita connection Γ^C_{AB} being

$$\Gamma^C_{AB} = \frac{1}{2}\{\partial_A h_{MB} + \partial_B h_{AM} - \partial_M h_{AB}\}h^{MC} \tag{7.2.39}$$

Hence, relying on the estimates previously obtained we can write

$$\partial_U(\partial_C h_{AB}) = X_{CAB}{}^{C'A'B'}(\partial_{C'}h_{A'B'}) + Y_{CAB} \tag{7.2.40}$$

where $|X_{CAB}{}^{C'A'B'}|$ and $|Y_{CAB}|$ are uniformly bounded. Using this system of first order ODEs and the uniform bound for $|\partial h|_{T_p}|_{h_F}$ at the initial slice T_p, we get directly the desired uniform boundedness of $|\partial_C h_{AB}|$.

Proving that for every $i \geq 0$ and $l \geq 0$, $|\partial^l_U\partial^{(i)}\lambda|_{h_F}$ and $|\partial^l_U\partial^{(i)}h|_{h_F}$ are uniformly bounded, is done by the iteration of the same arguments.

(II)-B. A 'GOOD' $S^1 \times S^1$-SYMMETRIC APPROXIMATION $\breve{\mathfrak{g}}$ OF $\mathfrak{g}$. We explain first how to define $\breve{\mathfrak{g}}$ and then we explain how well it does approximate $\mathfrak{g}$. Let p_0 be a point in T_p where the Gaussian curvature is zero. The choice of p_0 will play some role that we will explain later. Then define

$$\breve{\mathfrak{g}} = \breve{\lambda}^2 dU^2 + \breve{h} \tag{7.2.41}$$

where $\breve{\lambda}$ and $\breve{h}$ are, at every leaf T_U, simply the h_F-extensions of $\lambda(\zeta_{p_0}(U))$ and $h|_{\zeta_{p_0}(U)}$ respectively. Note, in particular, that $h - \breve{h}$ and $\lambda - \breve{\lambda}$ are zero all over $\zeta_{p_0}(U)$.

We prove now that for every $i \geq 0$ and $l \geq 0$ there is a uniform $C > 0$ such that

$$|\partial_U^l \partial^{(i)}(h - \breve{h})|_{h_F} \leq C \mathrm{diam}_{h_F}^{m^*}(T_p), \tag{7.2.42}$$

$$|\partial_U^l \partial^{(i)}(\lambda - \breve{\lambda})|_{h_F} \leq C \mathrm{diam}_{h_F}^{m^*}(T_p) \tag{7.2.43}$$

Fix i and l. In a coordinate patch (z_1, z_2, U) around $\zeta_{p_0} = p_0 \times I$, $(p_0 = (0,0))$, we have

$$\breve{h}_{AB}(z_1, z_2, U) = h_{AB}(0,0,U), \quad \breve{\lambda}(z_1, z_2, U) = \lambda(0,0,U) \tag{7.2.44}$$

for all (z_1, z_2, U). Taking ∂_U-derivatives we deduce that for every $l' \geq 0$, also $(\partial_U^{l'} \breve{h})|_{T_U}$ and $(\partial_U^{l'} \breve{\lambda})|_{T_U}$ are the h_F-extensions of $(\partial_U^{l'} h)|_{\zeta_{p_0}(U)}$ and $(\partial_U^{l'} \lambda)|_{\zeta_{p_0}(U)}$ respectively. Therefore $\partial_U^{l'}(h - \breve{h})$ and $\partial_U^{l'}(\lambda - \breve{\lambda})$ are zero at every point on $\zeta_{p_0}(U)$. If we prove that in addition for every $i' \geq 0$ and $l' \geq 0$, $|\partial^{(i')} \partial_U^{l'}(h - \breve{h})|_{h_F}$ is uniformly bounded then the $C_{h_F}^{i+m^*}$-norm of $\partial_U^l(h - \breve{h})$ on every T_U would be uniformly bounded. We could then use Proposition 7.2.2 at every tori T_U, (in Proposition 7.2.2 use $W = \partial_U^l(h - \breve{h})$, $k = i + m^*$ and $j = i$), to conclude (7.2.42) and (7.2.46). Let us prove then these bounds.

First, as $(\partial_U^{l'} \breve{h})|_{T_U}$ is the h_F extension of $(\partial_U^{l'} h)|_{\zeta_{p_0}(U)}$, then at every point q in a torus T_U we have $|\partial_U^{l'} \breve{h}|_{h_F}(q) = |\partial_U^{l'} \breve{h}|_{h_F}(\zeta_{p_0}(U)) = |\partial_U^{l'} h|_{h_F}(\zeta_{p_0}(U))$. But by (ii), for every $l' \geq 0$, $|\partial_U^{l'} h|_{h_F}$ is uniformly bounded, hence $|\partial_U^{l'}(h - \breve{h})|_{h_F} (\leq |\partial_U^{l'} h|_{h_F} + |\partial_U^{l'} \breve{h}|_{h_F})$, is uniformly bounded.

In second place, as $(\partial_U^{l'} \breve{h})|_{T_U}$ and $(\partial_U^{l'} \breve{\lambda})|_{T_U}$ are the h_F-extensions of $(\partial_U^{l'} h)|_{\zeta_{p_0}(U)}$ and $(\partial_U^{l'} \lambda)|_{\zeta_{p_0}(U)}$ respectively then for any $i' \geq 1$ we have $\partial^{(i')} \partial_U^{l'} \breve{h} = 0$ and $\partial^{(i')} \partial_U^{l'} \breve{\lambda} = 0$. Therefore,

$$\partial_U^l \partial^{(i')}(h - \breve{h}) = \partial_U^l \partial^{(i')} h \quad \text{and} \quad \partial_U^l \partial^{(i')}(\lambda - \breve{\lambda}) = \partial_U^l \partial^{(i')} \lambda \tag{7.2.45}$$

By the estimates (i) and (ii) in (I), the h_F-norm of the right hand side of each of these expressions is uniformly bounded. This concludes the proof of the bounds that we claimed above.

These estimates imply now, for any $i \geq 0$ and $l \geq 0$, we have

$$|\partial_U^l \partial^{(i)} DD(\lambda - \breve{\lambda})|_{h_F} \leq C_{li} \mathrm{diam}_{h_F}^{m^*}(T_p), \tag{7.2.46}$$

where the C_{li} are uniform (use that $D = \partial + \Gamma$). This is the estimate that will be used in (II)-C.

(II)-C. THE KASNER APPROXIMATION $\mathfrak{g}^{\mathbb{K}}$ OF $\mathfrak{g}$. In coordinates (z_1, z_2, U) the static equations are

$$\partial_U h_{AB} = 2\lambda\Theta_{AB}, \tag{7.2.47}$$
$$\partial_U \Theta_{AB} = -D_A D_B \lambda + \lambda(2\kappa h_{AB} - \theta\Theta_{AB} + 2\Theta_{AC}\Theta^C_{\ B}), \tag{7.2.48}$$
$$\partial_U \left(\frac{\sqrt{|h|}}{\lambda}\right) = 0, \tag{7.2.49}$$
$$\Theta_{AB}\Theta^{AB} - \theta^2 = -\frac{2}{\lambda^2} - 2\kappa, \tag{7.2.50}$$
$$D^A\Theta_{AB} = D_B\theta, \tag{7.2.51}$$

where, as earlier, $\theta = \Theta_A^{\ A}$. The equation (7.2.49) is the same as $\Delta U = 0$ and is equivalent to

$$\partial_U \lambda = \lambda^2\theta \tag{7.2.52}$$

We will use this equation instead of (7.2.49).

Evaluating (7.2.47), (7.2.48), (7.2.49), (7.2.52) and (7.2.51) at $\zeta_{p_0}(U)$ and (7.2.50) at p_0 we get,

$$\partial_U \breve{h}_{AB} = 2\breve{\lambda}\breve{\Theta}_{AB}, \tag{7.2.53}$$
$$\partial_U \breve{\Theta}_{AB} = \breve{\lambda}(2\overline{\kappa}\breve{h}_{AB} - \breve{\theta}\breve{\Theta}_{AB} + 2\breve{\Theta}_{AC}\breve{\Theta}^C_{\ B}) + O^\infty_{AB}(\mathrm{diam}^{m^*}_{h_F}(T_p)), \tag{7.2.54}$$
$$\partial_U \breve{\lambda} = \breve{\lambda}^2\breve{\theta}, \tag{7.2.55}$$
$$\left(\breve{\Theta}_{AB}\breve{\Theta}^{AB} - \breve{\theta}^2\right)\Big|_{p_0} = -\frac{2}{\breve{\lambda}^2}\Big|_{p_0}, \tag{7.2.56}$$
$$\breve{\partial}^A\breve{\Theta}_{AB} = \partial_B\breve{\theta}, \tag{7.2.57}$$

where $\overline{\kappa}$ is defined as

$$\overline{\kappa} = \left[-\frac{1}{\breve{\lambda}^2} - \frac{1}{2}\left(\breve{\Theta}_{AB}\breve{\Theta}^{AB} - \breve{\theta}^2\right)\right]\Big|_{p_0} \tag{7.2.58}$$

(and is not the Gaussian curvature of $\breve{h}$ which is zero) and where O^∞_{AB} is

$$O^\infty_{AB} = -D_A D_B \lambda. \tag{7.2.59}$$

This notation is to stress that, as was shown in (7.2.46), for any $l \geq 0$ we have

$$|\partial^l_U O^\infty_{AB}|_{h_F} \leq C_l \mathrm{diam}^{m^*}_{h_F}(T_p) \tag{7.2.60}$$

where C_l is uniform.

Consider now the metric

$$\mathfrak{g}^{\mathbb{K}} = (\lambda^{\mathbb{K}})^2 dU^2 + h^{\mathbb{K}}, \tag{7.2.61}$$

where $\lambda^{\mathbb{K}} = \lambda^{\mathbb{K}}(U)$ and $h^{\mathbb{K}} = h^{\mathbb{K}}(U)$ solve

$$\partial_U h^{\mathbb{K}}_{AB} = 2\lambda^{\mathbb{K}}\Theta^{\mathbb{K}}_{AB}, \tag{7.2.62}$$
$$\partial_U \Theta^{\mathbb{K}}_{AB} = \lambda^{\mathbb{K}}(-\theta^{\mathbb{K}}\Theta^{\mathbb{K}}_{AB} + 2\Theta^{\mathbb{K}}_{AC}\Theta^{\mathbb{K}C}_{\ \ B}), \tag{7.2.63}$$
$$\partial_U \lambda^{\mathbb{K}} = (\lambda^{\mathbb{K}})^2\theta^{\mathbb{K}} \tag{7.2.64}$$

subject to the initial data

$$h^{\mathbb{K}}_{AB}(0) = \breve{h}_{AB}(0), \quad \Theta^{\mathbb{K}}_{AB}(0) = \breve{\Theta}_{AB}(0) \quad \text{and} \quad \lambda^{\mathbb{K}}(0) = \breve{\lambda}(0). \tag{7.2.65}$$

Following the discussion in Section 6.2, we see that $(\lambda^{\mathbb{K}}(U), h^{\mathbb{K}}(U))$ satisfy (2.2.17), (2.2.18) and (7.2.64) for all U, and (2.2.19) at the initial time, hence is a Kasner solution. Hence

$$0 = -\frac{1}{(\lambda^{\mathbb{K}})^2} - \frac{1}{2}\big(\Theta^{\mathbb{K}}_{AB}\Theta^{\mathbb{K}AB} - (\theta^{\mathbb{K}})^2\big) \tag{7.2.66}$$

on each T_U. Thus, (7.2.63) is equivalent to

$$\partial_U \Theta^{\mathbb{K}}_{AB} = \lambda^{\mathbb{K}}(2\overline{\kappa}^{\mathbb{K}} - \theta^{\mathbb{K}}\Theta^{\mathbb{K}}_{AB} + 2\Theta^{\mathbb{K}}_{AC}\Theta^{\mathbb{K}C}_{\phantom{\mathbb{K}C}B}), \tag{7.2.67}$$

where $\overline{\kappa}^{\mathbb{K}}$ is the right hand side of (7.2.66) and is zero. Therefore, thought as ODE's, the system (7.2.53), (7.2.54), (7.2.55) is a perturbation of the system (7.2.62), (7.2.63), (7.2.64) where the 'perturbation' is O^{∞}_{AB} and should be thought as depending only on U. Both sistems have also the same initial data. Therefore, using (7.2.59) and standard ODE analysis we obtain

$$|\partial^l_U(\breve{h} - h^{\mathbb{K}})|_{h_F} \leq C^*_l \mathrm{diam}^{m^*}_{h_F}(T_p), \tag{7.2.68}$$

$$|\partial^l_U(\breve{\lambda} - \lambda^{\mathbb{K}})| \leq C^*_l \mathrm{diam}^{m^*}_{h_F}(T_p) \tag{7.2.69}$$

for any $l \geq 0$, where the C^*_l are uniform. Now note that because $\partial^{(i)}h_{\mathbb{K}} = \partial^{(i)}\breve{h} = 0$ then for every $i \geq 1$ we have

$$\partial^l_U\partial^{(i)}(h - h^{\mathbb{K}}) = \partial^l_U\partial^{(i)}(h - \breve{h}), \tag{7.2.70}$$

$$\partial^l_U\partial^{(i)}(\lambda - \lambda^{\mathbb{K}}) = \partial^l_U\partial^{(i)}(\lambda - \breve{\lambda}) \tag{7.2.71}$$

Thus, from (7.2.42) and (7.2.43) we obtain

$$|\partial^l_U\partial^{(i)}(h - h^{\mathbb{K}})|_{h_F} \leq C^*_{li}\mathrm{diam}^{m^*}_{h_F}(T_p), \tag{7.2.72}$$

$$|\partial^l_U\partial^{(i)}(\lambda - \lambda^{\mathbb{K}})|_{h_F} \leq C^*_{li}\mathrm{diam}^{m^*}_{h_F}(T_p) \tag{7.2.73}$$

where the C^*_{li} are uniform.

The estimates (7.2.30) claimed in (II) are equivalent to (7.2.68),(7.2.69), (7.2.72) and (7.2.73). This finishes the proof of the Proposition. □

Proposition 7.2.5. *Let $(\Sigma; \mathfrak{g}, U)$ be a static end, and let γ be a ray emanating from $\partial\Sigma$. Let $0 < \rho^* < 1/2$ and let m^* be an integer greater or equal than one. Let $\epsilon^*, \mu^*, r^*, C^*$ be as in Proposition 7.2.4. Then, there exist positive δ^*, ℓ^* and B^* such that for any point $p \in \gamma$ with $r = r(p) \geq r^*$ satisfying,*

(a) $d_{GH}\big((\mathcal{A}^c_r(p; 1/2, 2); d_r), ([1/2, 2]; |\ldots|)\big) \leq \epsilon^*$,

(b) $|\rho_r(p) - \rho^*| \leq \mu^*$,

(c) $|\theta(p) - 1| \leq \delta^*$,

(d) $\mathrm{diam}_{h_{\mathbb{K}}}(\phi(T_p)) \leq \ell^*$,

and for any point $p' \in \gamma$ with $r' := r(p') = a^*r$, then each of the following holds,

(I) $d_{GH}\big((\mathcal{A}^c_{a^*r}(p';1/2,2);d_{a^*r}),([1/2,2];|\ldots|)\big)\leq \epsilon^*/2$,

(II) $\mathrm{diam}_{g_{\mathbb{K}}}(T_{p'})\leq \mathrm{diam}_{g_{\mathbb{K}}}(T_p)/2$,

(III) $|\theta_{r'}(p')-1|\leq B^*\mathrm{diam}^2_{h_{\mathbb{K}}}(T_p)+|\theta_r(p)-1|/2$,

(IV) $|\rho_{r'_i}(p'_i)-\rho_{r_i}(p_i)|\leq B^*\mathrm{diam}^2_{h_{\mathbb{K}}}(T_{p_i})+|\theta_{r_i}(p_i)-1|/2$.

Proof. Proceeding by contradiction we assume that for each $\delta^*_i=1/i$, $\ell^*_i=1/i$ and $B^*_i=i$, there is $p_i\in\gamma$ with $r(p_i)\geq r^*$ satisfying (a)-(d), and there is $p'_i\in\gamma$ with $r'_i=r(p'_i)=a^*r(p_i)$ such that either (I), (II), (III) or (IV) does not hold.

We prove now that for $i\geq i_0$ with i_0 large enough, indeed all (I), (II), (III) and (IV) must hold.

(I) As $i\to\infty$, the metric distance between $(\mathcal{U}_{p_i};\mathfrak{g}_{r_i})$ and $(T_{p_i}\times I_i;\mathfrak{g}^{\mathbb{K}_i}_i)$ tends to zero and at the same time the spaces $(T_{p_i}\times I_i;\mathfrak{g}^{\mathbb{K}_i}_i)$ collapse metrically to a segment of length $(2a^*-1/(2a^*))$. Hence so does $(\mathcal{U}_{p_i};\mathfrak{g}_{r_i})$. As $\mathcal{U}_{p_i}$ contains $\mathcal{A}^c_{r_i}(p_i;1/(2a^*),2a^*)$ and therefore $\mathcal{A}^c_{r'_i}(p'_i;1/2,2)$, this set metrically collapses to a segment of length $2-1/2$. Hence (I) must hold for i sufficiently large.

(II) Let c_i be the Kasner coefficient of the Kasner space $\mathbb{K}_i$. Then by (b), for sufficiently large i we have $c_i\in[\rho^*/4,\rho^*/4+3/8]$ hence (II) must hold by the definition (7.2.26) of a^*.

(III) We write

$$|\theta_{r'_i}(p'_i)-1|\leq|\theta_{r'_i}(p'_i)-\theta^{\mathbb{K}_i}_{a^*}(p'_i)|+|\theta^{\mathbb{K}_i}_{a^*}(p'_i)-1| \tag{7.2.74}$$

where $\theta^{\mathbb{K}_i}_{a^*}(T_{p'_i})$ is the mean curvature of the slice $T_{p'_i}$ with respect to the Kasner metric $(1/a^*)^2\mathfrak{g}^{\mathbb{K}_i}$, namely $\theta^{\mathbb{K}_i}_{a^*}(T_{p'_i})=a^*\theta^{\mathbb{K}_i}(T_{p'_i})$. Similarly, as $r'_i=a^*r_i$ we have $\theta_{r'_i}(p'_i)=a^*\theta_{r_i}(p'_i)$. Therefore for the first term in the right hand side of (7.2.74) we can write

$$|\theta_{r'_i}(p'_i)-\theta^{\mathbb{K}}_{a^*}(p'_i)|=a^*|\theta_{r_i}(p'_i)-\theta^{\mathbb{K}}(p'_i)|\leq C^*_1\mathrm{diam}^2_{h_{\mathbb{K}_i}}(T_{p_i}) \tag{7.2.75}$$

where the last inequality is from (II) in Proposition 7.2.5 with $m^*=2$.

Write the Kasner metric $\mathfrak{g}^{\mathbb{K}_i}$ as

$$\mathfrak{g}^{\mathbb{K}_i}=dx^2+x^{2a_i}d\varphi_1^2+x^{2b_i}d\varphi_2^2=(\lambda^{\mathbb{K}_i})^2dU^2+h^{\mathbb{K}_i} \tag{7.2.76}$$

and let $x(p_i)=x_i$ and $x(p'_i)=x'_i$. Then,

$$\theta^{\mathbb{K}_i}(p_i)=\frac{1}{x_i},\quad\text{and}\quad\theta^{\mathbb{K}}(p'_i)=\frac{1}{x'_i} \tag{7.2.77}$$

and,

$$x'_i-x_i=\int\lambda^{\mathbb{K}_i}dU \tag{7.2.78}$$

where the integral is along any integral line of $\nabla^a U$.

On the other hand the $\mathfrak{g}_{r_i}$-length of the segment of γ between p_i and p'_i, is equal to a^*-1. This length is equal, up to an $O(\mathrm{diam}^2_{h^{\mathbb{K}_i}}(T_{p_i}))$ to the $\mathfrak{g}_{r_i}$-length of any integral line of $\nabla^a U$ between T_{p_i} and $T_{p'_i}$. So,

$$a^*-1=\int\lambda_{r_i}dU+O(\mathrm{diam}^2_{h^{\mathbb{K}_i}}(T_{p_i})) \tag{7.2.79}$$

But by Proposition 7.2.4 we have $|\lambda_{r_i} - \lambda^{\mathbb{K}_i}| \leq O(\text{diam}^2_{h_{\mathbb{K}_i}}(T_{p_i}))$. Substract (7.2.78) and (7.2.79) to get

$$x'_i = x_i + (a^* - 1) + O(\text{diam}^2_{h_{\mathbb{K}_i}}(T_{p_i})) \tag{7.2.80}$$

Thus

$$\theta^{\mathbb{K}_i}_{a^*}(p'_i) = \frac{a^*}{x_i + a^* - 1 + O(\text{diam}^2_{h_{\mathbb{K}_i}}(T_{p_i}))} \tag{7.2.81}$$

Then we calculate

$$|\theta^{\mathbb{K}_i}_{a^*}(p'_i) - 1| = \left|\frac{x_i - 1 + O(\text{diam}^2_{h_{\mathbb{K}_i}}(T_{p_i}))}{x_i + a^* - 1 + O(\text{diam}^2_{h_{\mathbb{K}_i}}(T_{p_i}))}\right| \tag{7.2.82}$$

$$\leq \frac{1}{2}\left|\frac{1}{x_i} - 1\right| + C^*_3 \text{diam}^2_{h_{\mathbb{K}_i}}(T_{p_i}) \tag{7.2.83}$$

where to obtain the bound we used that $x_i \to 1$ and that $a^* \geq 4$ (see definition of a^*). But

$$\frac{1}{x_i} = \theta^{\mathbb{K}_i}(p_i) = \theta^{\mathbb{K}_i}(p_{0i}) = \theta_{r_i}(p_{0i}) \tag{7.2.84}$$

where p_{0i} is the point over T_{p_i} that is used in the construction of $\mathfrak{g}^{\mathbb{K}_i}$ in (II)-C in Proposition 7.2.4. But again by Proposition 7.2.4 we have,

$$|\theta_{r_i}(p_i) - \theta_{r_i}(p_{0i})| \leq C^*_4 \text{diam}^2_{h_{\mathbb{K}_i}}(T_{p_i}) \tag{7.2.85}$$

and thus

$$\left|\frac{1}{x_i} - 1\right| = |\theta_{r_i}(p_{0i}) - 1| \leq |\theta_{r_i}(p_i) - 1| + C^*_4 \text{diam}^2_{h_{\mathbb{K}_i}}(T_{p_i}) \tag{7.2.86}$$

Combining now (7.2.74), (7.2.75), (7.2.82)-(7.2.83) and (7.2.86) we deduce that (III) also holds for i suficiently large.

(IV) This follows the same arguments as in (III). Write,

$$|\rho_{r'_i}(p'_i) - \rho_{r_i}(p_i)| \leq |\rho_{r'_i}(p'_i) - \rho^{\mathbb{K}_i}_{a^*}(p'_i)| + |\rho^{\mathbb{K}_i}(p_i) - \rho_{r_i}(p_i)| \tag{7.2.87}$$

$$+ |\rho^{\mathbb{K}_i}_{a^*}(p'_i) - \rho^{\mathbb{K}_i}(p_i)| \tag{7.2.88}$$

The two terms on the right hand side of (7.2.87) are bounded by $O(\text{diam}^2_{h_{\mathbb{K}_i}}(T_{p_i}))$ by Proposition 7.2.4 with $m^* = 2$. On the other hand following notation as in (III), write $U^{\mathbb{K}_i} = c_i \ln x$ with $c_i \to \rho^*$. Then the term in (7.2.88) is equal to

$$\left|a^* \frac{c_i}{x'_i} - \frac{c_i}{x_i}\right| \tag{7.2.89}$$

and using (7.2.80) we can easily manipulate this expression to obtain the bound

$$|x_i - 1|/2 + O(\text{diam}^2_{h_{\mathbb{K}_i}}(T_{p_i})) \tag{7.2.90}$$

because $a^* \geq 4$ and $\rho^* < 1/2$. Finally use (7.2.86) to bound this expression once more and obtain (IV). □

Theorem 7.2.6. (A characterisation of KA) *Let $(\Sigma; \mathfrak{g}, U)$ be a static end. Let γ be a ray and suppose that there is a sequence $p_i \in \gamma$ such that $\rho_{r_i}(p_i) \to \rho^*$, with $0 < \rho^* < 1/2$*

and that $(\mathcal{A}^c_{r_i}(p_i; 1/2, 2); \mathfrak{g}_{r_i})$ *metrically collapses to a segment* $([1/2, 2]; |\ldots|)$*. Then the end is asymptotically Kasner.*

Proof. For the ρ^* given in the hypothesis and for any integer $m^* \geq 1$ let ϵ^*, μ^*, r^* and C^* be as in Proposition 7.2.4, and let δ^*, ℓ^* and B^* be as in Proposition 7.2.5. We claim that there are $\mu^{**} \leq \mu^*$, $\delta^{**} \leq \delta^*$ and $\ell^{**} \leq \ell^*$ such that if for i big enough the point $p^0 := p_i$ is such that,

(a') $d_{GH}\big((\mathcal{A}^c_r(p^0; 1/2, 2); d_r), ([1/2, 2]; |\ldots|)\big) \leq \epsilon^*$,

(b') $|\rho_{r^0}(p^0) - \rho^*| \leq \mu^{**}$,

(c') $|\theta_{r^0}(p^0) - 1| \leq \delta^{**}$,

(d') $\mathrm{diam}_{h_{\mathbb{K}^0}}(\phi(T_{p^0})) \leq \ell^{**}$,

then for any $p^n \in \gamma$ such that $r(p^n) = (a^*)^n r(p_0)$ we have

(a) $d_{GH}\big((\mathcal{A}^c_{r^n}(p^n; 1/2, 2); d_{r^n}), ([1/2, 2]; |\ldots|)\big) \leq \epsilon^*$,

(b) $|\rho_{r^n}(p^n) - \rho^*| \leq \mu^*$,

(c) $|\theta_{r^n}(p^n) - 1| \leq \delta^*$,

(d) $\mathrm{diam}_{h_{\mathbb{K}^n}}(\phi(T_{p^n})) \leq \ell^* 2^{-n}$,

As by (d) $\mathrm{diam}_{h_{\mathbb{K}^n}}(\phi(T_{p^n})) \to 0$ and (a) and (b) hold for all p^n, then the end is asymptotically Kasner by Proposition 7.2.4.

To choose ϵ^{**}, δ^{**} and μ^{**} we make the following observation. If (a),(b),(c) and (d) hold for p^n, $n = 0, 1, 2, 3, \ldots, m \geq 1$ then, after using the conclusions (I),(II) and (III) in Proposition 7.2.5 m-times (each time use Prop 7.2.5 with $p = p^n, p' = p^{n+1}$) one obtains without difficulty the bounds,

$$\mathrm{diam}_{h_{\mathbb{K}^m}}(\phi(T_{p^m})) \leq \frac{\ell^{**}}{2^{m-1}}, \tag{7.2.91}$$

$$|\theta_{r^m}(p^m) - 1| \leq \frac{mB^*\ell^{**}}{2^{m-1}} + \frac{\delta^{**}}{2^m}, \tag{7.2.92}$$

$$|\rho_{r^n}(p^n) - \rho_{r^0}(p^0)| \leq \sum_{n=1}^{n=m} \left(\frac{B^*(\ell^{**})^2}{2^{2(n-1)}} + \frac{nB^*\ell^{**}}{2^n} + \frac{\delta^{**}}{2^{n+1}} \right) \tag{7.2.93}$$

With them at hand choose $\mu^{**} = \mu^*/4$, and δ^{**} and ℓ^{**} such that the right hand side of (7.2.92) is less or equal than $\delta^*/2$ for all $m \geq 1$ and, when in (7.2.93) we consider $m = \infty$ (i.e. the infinite sum), this sum is less or equal than $\mu^*/4$. Chosed that way it is then trivial that (a),(b),(c) and (d) in Proposition 7.2.5 indeed hold for all p^n, $n = 0, 1, 2, 3, \ldots, \infty$. □

7.2.2 The asymptotic of free S^1*-symmetric states*

Free S^1-symmetric ends have a well defined limit of U at infinity that we denoted by U_∞ (Proposition 6.4.6). In this section we study free S^1-symmetric ends with the property that

$$U(p) \leq U_\infty \tag{7.2.94}$$

for all p. We aim to prove the following theorem.

Theorem 7.2.7. *Let* $(\Sigma; \mathfrak{g}, U)$ *be a static free* S^1*-symmetric end such that* $U(p) \leq U_\infty$ *for all* $p \in \Sigma$*. Then, either the data set is locally M (flat) or is asymptotic is a Kasner different from A and C.*

Suppose $(\Sigma; \mathfrak{g}, U)$ is a data set as in the last proposition. If $U(p) = U_\infty$ at some $p \in \Sigma^\circ$ then U is constant by the maximum principle and the data set is flat. Due to this, from now on we are concerned with the case when $U < U_\infty$.

A large part of the proof of Theorem 7.2.7 is indeed quite general and is valid too for a class of data sets that will show up again crucially in the next section. They are the $\star$-static ends that we define below (the prefix "$\star$" is just a notation).

The level sets of U will be denoted as follows,

$$U_*^{-1} = \{p \in \Sigma : U(p) = U_*\} \tag{7.2.95}$$

For instance $U_1^{-1} = \{p \in \Sigma : U(p) = U_1\}$ and so forth.

As for the critical and regular values of U, it follows from Theorem 1 in [31] that the set of critical values of U is discrete. We will use this information below. Besides of this, the critical set $\{|\nabla U| = 0\}$ is well understood but this won't be necessary here, (see [1]).

Definition 7.2.8. *Let* $(\Sigma; \mathfrak{g}, U)$ *be a (non-necessarily free* S^1*-symmetric) static end. Then, we say that* $(\Sigma; \mathfrak{g}, U)$ *is a* $\star$*-static end iff*

1. *the limit of* U *at infinity exists (denote it by* $U_\infty \leq \infty$*),*
2. $U < U_\infty$ *everywhere,*
3. *there is a regular value* U_0 *of* U*, with* $U_0 > \sup\{U(p) : p \in \partial S\}$*, such that for any regular value* $U_1 \geq U_0$*,* U_1^{-1} *is a connected and compact surface of genus greater than zero.*

Note that $\star$-ends are non-flat. It is also easy to see that any two regular values $U_2 > U_1$ greater or equal than U_0, enclose a compact region Ω_{12}, that is $\partial\Omega_{12} = U_1^{-1} \cup U_2^{-1}$.

The proof of Theorem 7.2.7 follows by from the next three propositions.

Proposition 7.2.9. *Let* $(\Sigma; \mathfrak{g}, U)$ *be a static free* S^1*-symmetric end such that* $U(p) < U_\infty$ *for all* p*. Then* $(\Sigma; \mathfrak{g}, U)$ *is a* $\star$*-static end and has a simple cut* $\{\mathcal{S}_j\}$*.*

Proposition 7.2.10. *Let* $(\Sigma; \mathfrak{g}, U)$ *be a static free* S^1*-symmetric end such that* $U(p) < U_\infty$ *for all* p*. Then the end is asymptotic to a Kasner different from A and C, or has sub-quadratic curvature decay.*

Proposition 7.2.11. *Let* $(\Sigma; \mathfrak{g}, U)$ *be a* $\star$*-static end and let* γ *be a ray. Suppose that the data set has a simple cut* $\{\mathcal{S}_i\}$*. Then the curvature does not decay sub-quadratically along* $\gamma \cup (\cup_j \mathcal{S}_j)$*.*

Proof of Theorem 7.2.7. Direct from Propositions 7.2.9, 7.2.10 and 7.2.11. □

Propositions 7.2.9 and 7.2.10 concern only free S^1-symmetric ends and are simple to prove.

Proof of Proposition 7.2.9. To prove that the data is a $\star$-static data, we need to show only 2 of Definition 7.2.8, items 1 and 2 are verified by hypothesis. Without loss of generality we can assume that the quotient manifold S is diffeomorphic to $\mathrm{S}^1 \times [0, \infty)$

(Propositions 6.4.2, 6.4.4). We work on $(S; q, U, V)$ in particular we think U as a function from S into $\mathbb{R}$. Clearly there is a regular value U_0 such that for any regular value $U_1 \geq U_0$, U_1^{-1} is compact, that is, a collection of circles. None of such circles can be contractible otherwise we would violate the maximum principle. But if there are two such circles, then they enclose a compact manifold (finite cylinder) hence the maximum principle would be also violated. Therefore U_1^{-1} is just diffeomoprhic to S^1. Now thinking U as a function from Σ to $\mathbb{R}$, we have that U_1^{-1} is diffeomorphic to a torus. The existence of a simple cut $\{\mathcal{S}_i\}$ was shown in Proposition 6.4.5. □

Proof of Proposition 7.2.10. We work on $(S; q, U, V)$. Let $\mu := \lim A(B(\partial S, r))/r^2$. If $\mu > 0$ then $(S; q)$ is asymptotic to a cone. Hence κ decays sub-quadratically and therefore so does $|\nabla U|^2$ by (6.1.9). Suppose now that $\mu = 0$. Let γ be a ray from ∂S. If $\mu = 0$ then any sequence of annuli $(\mathcal{A}_{r_i}(p_i; 1/2, 2); q_{r_i})$, with $p_i \in \gamma$, metrically collapses to the segment $[1/2, 2]$. For this reason, if $|\nabla U|^2$ decays sub-quadratically along any sequence $p_i \in \gamma$ then indeed $|\nabla U|^2$ decays sub-quadratically along the end. On the other hand if for a certain sequence p_i, $|\nabla U|^2_{r_i}(p_i) \geq \rho_* > 0$ (ρ_* a given constant), then the end $(\Sigma; \mathfrak{g}, U)$ is indeed asymptotic to a Kasner different from A and C by Proposition 7.2.6. (There is a caveat here. Proposition 7.2.6 requires that for i large enough, the annulus $(\mathcal{A}_{r_i}(p_i; 1/2, 2); \mathfrak{g}_{r_i})$ (annulus in Σ) to be metrically close to the segment $[1/2, 2]$. For i large enough the annulus $(\mathcal{A}_{r_i}(p_i; 1/2, 2); q_{r_i})$ (annulus in S) is close to the segment $[1/2, 2]$, then, if necessary, just make a scaling as in (6.1.8), with $\lambda_i = 1, \mu_i = 0$ and with ν_i small enough that also the annulus $(\mathcal{A}_{r_i}(p_i; 1/2, 2); \mathfrak{g}_{r_i})$ is close to $[1/2, 2]$. Note that such scaling only changes the $\mathfrak{g}$-length of the S^1-fibers in Σ and so doesn't affect the norm $|\nabla U|^2$). □

The proof of Proposition 7.2.11 will be carried out through several steps (Proposition 7.2.12, 7.2.13, 7.2.14, Corollary 7.2.15, and Proposition 7.2.16).

Proposition 7.2.12. *Let $(\Sigma; \mathfrak{g}, U)$ be a $\star$-static end. Let U_0 be a regular value as in Definition 7.2.8 and consider another regular value $U_1 \geq U_0$. Then, the set of points in U_0^{-1} reaching U_1^{-1} in time $U_1 - U_0$ under the flow of ∂_U is a set of total measure in U_0^{-1} and its image is a set of total measure in U_1^{-1}.*

Proof. Denote by Ω_{02} the manifold enclosed by U_0^{-1} and U_2^{-1}. Let $\mathcal{C} = \{p : \nabla U(p) = 0\} \cap \Omega_{12}$ be the set of critical points in Ω_{12}°. The closed set of points C in U_1^{-1} that do not reach U_2^{-1} in time $U_2 - U_1$ under the flow of $\partial_U = \nabla^i U/|\nabla U|^2$, end in a smaller time at a point in $\mathcal{C}$. Let $\phi(x, t) : C \times [0, \infty) \to \Omega_{02}$ be the map generated by the flow of the vector field $\nabla^i U$, (not the collinear field ∂_U), that is, that takes a point x in C and moves it a time t by the flow of $\nabla^i U$ (note that indeed if $x \in C$, then the orbit under the flow of $\nabla^i U$ is defined for all time). Suppose that the area of C is positive. Then the set

$$C_1 = \{\phi(x, t) : x \in C, 0 \leq t \leq 1\} \tag{7.2.96}$$

has positive volume $V(C_1)$. But as U is harmonic the flow of $\nabla^i U$ preserves volume and so we have $V(\phi(C_1, t)) = V(\phi(C_1, 0))$ for all $t \geq 0$. Let $\epsilon > 0$ be small enough that

$$V(B(\mathcal{C}, \epsilon) \setminus \mathcal{C}) < V(C_1)/2 \tag{7.2.97}$$

where $B(\mathcal{C}, \epsilon)$ is the ball of points at a distance less than epsilon from $\mathcal{C}$. Then a contradiction is reached by choosing t large enough that $\phi(C_1, t) \subset B(\mathcal{C}, \epsilon) \setminus \mathcal{C}$ because then it would be

$$V(C_1) = V(\phi(C_1,t)) \leq V(B(\mathcal{C},\epsilon)\setminus \mathcal{C}) < V(C_1)/2 \tag{7.2.98}$$

To show that the image of $U_0^{-1}\setminus C$ under the flow of ∂_U is a set of total measure in U_1^{-1} just reverse the argument using the flow of $-\partial_U$ from U_1^{-1} to U_0^{-1}. □

The following function of the level sets of U, $(U \geq U_0)$, will be central in the analysis later,

$$G(U) = \int_{U^{-1}} |\nabla U|^2 dA \tag{7.2.99}$$

The function $G(U)$ is well defined at least for regular values of U. It is also well defined at the critical values but this won't be needed. As mentioned before Definition 7.2.8, critical values of U are discrete and, as we will show next, the lateral limits of $G(U)$ at any critical value U_c coincide (and are finite). Let us see this property. Let $U_2 > U_1$ be any two regular values with $U_2 > U_c > U_1 \geq U_0$ and let Ω_{12} be the region enclosed them. As in Proposition 7.2.12 let C be the closed set of points in U_1^{-1} that do not reach U_2^{-1} in time $U_2 - U_1$ under the flow of ∂_U. For any $\epsilon > 0$ small enough let $R(\epsilon)$ be an open region in U_1^{-1}, with smooth boundary, containing C, and inside the ball $B(C,\epsilon)$. Let $C_1(\epsilon) = U_1^{-1}\setminus R(\epsilon)$. Let $\Omega_{12}(\epsilon)$ be the union of the set of integral curves (inside Ω_{12}) of ∂_U starting from points in $C_1(\epsilon)$ and ending in U_2^{-1}, and let $C_2(\epsilon)$ be the union of the end-points in U_2^{-1} of these integral curves. Then the divergence theorem gives

$$\int_{C_1(\epsilon)} |\nabla U|^2 dA - \int_{C_2(\epsilon)} |\nabla U|^2 dA = \int_{\Omega_{12}(\epsilon)} \langle \nabla\nabla U, \frac{\nabla U}{|\nabla U|}\nabla U\rangle dV \tag{7.2.100}$$

Take the limit $\epsilon \to 0$ and use Proposition 7.2.12 to deduce,

$$G(U_2) - G(U_1) = \int_{\Omega'_{12}} \langle \nabla\nabla U, \frac{\nabla U}{|\nabla U|}\nabla U\rangle dV \tag{7.2.101}$$

where Ω'_{12} the union of the set of integral curves of ∂_U starting from points in $U_1^{-1}\setminus C$ and ending in U_2^{-1} and is equal to Ω'_{12} minus a set of measure zero. Observe that the integrand is bounded. Take finally the limit $U_1 \uparrow U_c$ and $U \downarrow U_c$ and note that the volume of Ω_{12} tends to zero to get

$$\lim_{U_1\uparrow U_c} G(U) = \lim_{U_2 \downarrow U_c} G(U) \tag{7.2.102}$$

as claimed.

The function $G(U)$ will be thought as defined for all $U \geq U_0$, continuous everywhere and diferentiable except on a discrete set (the critical values of U). The continuity will be used implicitly several times in what follows.

Proposition 7.2.13. *Let $(\Sigma; \mathfrak{g}, U)$ be a $\star$-static end. Let U_0 be a regular value as in Definition 7.2.8. Then for any two regular values $U_2 > U_1 \geq U_0$ we have,*

$$G'(U_2) \geq G'(U_1) \tag{7.2.103}$$

where $G' = dG/dU$.

Proof. Let U_* be a regular value. Identify nearby level sets U^{-1} to U_*^{-1} through the flow of $\partial_U := \nabla^i U/|\nabla U|^2 = n/|\nabla U|$ where n is the unit normal to U^{-1}. As U is harmonic, the form $|\nabla U| dA$ is preserved. With an abuse of notation we write $|\nabla U| dA = |\nabla U_*| dA_*$. Thus

$$G(U) = \int_{U^{-1}} |\nabla U||\nabla U_*| dA_* \tag{7.2.104}$$

Therefore

$$G'(U) = \int_{U^{-1}} (\nabla_n |\nabla U|) \frac{|\nabla U_*|}{|\nabla U|} dA_* = \int_{U^{-1}} \nabla_n |\nabla U| dA \tag{7.2.105}$$

Let Ω_{12} be the region enclosed by U_1^{-1} and U_2^{-1}. Now let $\epsilon > 0$ be a regular value of $|\nabla U|$ smaller than the minimum of $|\nabla U|$ over U_1^{-1} and U_2^{-1}. Let $E = \{p \in \Omega_{12} : |\nabla U|(p) \leq \epsilon\}$. The divergence theorem gives us

$$\int_{U_2^{-1}} \nabla_n |\nabla U| dA = \int_{U_1^{-1}} \nabla_n |\nabla U| dA + \int_{\Omega_{12} \setminus E^\circ} \Delta |\nabla U| dV + \int_{\partial E} \nabla_n |\nabla U| dA \tag{7.2.106}$$

The last term on the right hand side is positive, and the second from last is non-negative because $\Delta |\nabla U| \geq 0$ (use Bochner or just see [3] Lemma 3.5). The proposition follows. ☐

Proposition 7.2.14. *Let $(\Sigma; \mathfrak{g}, U)$ be a $\star$-static end. Let U_0 be a regular value as in Definition 7.2.8. Then, for any two regular values $U_2 \geq U_1 \geq U_0$, we have*

$$\left(\frac{G'}{G}\right)(U_2) \geq \left(\frac{G'}{G}\right)(U_1) \tag{7.2.107}$$

where $G' = dG/dU$.

Proof. First, recall that the set of critical values of U is discrete. We start proving that for any two regular values $U_2 > U_1$ with no critical value in between, the inequality (7.2.107) holds.

We write

$$\mathfrak{g} = \frac{1}{|\nabla U|^2} dU^2 + h \tag{7.2.108}$$

where $\star$ is a two-metric over the leaves U^{-1} between U_1^{-1} and U_2^{-1}. Denote with a prime ($'$) the derivative with respect to $\partial_U = \nabla^i U/|\nabla U|^2$. We will use again the notation $\lambda := 1/|\nabla U|$. Let Θ and θ be the second fundamental form and mean curvature respectively of the leaves U^{-1}.

Fix a leaf U_*^{-1}. Identify the leaves U^{-1} to U_*^{-1} through the flow of ∂_U. As U is harmonic we have $|\nabla U| dA = |\nabla U_*| dA_*$. Hence

$$G = \int_{U^{-1}} |\nabla U|^2 dA = \int_{U^{-1}} \frac{1}{\lambda} |\nabla U_*| dA_*. \tag{7.2.109}$$

As $dA = \lambda |\nabla U_*| dA_*$ and $\theta = (\partial_n dA)/dA$ we deduce $\theta = -(1/\lambda)'$. Thus,

$$G' = -\int_{U^{-1}} \theta |\nabla U_*| dA_* \tag{7.2.110}$$

$$G'' = -\int_{U^{-1}} \theta' |\nabla U_*| dA_* = -\int_{U^{-1}} \frac{\theta'}{\lambda} dA \tag{7.2.111}$$

We use now that in dimension three θ' has the standard expression,

$$\theta' = -\Delta\lambda - (-2\kappa + tr_h Ric + \theta^2)\lambda \tag{7.2.112}$$

to deduce,

$$G'' = -4\pi\chi + \int_{U^1} (\frac{|\nabla\lambda|^2}{\lambda^2} + tr_h Ric) dA + \int_{U^{-1}} \theta^2 dA \tag{7.2.113}$$

where χ is the Euler characteristic of the leaves U^{-1}. On the right hand side of this expression the first two terms are non-negative. For the last term we have

$$\int_{U^{-1}} \theta^2 dA = \int_{U^{-1}} \theta^2 \lambda |\nabla U_*| dA_* \geq \frac{\left(\int_{U^{-1}} \theta |\nabla U_*| dA_* \right)^2}{\int_{U^{-1}} \frac{1}{\lambda} |\nabla U_*| dA_*} = \frac{G'^2}{G} \tag{7.2.114}$$

Therefore,

$$G'' \geq \frac{G'^2}{G} \tag{7.2.115}$$

which is equivalent to $(G'/G)' \geq 0$ from which (7.2.107) follows.

We prove now that (7.2.107) also holds when $U_2 > U_1$ are two regular values, and between them there is only one critical value U_c. This would complete the proof of the proposition. To see this we just compute,

$$\left(\frac{G'}{G}\right)(U_2) \geq \lim_{U \to U_c^+} \left(\frac{G'}{G}\right)(U) = \left(\frac{\lim_{U \to U_c^+} G'(U)}{G(U_c)}\right) \tag{7.2.116}$$

$$\geq \left(\frac{\lim_{U \to U_c^-} G'(U)}{G(U_c)}\right) = \lim_{U \to U_c^-} \left(\frac{G'}{G}\right)(U) \tag{7.2.117}$$

$$\geq \left(\frac{G'}{G}\right)(U_1) \tag{7.2.118}$$

where to pass from (7.2.116) to (7.2.117) we use Proposition 7.2.13 (note $G(U) > 0$ for all U). □

Corollary 7.2.15. *Let $(\Sigma; \mathfrak{g}, U)$ be a $\star$-static end. Then, there is a divergent sequence of points p_i, and constants $C > 0$ and $D > 0$ such that*

$$|\nabla e^{CU}|(p_i) \geq D \tag{7.2.119}$$

Proof. From Proposition 7.2.14 we get

$$G(U) \geq G(U_0) e^{-C(U-U_0)} \tag{7.2.120}$$

where $C = -G'(U_0)/G(U_0)$. If $C \leq 0$ then $G(U) \geq G(U_0)$. But

$$G(U) = \int_{U^{-1}} |\nabla U| |\nabla U_0| dA_0 \tag{7.2.121}$$

which has a fixed integration measure $|\nabla U_0| dA_0$. It follows that there must be a divergent sequence of points p_i for which $|\nabla U|(p_i)$ is bounded away from zero (which is not the case). Thus $C > 0$. In this case we have

$$G(U)e^{CU} \geq G(U_0)e^{CU_0} > 0 \tag{7.2.122}$$

But as

$$G(U)e^{CU} = \int_{U^{-1}} |\nabla e^{CU}||\nabla U_0| dA_0 \tag{7.2.123}$$

again we conclude that there must be a divergent sequence of points p_i and a constant $D > 0$ for which (7.2.119) holds. □

Proposition 7.2.16. *Let $(\Sigma; \mathfrak{g}, U)$ be a $\star$-static end and let γ be a ray. Suppose that the data set has a simple cut $\{\mathcal{S}_i\}$ and that the curvature decays sub-quadratically along $\gamma \cup (\cup_j \mathcal{S}_j)$. Then, for any constant $C > 0$, $|\nabla e^{CU}|$ tends to zero at infinity.*

Proof. Let $\gamma(s)$ be a ray from $\partial\Sigma$ and parametrized by arc-length s, (i.e. $d(\gamma(s), \partial\Sigma) = s$). As we have done before, we will use the notation $r(p) = d(p, \partial\Sigma)$, for $p \in \Sigma$. Thus $r(\gamma(s)) = s$.

As $|\nabla U|^2$ decays faster than quadratically along γ we have,

$$r|\nabla U|(r) \to 0 \quad \text{as} \quad r \to \infty, \tag{7.2.124}$$

where we have denoted $|\nabla U|(\gamma(r))$ by $|\nabla U|(r)$. Let r_0 be such that for all $r \geq r_0$ we have $|\nabla U|(r) \leq 1/(2Cr)$. Integrating we obtain

$$|U(r) - U(r_0)| \leq \frac{1}{2C} \ln \frac{r}{r_0} \tag{7.2.125}$$

where to simplify notation we made $U(r) := U(\gamma(r))$. Thus,

$$e^{CU(r)} \leq c_1 r^{1/2} \tag{7.2.126}$$

We will use this inequality below.

The ray γ intersects $\mathcal{S}_j$ and $\mathcal{S}_{j+1}$. So let $\alpha_{j,j+1}$ be the segment of γ intersecting $\mathcal{S}_j$ and $\mathcal{S}_{j+1}$ only at its end points. Let r_j be the number such that $\gamma(r_j)$ is the end point of $\alpha_{j,j+1}$ in $\mathcal{S}_j$. The connected set

$$Z_j = \mathcal{S}_j \cup \alpha_{j,j+1} \cup \mathcal{S}_{j+1} \tag{7.2.127}$$

is included inside $\mathcal{A}(2^{1+2j}, 2^{4+2j})$. So by Proposition 4.4.5 (with $Z = Z_j$) we deduce,

$$U(q) \leq \eta + U(\gamma(r_j)) \tag{7.2.128}$$

for any q in $\mathcal{S}_j \cup \mathcal{S}_{j+1}$, and where η does not depend on j.

Let $\mathcal{U}_{j,j+1}$ be the compact manifold enclosed by $\mathcal{S}_j$ and $\mathcal{S}_{j+1}$. By the maxumum principle, the maximum of U on $\mathcal{U}_{j,j+1}$ takes place at a point, say x_j, in $\mathcal{S}_j \cup \mathcal{S}_{j+1}$. So,

$$U(x) \leq U(x_j) \tag{7.2.129}$$

for any $x \in \mathcal{U}_{j,j+1}$. Combining this with (7.2.128) with $q = x_j$ we obtain,

$$e^{CU(x)} \leq c_2 e^{CU(\gamma(r_j))} \tag{7.2.130}$$

for any $x \in \mathcal{U}_{j,j+1}$ and where the constant c_2 does not depend on j.

Now, $\mathcal{S}_j$ is included in $\mathcal{A}(2^{1+2j}, 2^{2+2j})$ and so we have,

$$r_j \leq 2^{2+2j} \tag{7.2.131}$$

which plugged in (7.2.126) gives

$$e^{CU(\gamma(r_j))} \leq c_1 2^{1+j} \tag{7.2.132}$$

Combining this bound and (7.2.130) we deduce

$$e^{CU(x)} \leq c_4 2^j \tag{7.2.133}$$

for any $x \in \mathcal{U}_{j,j+1}$ and where c_4 does not depend on j.

On the other hand we also have $\Delta|\nabla U|^2 \geq 0$ and thus the maximum of $|\nabla U|^2$ over $\mathcal{U}_{j,j+1}$ is reached again at $\mathcal{S}_j \cup \mathcal{S}_{j+1}$. From this fact we conclude that for every point $x \in \mathcal{U}_{j,j+1}$ it must be,

$$|\nabla U|(x) \leq \max\{|\nabla U|(q) : q \in \mathcal{S}_j \cup \mathcal{S}_{j+1}\} \leq \frac{c_5}{2^{2j}} \tag{7.2.134}$$

where the constant c_5 does not depend on j and where to obtain the last inequality it was used that $|\nabla U|(q) \leq K/r(q)$ (Anderson's estimate) and the bound $r(q) \geq 2^{1+2j}$ for any $q \in \mathcal{S}_j \cup \mathcal{S}_{j+1}$ because $\mathcal{S}_j \cup \mathcal{S}_{j+1}$ is included in $\mathcal{A}(2^{1+2j}, 2^{4+2j})$.

Let p_j be any divergent sequence such that $p_j \in \mathcal{U}_{j,j+1}$ for each j. Then, using (7.2.133) and (7.2.134) we reach,

$$|\nabla e^{CU}|(p_j) = Ce^{CU(p_j)}|\nabla U|(p_j) \leq \frac{c_6}{2^j} \tag{7.2.135}$$

where c_6 does not depend on j. Thus $|\nabla e^{CU}|(p_j)$ tends to zero as j goes to infinity. As the sequence p_j is arbitrary we have proved the proposition. □

7.2.3 Proof of the KA of static black hole ends

In this section we aim to prove finally Theorem 7.2.1 stating that a static black hole data set with sub-cubic volume growth is indeed AK.

Let Σ be the manifold of a static black hole data. An embedded connected surface $\mathcal{S}$ is *disconnecting* if $\Sigma \setminus \mathcal{S}$ has two connected components one of which contains $\partial\Sigma$ and the other infinity. The closure of the component of $\Sigma \setminus \mathcal{S}$ containing $\partial\Sigma$ is denoted by $\Omega(\partial\Sigma, \mathcal{S})$. For instance, the surfaces $\mathcal{S}_j$ of a simple cut are disconnecting.

For any disconnecting surface S we have,

$$\max\{U(p) : p \in \Omega(\partial\Sigma, S)\} = \max\{U(p) : p \in S\} \tag{7.2.136}$$

by the maximum principle. We will use this simple fact in the proof of the next proposition.

Proposition 7.2.17. *Let $(\Sigma; \mathfrak{g}, U)$ be a static black hole end with sub-cubic volume growth. Let γ be a ray and let $\{\mathcal{S}_j\}$ be a simple cut. Then the end is either asymptotically Kasner or the curvature decays sub-quadratically along the set $\gamma \cup (\cup_j \mathcal{S}_j)$.*

Proof. Suppose that for every n there is a point $p_n \in \gamma \cup (\cup_j \mathcal{S}_j)$ such that

$$|\nabla U|_{r_n}(p_n) \geq \rho_* \tag{7.2.137}$$

for some $\rho_* > 0$. If a subsequence of the annuli $(\mathcal{A}^c_{r_n}(p_n, 1/2, 2); \mathfrak{g}_{r_n})$ collapses to a segment then γ must pass through the annuli $\mathcal{A}^c_{r_n}(p_n, 1/2, 2)$ and the end must be asymptotically Kasner by Theorem 7.2.6. If no subsequence of these annuli metrically collapses to a segment then one can find a subsequence (also indexed by n) and neighbourhoods $\mathcal{B}_n$ of $\mathcal{A}^c_{r_n}(p_n, 1/2, 2)$ such that $(\mathcal{B}_n; \mathfrak{g}_{r_n})$ collapses to a two-dimensional orbifold. Having this, by a diagonal argument, one can find a subsequence of it (also indexed by n) and neighbourhoods $\mathcal{B}_{k_n}$ of $\mathcal{A}^c_{r_n}(p_n; 1/2, 2^{k_n})$, with $k_n \to \infty$, and collapsing to a two-dimensional orbifold $(S_\infty; q_\infty)$. As the collapse is along S^1-fibers (hence defining asymptotically a symmetry), we obtain, in the limit, a well defined reduced data $(S; q, \bar{U}, V)$ where $\overline{U}$ is obtained as the limit of $U_n := U - U(p_n)$. This data has $|\nabla \overline{U}|_q \not\equiv 0$ by (7.2.137) and therefore is non flat. Moreover it has at least one end containing a limit, say $\overline{\gamma}$, of the ray γ. Let us denote that end by $S_{\overline{\gamma}}$.

As observed in Section **??** the limit orbifold has only a finite number of conic points, therefore the basic structure of the asymptotic of the reduced data on the end $S_{\overline{\gamma}}$ is described by Propositions **??** and **??**. Furthermore $\overline{U}$ has a limit value $\overline{U}_\infty \leq \infty$ at infinity by Proposition **??**.

We claim that we must have $\overline{U} \leq \overline{U}_\infty$. Let us see this. Let j_n be an integer such that $j_n \leq r_n = r(p_n) \leq j_n + 1$. As γ intersect all the surfaces $\mathcal{S}_j$, then fixed an integer $k \geq 1$, the surfaces $\mathcal{S}_{j_n+k}$ "collapse into sets" in $S_{\overline{\gamma}}$ as $n \to \infty$. The bigger k is, the farther away the sets "collapse". As $\overline{U} \to \overline{U}_\infty$ over the end $S_{\overline{\gamma}}$ then one can find a sequence $k_n \to \infty$ such that U_n converges to $\overline{U}_\infty$ (as $n \to \infty$) when restricted to the surfaces $S_{j_n+k_n}$. Then, by (7.2.136), we have

$$\max\{U(p) : p \in \Omega(\partial\Sigma, S_{j_n+k_n})\} = \max\{U(p) : p \in S_{j_n+k_n}\} \to \overline{U}_\infty \tag{7.2.138}$$

and the claim follows because if $\overline{U}(q) \geq \overline{U}_\infty + \epsilon$ for some $\epsilon > 0$ and for some $q \in S_{\overline{\gamma}}$ then there is a sequence of points $q_n \in \Omega(\partial\Sigma, S_{j_n+k_n})$ with $U_n(q_n) > \overline{U}_\infty + \epsilon/2$ if $n \geq n_0$, that would eventually violate (7.2.138).

As $(S_{\overline{\gamma}}; q, \overline{U}, V)$ is non-flat then it has to be AK different from the Kasner A and C by Proposition 7.2.7. Therefore one can find a sequence k_n such that the annuli

$$(\mathcal{A}^c(\gamma(r_n 2^{k_n}); r_n 2^{k_n-1}, r_n 2^{k_n+1}); \mathfrak{g}_{r_n 2^{k_n}}), \tag{7.2.139}$$

neighbouring the points $\gamma(r_n 2^{k_n})$, collapse to a segment $[1/2, 2]$ while having

$$|\nabla U|_{\mathfrak{g}_{r_n 2^{k_n}}}(\gamma(r_n 2^{k_n})) \geq \rho^{**} \tag{7.2.140}$$

for some $\rho^{**} > 0$. Then the end must be asymptotically Kasner by Theorem 7.2.6. We reach thus a contradiction. Hence, the curvature decays sub-quadratically along the set $\gamma \cup (\cup_j \mathcal{S}_j)$. □

Corollary 7.2.18. *Let $(\Sigma; \mathfrak{g}, U)$ be a static black hole data set with sub-cubic volume growth that is not AK. Then*

$$(\max\{U(p) : p \in \mathcal{S}_j \cup \mathcal{S}_{j+1}\} - \min\{U(p) : p \in \mathcal{S}_j \cup \mathcal{S}_{j+1}\}) \to 0 \tag{7.2.141}$$

where $\{\mathcal{S}_j\}$ is a simple cut.

Proof. If the data is not AK, then we deduce by Proposition 7.2.17 that for any sequence of points $p_j \in \mathcal{S}_j$ we have

$$|\nabla U|_{r_j}(p_j) \to 0, \tag{7.2.142}$$

where $r_j = r(p_j)$ as usual. Now, if $p_j \in \mathcal{S}_j$ then $2^{1+2j} \leq r_j \leq 2^{4+2j}$, thus (7.2.142) implies right away that,

$$\max\{|\nabla U|_{\hat{r}_j}(q) : q \in \mathcal{S}_{j-1} \cup \mathcal{S}_{j+2}\} \to 0, \qquad (7.2.143)$$

as $j \to \infty$, where we made $\hat{r}_j = 2^{2j}$. Now, as the maximum of $|\nabla U|_{\hat{r}_j}$ on $\mathcal{U}_{j-1,j+2}$ is reached at $\mathcal{S}_{j-1} \cup \mathcal{S}_{j+2}$ we conclude that,

$$\max\{|\nabla U|_{\hat{r}_j}(q) : q \in \mathcal{U}_{j-1,j+2}\} \to 0 \qquad (7.2.144)$$

as $j \to \infty$. Observe that because $\mathcal{S}_j$ and $\mathcal{S}_{j+1}$ are intersected by any ray γ ($\{\mathcal{S}_j\}$ is a simple cut), they belong to the same connected component of $\mathcal{A}(2^{1+2j}, 2^{4+2j}) = \mathcal{A}_{\hat{r}_j}(2,4)$. Denote that component by $\mathcal{A}^c_{\hat{r}_j}(2,4)$. We have,

$$\mathcal{S}_j \cup \mathcal{S}_{j+1} \subset \mathcal{A}^c_{\hat{r}_j}(2,4) \subset \mathcal{A}^c_{\hat{r}_j}(1/2, 2^6) \subset \mathcal{U}_{j-1,j+2} \qquad (7.2.145)$$

and remember that the by (7.2.144) the maximum of $|\nabla U|_{\hat{r}_j}$ over $\mathcal{A}^c_{\hat{r}_j}(1/2, 2^6)$ tends to zero. So (7.2.141) is exactly item 2 in Proposition 4.4.5 with $a = 2$, $b = 4$ and $Z_j = \mathcal{S}_j \cup \mathcal{S}_{j+1}$. □

Proposition 7.2.19. *Let $(\Sigma; \mathfrak{g}, U)$ be a static black hole data set with sub-cubic volume growth. Then U tends uniformly to a constant $U_\infty \leq \infty$ at infinity.*

Proof. The claim is obviously true if the end is AK. Let us assume then that the end is not AK. Let $\{\mathcal{S}_j\}$ be a simple cut and γ a ray. By Corollary 7.2.18 we have,

$$(\max\{U(p) : p \in \mathcal{S}_j \cup \mathcal{S}_{j+1}\} - \min\{U(p) : p \in \mathcal{S}_j \cup \mathcal{S}_{j+1}\}) \to 0 \qquad (7.2.146)$$

And by the maximum principle,

$$\max\{U(p) : p \in \mathcal{S}_j \cup \mathcal{S}_{j+1}\} \geq \max\{U(p) : p \in \mathcal{U}_{j,j+1}\} \geq \qquad (7.2.147)$$

$$\geq \min\{U(p) : p \in \mathcal{U}_{j,j+1}\} \geq \min\{U(p) : p \in \mathcal{S}_j \cup \mathcal{S}_{j+1}\} \qquad (7.2.148)$$

Therefore the function U is becoming more and more constant over the manifolds $\mathcal{U}_{j,j+1}$ enclosed by $\mathcal{S}_j$ and $\mathcal{S}_{j+1}$. A simple application of this fact is that if there is a sequence of manifolds $\mathcal{U}_{j_i,j_i+1}$ over which U tends to infinity then U must tend to infinity over any other sequence $\mathcal{U}_{j'_i,j'_i+1}$, as if not then for some $i_1 < i_2$ the minimum of U over the manifold $\mathcal{U}_{j_{i_1},j_{i_2}}$ enclosed by $\mathcal{S}_{j_{i_1}}$ and $\mathcal{S}_{j_{i_2}}$ would not be reached at a point on either $\mathcal{S}_{j_{i_1}}$ or $\mathcal{S}_{j_{i_2}}$, but rather at a point on a manifold $\mathcal{U}_{j,j+1}$ with $j_{i_1} < j$ and $j + 1 < j_{i_2}$. This would violate the maximum principle. For the same reason if U tends to a finite constant over a sequence of manifolds $\mathcal{U}_{j,j+1}$ then it must tend also to the same constant over any other sequence. □

Proposition 7.2.20. *Let $(\Sigma; \mathfrak{g}, U)$ be a static black hole data set with sub-cubic volume growth. Then, there is a divergent sequence of disconnecting tori embedded in Σ and enclosing solid tori in $\Sigma \cup \mathrm{B}$.*

Proof. By the result of Galloway [?], it is enough to find a divergent sequence of disconnecting tori T_i having outwards mean curvature positive. Let us prove this.

Let $\{\mathcal{S}_j\}$ be a simple cut and let p_j be for each j a point in $\gamma \cap \mathcal{S}_j$. If for a subsequence p_{j_i} the annuli $(\mathcal{A}^c_{r_{j_i}}(p_{j_i}; 1/2, 2); g_{r_{j_i}})$ collapse to a segment $[1/2, 2]$ then there are neighbourhoods $\mathcal{B}_i$ of $\mathcal{A}^c_{r_i}(p_{j_i}; 1/2, 2)$ and finite coverings $\tilde{\mathcal{B}}_i$ such that the sequence $(\tilde{\mathcal{B}}_i; g_{r_{j_i}})$ converges to a $\mathrm{S}^1 \times \mathrm{S}^1$-symmetric space $([1/2, 2] \times T; g_F)$. As by Proposition **??**

Let T_i be a sequence of embedded tori in $\mathcal{B}_i$ such that the coverings $\tilde{T}_i$ converge (in C^2) to the torus $\{1\} \times T$ on $[1/2, 2] \times T$. Observe that as the disconnecting surfaces $\mathcal{S}_{j_i}$ are embedded in $\mathcal{B}_i$ the tori T_i are also disconnecting. If the outwards mean curvature of the torus $\{1\} \times T$ is negative, then so is the mean curvature of the torus T_i for i suficiently large. But this is not possible because as $Ric \geq 0$ any ray from T_i would develop a focal point at a finite distance from T_i. On the other hand if the outwards mean curvature is positive, then for $i \geq i_0$ with i_0 large enough the tori T_i conform the tori we are looking for. So let us suppose that the mean curvature of the torus $\{1\} \times T$ is zero. Then....

□

Proposition 7.2.21. *Let $(\Sigma; \mathfrak{g}, U)$ be a static black hole data set with sub-cubic volume growth. Then, there is a regular value $U_0 < U_\infty$, such for any regular value U_1 of U with $U_\infty > U_1 \geq U_0$, U_1^{-1} is a compact connected surface of genus greater than zero.*

Proof. Suppose on the contrary that there is a sequence of regular values U_i tending to U_∞ such that each U_i^{-1} is a sphere. Clearly such sequence of spheres is divergent (i.e. scapes any compact set). Also, by Proposition 7.2.20, every sphere is embedded inside a solid torus in $\Sigma \cup \mathrm{B}$. Hence, every U_i^{-1} bounds a ball. Thus $\Sigma \cup \mathrm{B}$ must be diffeomorphic to $\mathbb{R}^3$. Hence, the complement of an open set of Σ is diffeomorphic to $\mathrm{S}^2 \times \mathbb{R}^+_0$ and the end must have cubic-volume growth by [28] which is against the hypothesis. □

The next Corollary is direct from Propositions 7.2.19, 7.2.21.

Corollary 7.2.22. *Let $(\Sigma; \mathfrak{g}, U)$ be a static black hole data set with sub-cubic volume growth. Then $(\Sigma; \mathfrak{g}, U)$ is a $\star$-static end.*

We are now ready to prove the Theorem 7.2.1.

Proof of Theorem 7.2.1. Suppose that the data is not AK. Let $\{\mathcal{S}_i\}$ be a simple cut and let γ be a ray. Then, by Proposition 7.2.17, the curvature decays sub-quadratically along $\gamma \cup (\cup \mathcal{S}_i)$. By Corollary 7.2.22 the data is $\star$-static and by Proposition 7.2.11 the curvature cannot decay sub-quadratically along $\gamma \cup (\cup \mathcal{S}_i)$. We obtain a contradiction. Therefore the data is AK. □

8 The classification theorem

Proof of the classification theorem 2.1.6. Let $(\Sigma; g, N)$ be a static black hole data set. By Proposition 4.4.3 we know that one of the following holds,

1. $\partial\Sigma = H$, where H is a two-torus, or,
2. $\partial\Sigma = H_1 \cup \ldots \cup H_h$, $h \geq 1$, where each H_j is a two-sphere, and $(\Sigma; \mathfrak{g})$ has cubic volume growth, or,
3. $\partial\Sigma = H_1 \cup \ldots \cup H_h$, $h \geq 1$, where each H_j is a two-sphere, and $(\Sigma; \mathfrak{g})$ has less than cubic volume growth.

Then depending on whether $1, 2$ or 3 holds, we can conclude the following,

1. If $\partial\Sigma = H$, then the solution is a Boost as explained in Proposition 4.4.3.
2. In this case the state is asymptotically flat (with Schwarzschildian fall off), as discussed in Section 7.1. By Galloway's theorem **??**, Σ is diffeomorphic to $\mathbb{R}^3$ minus $\star$-balls and the uniqueness theorem of Israel-Robinson-Bunting-Masood-um-Alam, shows that the solution is Schwarzschild.
3. By Theorem 7.2.1 the solution is asymptotically Kasner. Moreover, by Galloway's theorem **??**, Σ is diffeomorphic to a solid torus minus a finite number of open balls and by Proposition 4.4.3 the horizons are weakly outermost. Thus, according to Definition 2.1.5, $(\Sigma; g, N)$ is of Korotkin-Nicolai type.

□

References

[1] Virginia Agostiniani and Lorenzo Mazzieri. On the geometry of the level sets of bounded static potentials. 2015.

[2] Michael T. Anderson. On stationary vacuum solutions to the Einstein equations. *Ann. Henri Poincaré*, 1(5):977–994, 2000.

[3] Michael T. Anderson. On the structure of solutions to the static vacuum einstein equations. *Ann. Henri Poincaré*, 1(6):995–1042, 2000.

[4] Gary L. Bunting and A. K. M. Masood-ul Alam. Nonexistence of multiple black holes in asymptotically Euclidean static vacuum space-time. *Gen. Relativity Gravitation*, 19(2):147–154, 1987.

[5] Robinson. D. C. Four decades of black hole uniqueness. In M. Scott D. L. Wiltshire, M. Visser, editor, *The Kerr spacetime: Rotating black holes in General Relativity*, pages 115–143. Cambridge University Press, 2009.

[6] Jeffrey S. Case. The nonexistence of quasi-Einstein metrics. *Pacific J. Math.*, 248(2):277–284, 2010.

[7] Sergio Dain. Axisymmetric evolution of Einstein equations and mass conservation. *Class.Quant.Grav.*, 25:145021, 2008.

[8] Gregory J. Galloway. On the topology of black holes. *Comm. Math. Phys.*, 151(1):53–66, 1993.

[9] Gregory J. Galloway. Rigidity of marginally trapped surfaces and the topology of black holes. *Communications in Analysis and Geometry*, 16(1):217–229, 1 2008.

[10] H. Müller Zum Hagen, David C. Robinson, and H. J. Seifert. Black holes in static vacuum space-times. *General Relativity and Gravitation*, 4(1):53–78, 1973.

[11] Israel. Event horizons in static vacuum space-times. *Phys. Review*, vol.164:5:1776–1779, 1967.

[12] Pascual Jordan, Jürgen Ehlers, and Wolfgang Kundt. Republication of: Exact solutions of the field equations of the general theory of relativity. *General Relativity and Gravitation*, 41(9):2191–2280, 2009.

[13] Edward Kasner. An algebraic solution of the Einstein equations. *Trans. Amer. Math. Soc.*, 27(1):101–105, 1925.

[14] Edward Kasner. Solutions of the Einstein equations involving functions of only one variable. *Trans. Amer. Math. Soc.*, 27(2):155–162, 1925.

[15] D. Korotkin and H. Nicolai. The ernst equation on a riemann surface. *Nuclear Physics B*, 429(1):229 – 254, 1994.

[16] D. Korotkin and H. Nicolai. A periodic analog of the schwarzschild solution. arxiv:gr-qc/9403029v1, 1994.

[17] H. P. Knzle. On the spherical symmetry of a static perfect fluid. *Comm. Math. Phys.*, 20(2):85–100, 1971.

[18] T. Levi-Civita. Republication of: Einsteinian ds 2 in newtonian fields. ix: The analog of the logarithmic potential. *General Relativity and Gravitation*, 43(8):2321–2330, 2011.

[19] T. Levi-Civita. Republication of: The physical reality of some normal spaces of bianchi. *General Relativity and Gravitation*, 43(8):2307–2320, 2011.

[20] Marc Mars and Martin Reiris. Global and uniqueness properties of stationary and static spacetimes with outer trapped surfaces. *Comm. Math. Phys.*, 322(2):633–666, 2013.

[21] R. C. Myers. Higher-dimensional black holes in compactified space-times. *Phys. Rev. D*, 35:455–466, Jan 1987.

[22] Peter Petersen. *Riemannian geometry*, volume 171 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2006.

[23] Martin Reiris. Static solutions of the Einstein-Scalar field equations. arxiv:1507.04570.

[24] Martin Reiris. Static solutions from the point of view of comparison geometry. *J. Math. Phys.*, 53(1):012501, 31, 2012.

[25] Martin Reiris. Stationary solutions and asymptotic flatness I. *Classical Quantum Gravity*, 31(15):155012, 33, 2014.

[26] Martin Reiris. Stationary solutions and asymptotic flatness II. *Classical Quantum Gravity*, 31(15):155013, 18, 2014.

[27] Martin Reiris. The asymptotic of static isolated systems and a generalized uniqueness for schwarzschild. *Classical and Quantum Gravity*, 32(19):195001, 2015.

[28] Martin Reiris. On Ricci curvature and volume growth in dimension three. *J. Differential Geom.*, 99(2):313–357, 2015.

[29] Martín Reiris. On static solutions of the einstein-scalar field equations. *General Relativity and Gravitation*, 49(3):46, 2017.

[30] D.C. Robinson. A simple proof of the generalization of israel's theorem. *General Relativity and Gravitation*, 8(8):695–698, 1977.

[31] J. Souček and V. Souček. Morse-Sard theorem for real-analytic functions. *Comment. Math. Univ. Carolinae*, 13:45–51, 1972.

[32] Robert M. Wald. *General Relativity*. University of Chicago Press, Chicago, IL, 1984.

[33] Guofang Wei and Will Wylie. Comparison geometry for the Bakry-Emery Ricci tensor. *J. Differential Geom.*, 83(2):377–405, 2009.

[34] Hermann Weyl. Zur gravitationstheorie. *Annalen der Physik*, 359(18):117–145, 1917.