

# Linear Perturbations for the Vacuum Axisymmetric Einstein Equations

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**Abstract.** In axial symmetry, there is a gauge for Einstein equations such that the total mass of the spacetime can be written as a conserved, positive definite, integral on the spacelike slices. This property is expected to play an important role in the global evolution. In this gauge the equations reduce to a coupled hyperbolic–elliptic system which is formally singular at the axis. Due to the rather peculiar properties of the system, the local in time existence has proved to resist analysis by standard methods. To analyze the principal part of the equations, which may represent the main source of the difficulties, we study linear perturbation around the flat Minkowski solution in this gauge. In this article we solve this linearized system explicitly in terms of integral transformations in a remarkable simple form. This representation is well suited to obtain useful estimates to apply in the non-linear case.

## 1. Introduction

The study of the gravitational systems with axisymmetry is particularly appealing for at least a pair of important reasons. Firstly, quite a large number of interesting physical phenomena are included and can be described inside this category. Most notably the Kerr family and all its derived physics and mathematics belongs to it. Secondly because certain of its mathematical formulations enjoy a interesting number of remarkable mathematical properties. In particular there is a gauge, called the *maximal-isothermal gauge* in the system reduced by the axisymmetric Killing field where rather important properties, as the positivity of mass, are explicitly manifest and, presumably, would become important in the mathematical investigations of these axisymmetric systems (see [1–3] and references therein).

To take advantages of all this one needs to show, naturally, that the reduced Einstein equations in the maximal-isothermal gauge is a mathematically well posed theory. As it usually happens in coordinates systems adapted

to axial symmetry, the equations are formally singular at the axis. It is the particular combination of such formally singular terms inside the whole system of equations what makes it difficult to treat. Until now no well posedness result has been given in the literature.

The analysis of the linearized equations inherit similar difficulties and therefore has proved to be non-trivial (see the discussion in [4], where this system was analyzed numerically). In this article we solve precisely this linear problem. The remarkable algebraic properties that this system has, allows us to solve it by a combination of integral transformations suitable adapted to these equations. This solution is the perfect analog to the solution in terms of Fourier transform of a constant coefficient equation. The construction appears to be finely adapted to this particular kind of singular hyperbolic–elliptic systems.

We expect that this representation will be useful in the future to obtain relevant estimates for the non-linear case. In this sense, we believe that the present result opens the possibility to analyze the full axially symmetric Einstein equations in the maximal-isothermal gauge.

The plan of the article is the following. In Sect. 1.1 we present our main result. At the end of this section we present the strategy of the proof, which is split in three main parts discussed in Sects. 2.1, 2.2 and 2.3, respectively. Finally, we include an Appendix in which we collect some properties of Bessel functions used in this article.

### 1.1. Statement of the Main Result

In the maximal-isothermal gauge, the linearized Einstein equations with respect to a flat background (in the twist-free case) reduce to the following system of equations for the functions  $v$  and  $\beta$

$$\ddot{v} = \Delta v - \frac{\partial_\rho v}{\rho} + \rho \partial_\rho \left( \frac{\beta}{\rho} \right), \quad (1)$$

$$\Delta \beta = \frac{2}{\rho} \left( \Delta v - \frac{\partial_\rho v}{\rho} \right). \quad (2)$$

See [4] for the deduction and physical discussion of these equations.<sup>1</sup>

In these equations the coordinates are  $(t, \rho, z)$ . The relevant domain is the half plane  $0 \leq \rho, -\infty < z < \infty$ , which is denoted by  $\mathbb{R}_+^2$ . The axis is given by  $\rho = 0$  and it defines the boundary of the domain  $\mathbb{R}_+^2$ . A dot denotes time derivative,  $\Delta$  is the flat Laplacian in two dimensions

$$\Delta v = \partial_\rho^2 v + \partial_z^2 v, \quad (3)$$

and  $\partial$  denotes partial derivative with respect to the spatial coordinates  $\rho$  and  $z$ .

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<sup>1</sup> We have slightly changed the notation with respect to this reference where the function  $\beta$  was denoted by  $\beta^\rho$  to point out that it is the  $\rho$  component of the shift vector. Since in this article we will not discuss the other components of the linear perturbation (which can all be written in terms of  $v$  and  $\beta$ ) to simplify the notation we drop the index  $\rho$ .

The solutions  $(v(t, \rho, z), \beta(t, \rho, z))$  we seek for in this article are  $C^\infty$  functions of  $\mathbb{R} \times \mathbb{R}_+^2$  (meaning there is a  $C^\infty$  extension of  $(v, \beta)$  into a open neighborhood of  $\mathbb{R} \times \mathbb{R}_+^2$  in  $\mathbb{R} \times \mathbb{R}^2$ ). For linearized gravity it is necessary to impose the following boundary and asymptotic conditions (see [4]).

For  $\beta$  it is required

1.  $\beta(t, 0, z) = 0$  and for every fixed  $(t, z), \beta(t, \rho, z)$  is an odd function of  $\rho$ ,
2. For every fixed time  $(t), \beta = O(r^{-1})$ , and  $\partial^k \beta = O(r^{-1-k})$ , where  $r = \sqrt{\rho^2 + z^2}$ .

For  $v$  it is required

1.  $v(t, 0, z) = (\partial_\rho v)(t, 0, z) = 0$  and for every fixed  $(t, z), v(t, \rho, z)$  is an even function of  $\rho$ .
2. For every fixed time  $(t), v = O(r^{-2})$  and  $\partial^k v = O(r^{-2-k})$ .

Equation (1) is a wave equation for  $v$  and so it is necessary to prescribe as initial data, roughly speaking, the *position* and *velocity* at the initial time ( $t = 0$ ) which we will denote as  $v|_{t=0}$  and  $\dot{v}|_{t=0}$ .

*Remark 1.* From the series expansion argument presented in [4] it follows that, for analytic solutions, the requirement that  $v$  is even (in  $\rho$ ) and  $\beta$  odd (for all times), follows only from the condition  $\beta(t, 0, z) = 0$  and  $v(0, 0, z) = (\partial_\rho v)(0, 0, z) = 0$ .

The strategy to solve the system (1)–(2) is to use an appropriate integral transformation to the whole set of equations to obtain simpler ones in the transformed variables. This integral transform is a combination of a Fourier transformation in the  $z$  coordinate and a Hankel transform in the  $\rho$  coordinate (see the Appendix A for a definition of the Hankel transform). The explicit form of the integral transform and its inverse are given by

$$w(t, \rho, z) = \int_{\mathbb{R}_+^2} \hat{w}(t, k, \lambda) (k|\lambda|\rho)^{\frac{1}{2}} J_1(k|\lambda|\rho) e^{2\pi i \lambda z} dk d\lambda, \tag{4}$$

$$\frac{\hat{w}(t, k, \lambda)}{|\lambda|} = \int_{\mathbb{R}_+^2} w(t, \rho, z) (k|\lambda|\rho)^{\frac{1}{2}} J_1(k|\lambda|\rho) e^{-2\pi i \lambda z} d\rho dz, \tag{5}$$

where  $J_1$  is the Bessel function of the first kind of order one. The ranges for the variables are  $0 \leq k < \infty$  and  $-\infty < \lambda < \infty$ . They define the same domain of integration as the variables  $(\rho, z)$  and hence we denote it by the same symbol  $\mathbb{R}_+^2$ . By analogy to standard physical terminology we will call the space comprised by  $(\rho, z)$  the *physical space*, while the one comprised by  $(k|\lambda|, \lambda)$  will be called the *momentum space*.

**Theorem 1** (Representation of solutions). *Let  $F(k, \lambda)$  and  $G(k, \lambda)$  be two arbitrary smooth functions of compact support in  $\mathbb{R}_+^2$ . Define  $\hat{w}$  and  $\hat{\gamma}$  by*

$$\begin{aligned} \hat{w}(t, k, \lambda) = & \frac{k^{\frac{3}{2}}}{(1+k^2)^{\frac{1}{2}}} \int_k^\infty \left[ F(\bar{k}, \lambda) \cos\left(\lambda t(1+\bar{k}^2)^{\frac{1}{2}}\right) \right. \\ & \left. + \frac{G(\bar{k}, \lambda)}{\lambda(1+\bar{k}^2)^{\frac{1}{2}}} \sin\left(\lambda t(1+\bar{k}^2)^{\frac{1}{2}}\right) \right] d\bar{k}, \end{aligned} \quad (6)$$

and

$$\hat{\gamma}(t, k, \lambda) = 2\hat{w}(t, k, \lambda) + \frac{2k^{\frac{3}{2}}}{(1+k^2)^{\frac{3}{2}}} \int_k^\infty \frac{(1+\bar{k}^2)^{\frac{1}{2}}}{\bar{k}^{\frac{1}{2}}} \hat{w}(t, \bar{k}, \lambda) d\bar{k}. \quad (7)$$

For  $\hat{w}$  and  $\hat{\gamma}$  define  $w$  and  $\gamma$  by the integral transformation (4). Then, the functions  $v = \rho^{\frac{1}{2}}w$  and  $\beta = \rho^{-\frac{1}{2}}\gamma$  define a solution of Eqs. (1)–(2), whose initial data  $v|_{t=0}$  and  $\dot{v}|_{t=0}$  is given from  $F$  and  $G$  by

$$v|_{t=0} = \sqrt{\rho} \int_{\mathbb{R}_+^2} \frac{k^{\frac{3}{2}}}{(1+k^2)^{\frac{1}{2}}} \left( \int_k^\infty F(\bar{k}, \lambda) d\bar{k} \right) (k|\lambda\rho)^{\frac{1}{2}} J_1(k|\lambda\rho) e^{2\pi i \lambda z} dk d\lambda, \quad (8)$$

$$\dot{v}|_{t=0} = \sqrt{\rho} \int_{\mathbb{R}_+^2} \frac{k^{\frac{3}{2}}}{(1+k^2)^{\frac{1}{2}}} \left( \int_k^\infty G(\bar{k}, \lambda) d\bar{k} \right) (k|\lambda\rho)^{\frac{1}{2}} J_1(k|\lambda\rho) e^{2\pi i \lambda z} dk d\lambda. \quad (9)$$

In this representation of solutions the boundary conditions required for  $\beta$  and  $v$  at the axis (*items 1* above) are automatically satisfied.

In Theorem 1 we have used arbitrary functions  $F$  and  $G$  of compact support in the space  $(k, \lambda)$ . This family of generating functions is at the same time simple and rich, and allows us to avoid technical lengthy developments in the proof. The main purpose of the present article is to introduce the generating formulas (6) and (7) in the most direct and comprehensive fashion.

There exists, however, a number of important issues that remains to be studied. It is desirable to have a characterization in physical space of the fall properties of the solution generated by a compactly supported  $F$  and  $G$ . Such characterization turns out to be more complicated than we have expected. The reason is that in the formula for the Fourier–Hankel transform the two variables  $k$  and  $\lambda$  (corresponding to Fourier and Hankel, respectively) appear entangled. That is, we can not analyze the behavior of this transform using standard theorems for Fourier and Hankel separately. We need to develop new kind of isomorphism theorems and a new kind of functional spaces (with appropriate fall off properties) for the Fourier–Hankel transform.

In particular, this analysis should allow us to prove that the solutions constructed in Theorem 1 satisfy the asymptotic conditions at infinity discussed in *items 2* above. We expect that these conditions are satisfied for solutions

generated with  $F$  and  $G$  of compact support and also for a much wider kind of functions. They appear, however, to be difficult to establish, they will require extensive technical analysis.

The asymptotic behavior of the solutions are connected also with another important issue not covered by Theorem 1, namely uniqueness. For solutions with decay given by *items 2*, uniqueness follows from the mass conservation formula. In order to define the mass we need to construct an additional function  $\sigma$  defined in terms of  $v$  by the following elliptic equation

$$\Delta\sigma + \frac{\partial_\rho\sigma}{\rho} = -\Delta v, \quad (10)$$

The boundary boundary conditions for Eq. (10) are the following. At the axis we require

$$\partial_\rho\sigma|_{\rho=0} = 0, \quad (11)$$

and at infinity

$$\sigma = O(r^{-1}). \quad (12)$$

The total mass of the system is given by the following integral

$$m = \frac{1}{16} \int_{\mathbb{R}_+^2} \left( 4 \frac{|\partial v|^2}{\rho^2} + (\Delta v)^2 + |\partial\sigma|^2 \right) \rho d\rho dz. \quad (13)$$

where we used the notation  $|\partial v|^2 = (\partial_\rho v)^2 + (\partial_z v)^2$ .

For solutions with the decay given in *items 2*, the integral (13) is conserved. That is,

$$\dot{m} = 0. \quad (14)$$

See [4] for a proof of this result for the this system and [1] for the analog result for the full Einstein equations. From the conservation (14), it follows directly a uniqueness result for solutions with finite mass.

Finally, we want to make some comments regarding the general strategy of the proof of Theorem 1. The proof is divided in three steps. In the first one, we use the scale invariance of the system to introduce new rescaled variables. The structure of the equations simplify considerable in these new variables. The essential point is that the original variables  $v$  and  $\beta$  have different scale behavior, the new rescaled variables are constructed in such a way that both unknowns have the same scale behavior. This is done in Sect. 2.1. The second step (Sect. 2.2) is to use the standard Fourier transform in the  $z$  coordinate that essentially eliminates the  $z$  dependence of the equations. Finally, in Sect. 2.3 we analyze the  $\rho$  dependence of the equations using the Hankel transform. This is the most important part of the article. The proof of Theorem 1 is given thereafter.

There is one key point in the proof which we want to clearly highlight here. The fact that the Fourier and Hankel transform can play an useful role in this system of equations can be expected a priori. For the Fourier transform in  $z$  this is obvious since the coefficients of the equations do not depend on  $z$ .

The differential operator in  $\rho$  acting on  $v$  in Eqs. (1)–(2) is very similar to the Laplace operator in cylindrical coordinates. This certainly suggests that the appropriate integral transform in  $\rho$  is the Hankel transform. The behavior of the Hankel transform on the differential operator acting on  $\beta$  is less obvious. It turns out that it can be characterized by a remarkable simple integral expression provided in Lemma 1. But in any case, even without such characterization, it is clear that for this kind of operators (namely, derivatives with respect to  $\rho$  times some power of  $\rho$ ) it is possible to compute the Hankel transform in some way. From these considerations, it would appear that an analog of Theorem 1 will hold for a whole family of similar equations. However, this is not the case. This is an important point, which makes the system of Eqs. (1)–(2) very peculiar. The reason is the following.

The elliptic equation (2) can be solved, for a given function  $v$ , using a Green function. We can insert the solution  $\beta$  into Eq. (1) to reduce the system to only one integro-differential equation for  $v$ . It is a priori not clear at all how to solve this integro-differential equation due to the singular behavior at the axis. If we take an integral transform to the system, we get also an integro-differential equation in momentum space (see Eq. (66)). This equation is, in principle, as complicated as the original equation in physical space. There is no reason a priori to expect a simplification in momentum space of this problem. Remarkably enough, for the system (1)–(2) this integro-differential equation in momentum space can be reduced to a pure differential equation (see Eq. (68)). This is possible because a cancellation occurs in the equations (see the discussion after Eq. (68)). This cancellation depends on the particular coefficients present in the system (1)–(2). Namely, if we take a similar system with the same differential operators but with different constant coefficients, this cancellation will not occur and hence an analogous of theorem 1 can not be proved in that case (see, however, the remark in footnote 2). It appears that this cancellation describes a very particular kind of hyperbolic–elliptic systems.

## 2. The Equations in Momentum Space

### 2.1. Scaling Symmetry

Equations (1)–(2) enjoy scaling symmetry (see [4]). This symmetry will play a fundamental role in the analysis of these equations. Let us describe this property. For a given solution  $v(t, \rho, z)$  and  $\beta(t, \rho, z)$  of Eqs. (1)–(2) we define the rescaled functions as

$$v_s(\hat{t}, \hat{\rho}, \hat{z}) = v\left(\frac{t}{s}, \frac{\rho}{s}, \frac{z}{s}\right), \quad \beta_s = \frac{1}{s}\beta\left(\frac{t}{s}, \frac{\rho}{s}, \frac{z}{s}\right), \quad (15)$$

where

$$\hat{t} = \frac{t}{s}, \quad \hat{\rho} = \frac{\rho}{s}, \quad \hat{z} = \frac{z}{s}. \quad (16)$$

and  $s$  is a positive real number. Then,  $v_s$  and  $\beta_s$  define also a solution of Eqs. (1)–(2) written in terms of the rescaled coordinates  $(\hat{t}, \hat{\rho}, \hat{z})$ .

Note that  $v$  and  $\beta$  have different scaling. This difference manifests also in the power expansion series of these functions.

To analyze the equations it is very convenient to introduce rescaled functions in such a way that both unknowns have the same scaling and the same behavior near the axis. Let us consider  $w$  and  $\gamma$  defined by

$$w = \frac{v}{\sqrt{\rho}}, \quad \gamma = \sqrt{\rho}\beta. \quad (17)$$

In terms of these variables, Eqs. (1)–(2) are given by

$$\ddot{w} = \Delta w - \frac{3}{4} \frac{w}{\rho^2} + \sqrt{\rho} \partial_\rho \left( \gamma \rho^{-3/2} \right), \quad (18)$$

$$\Delta \gamma - \frac{\partial_\rho \gamma}{\rho} + \frac{3}{4} \frac{\gamma}{\rho^2} = 2 \left( \Delta w - \frac{3}{4} \frac{w}{\rho^2} \right). \quad (19)$$

The functions  $(w, \gamma)$  are scale invariant in the following sense. For a given solution  $(w, \gamma)$  of Eqs. (18)–(19) the rescaled functions

$$w_s(\hat{t}, \hat{\rho}, \hat{z}) = w \left( \frac{t}{s}, \frac{\rho}{s}, \frac{z}{s} \right), \quad \gamma_s(\hat{t}, \hat{\rho}, \hat{z}) = \gamma \left( \frac{t}{s}, \frac{\rho}{s}, \frac{z}{s} \right), \quad (20)$$

define also a solution of these equations in terms of the rescaled coordinates (16). Also, for a given solution, we expect  $w$  and  $\gamma$  to have the same behavior at the axis, namely  $w = \gamma = O(\rho^{3/2})$ .

*Remark 2.* The important part of the rescaling (17) is the relative power of  $\rho$  between  $v$  and  $\beta$  which compensates the different scale behavior. We could consider scalings of the form  $v = \rho^\alpha w, \beta = \rho^{\alpha-1} \gamma$  for any arbitrary real number  $\alpha$ . The specific power chosen in (17) is motivated by the Hankel transform (see Eqs. (40)–(41) in Sect. 2.3). This choice makes the appropriate integral transformation symmetric with respect to its inverse. Another possible choice is  $\alpha = 2$ . This power has the advantage that both functions  $w$  and  $\gamma$  are even in  $\rho$ . However, the related Hankel transform is not symmetric with respect to its inverse and the formula does not coincide with the definition used in the literature. Hence to apply standard results concerning the Hankel transform we need to translate them into this new definition. This makes the proofs laborious, but there is no essential difficulty.

The differential operators in the spatial coordinates involved in this equation are given by

$$\mathbf{P}(w) = \Delta w - \frac{3}{4} \frac{w}{\rho^2}, \quad (21)$$

and

$$T(\gamma) = \sqrt{\rho} \partial_\rho \left( \gamma \rho^{-3/2} \right). \quad (22)$$

The distinction in the notation between  $\mathbf{P}$  and  $T$  (boldface for  $\mathbf{P}$ ) is to emphasize that the differential operator  $\mathbf{P}$  acts in both coordinates  $\rho$  and  $z$  while  $T$  does only in the  $\rho$  coordinate. Later on, we will define the operator  $P$  as the

$\rho$  part of  $\mathbf{P}$  (see Eq. (29)). As we will see, all the important features of the equations are contained in the  $\rho$  dependence.

A remarkable fact is that the operators (21) and (22) are also the natural operators for the second equation (19). Namely, both equations (18)–(19) are written in terms of  $\mathbf{P}$  and  $T$  as follows

$$\ddot{w} = \mathbf{P}(w) + T(\gamma), \quad (23)$$

$$\mathbf{P}(\gamma) - T(\gamma) = 2\mathbf{P}(w). \quad (24)$$

Note that this symmetry between both equations is not evident in terms of the original variables  $v$  and  $\beta$ . The fact that in the second equation (24) appears precisely this combination of  $\mathbf{P}$  and  $T$  will be crucial. *Our proofs will not work if we insert different (constant) coefficients in front of  $\mathbf{P}$  and  $T$  in these equations.*<sup>2</sup>

*Remark 3.* The operator  $\mathbf{P}$  is singular at the axis. However, this kind of singular behavior is essentially the same as the one of the Laplace operator in  $n$ -dimensions for axially symmetric functions written in terms of cylindrical coordinates (see, for example, the introduction of [6]). A standard trick to avoid this problem is precisely to work in a higher dimensional space in which the operator is regular. This can be done also in the case of the operator  $\mathbf{P}$ . However, the operator  $T$  will not be regular in that higher dimensional space. It appears not to be possible to write Eqs. (23)–(24) as regular equations on a single higher dimensional space.

It is the presence of the operator  $T$  in these equations which makes them so peculiar. The operator  $T$  is, outside the axis, a first order operator but at the axis it is a second order operator (due to L'Hopital rule). This behavior indicates that we can not decompose (23)–(24) as a principal part (containing only the second order operator  $\mathbf{P}$ ) plus some lower order terms (containing only the operator  $T$ ). This kind of decomposition is essential to construct an iteration scheme in which each of the equations is solved in alternative steps of the iteration. Outside the axis this iteration scheme can be constructed, but it appears not to be possible to include the axis (see the heuristic discussion in [4]). In fact, our analysis suggests that the system (23)–(24) should be viewed as a unity that can not be further decomposed.

## 2.2. The Fourier Transform in $z$

In Eqs. (23)–(24) the  $z$  dependence is clearly simpler than the  $\rho$  dependence. The equations are regular in  $z$  and the coefficients of the differential operators do not depend on  $z$ . Hence, in order to factor out the  $z$  dependence we can

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<sup>2</sup> What is remarkable about this particular choice of coefficients is that, after a Fourier transform in  $z$  (Sect. 2.2) and a Hankel transform in  $\rho$  (Sect. 2.3), the resulting system can be reduced into a ordinary harmonic oscillator equation (in time) as is displayed in Eq. (68). It is worth noting, however, that with other constant coefficients the system (23)–(24) could be further explored by separation of variables. We would like to thank the referee for pointing out this remark.



use the Fourier transform in this coordinate defined in the standard way by

$$\mathfrak{F}(f) = \tilde{f}(\lambda) = \int_{-\infty}^{\infty} f(z)e^{-2\pi i\lambda z} dz, \tag{25}$$

$$\mathfrak{F}^{-1}(\tilde{f}) = f(z) = \int_{-\infty}^{\infty} \tilde{f}(\lambda)e^{2\pi i\lambda z} d\lambda. \tag{26}$$

Taking the Fourier transform in  $z$  to Eqs. (23)–(24), we obtain the following equations for the transformed functions  $\tilde{w}(t, \rho, \lambda), \tilde{\gamma}(t, \rho, \lambda)$

$$\ddot{\tilde{w}} = P(\tilde{w}) - \lambda^2\tilde{w} + T(\tilde{\gamma}), \tag{27}$$

$$P(\tilde{\gamma}) - \lambda^2\tilde{\gamma} - T(\tilde{\gamma}) = 2(P(\tilde{w}) - \lambda^2\tilde{w}), \tag{28}$$

where we have defined  $P$  as the  $\rho$  part of the operator  $\mathbf{P}$ , namely

$$P(\tilde{w}) = \tilde{w}'' - \frac{3\tilde{w}}{4\rho^2}. \tag{29}$$

A prime denotes derivative with respect to  $\rho$ .

We use the scaling symmetry to reduce these equations to the case  $\lambda = 1$  in the following way. Define rescaled variables

$$\tilde{t} = t\lambda, \quad \tilde{\rho} = \rho|\lambda|, \tag{30}$$

then the rescaled functions  $\tilde{w}_1(\tilde{t}, \tilde{\rho}), \tilde{\gamma}_1(\tilde{t}, \tilde{\rho})$  (no  $\lambda$  dependence) satisfy the equations

$$\ddot{\tilde{w}}_1 = P(\tilde{w}_1) - \tilde{w}_1 + T(\tilde{\gamma}_1), \tag{31}$$

$$P(\tilde{\gamma}_1) - \tilde{\gamma}_1 - T(\tilde{\gamma}_1) = 2(P(\tilde{w}_1) - \tilde{w}_1). \tag{32}$$

In these equations the derivatives are taken with respect to the rescaled coordinates (30). If we have a solution  $\tilde{w}_1(\tilde{t}, \tilde{\rho}), \tilde{\gamma}_1(\tilde{t}, \tilde{\rho})$  of Eqs. (31)–(32), the solution of the original Eqs. (27)–(28) is given by

$$\tilde{w}(t, \rho, \lambda) = \tilde{w}_1(t\lambda, \rho|\lambda|), \quad \tilde{\gamma}(t, \rho, \lambda) = \tilde{\gamma}_1(t\lambda, \rho|\lambda|. \tag{33}$$

The set of reduced Eqs. (31)–(32) constitute our main equations. They encode all the main difficulties of the original equations. They will be solved in the next section.

### 2.3. The Hankel Transform in $\rho$

To simplify the notation, let us write Eqs. (31)–(32) without the tilde and without the subscript 1, namely

$$\ddot{w} = P(w) - w + T(\gamma), \tag{34}$$

$$P(\gamma) - \gamma - T(\gamma) = 2(P(w) - w). \tag{35}$$

The strategy to solve these equations is to expand the solution in terms of eigenfunctions of the operator  $P$ . That is, as a first step we look for a solution  $j$  of the eigenvalue equation for  $P$

$$P(j) = -k^2j. \tag{36}$$

By the same scaling argument used in the previous section (with  $\lambda$  replaced by  $k$ ), it is enough to consider the case  $k = 1$

$$P(j) = -j. \quad (37)$$

Set  $j = \sqrt{\rho}J$ , then Eq. (37) in terms of  $J$  is given by

$$\rho^2 J'' + \rho J' + (\rho^2 - 1)J = 0. \quad (38)$$

This is the Bessel equation (see Eq. (72) in the Appendix). The behavior at the axis fixes the solution  $J$  to be, up to a factor, the Bessel function of the first kind of order one, denoted by  $J_1(\rho)$ . Thus,  $j = c\sqrt{\rho}J_1$  we take  $c = \sqrt{k}$ . After rescaling, we have that the eigenfunctions of (36) are given by

$$j = \sqrt{k\rho}J_1(k\rho). \quad (39)$$

The solutions of Eqs. (34)–(35) will be constructed as a linear superposition of the eigenfunctions  $j$ . As in the case of the Fourier transform with respect to the plane waves  $e^{2\pi i\lambda z}$ , the superposition of the eigenfunctions (39) lead to the following integral transforms

$$\mathfrak{H}(f) = \hat{f}(k) = \int_0^\infty f(\rho)\sqrt{k\rho}J_1(k\rho) d\rho, \quad (40)$$

$$\mathfrak{H}^{-1}(\hat{f}) = f(\rho) = \int_0^\infty \hat{f}(k)\sqrt{k\rho}J_1(k\rho) dk. \quad (41)$$

These are Hankel transformations of first order (see Eqs. (78)–(79) and also [7] for further properties of the Hankel transform). The orthogonality property of the Bessel function (see Eq. (74)) ensure that  $\mathfrak{H}^{-1}\mathfrak{H}(f) = f$  for  $f$  in an appropriate functional space (see the Appendix). We will call the space of  $f(\rho)$  the “physical space” and the space of  $\hat{f}(k)$  the “momentum space”.

The rescaling (17) has been tailored to obtain precisely this form of the Hankel transform. Other rescaling will produce integral transforms which are not symmetric with respect to their inverse. They will have different weights in  $\rho$ .

Let  $\hat{f} = \mathfrak{H}(f)$  and  $\hat{g} = \mathfrak{H}(g)$ , then we have the following Parseval type of identity

$$\int_0^\infty f(\rho)g(\rho) d\rho = \int_0^\infty \hat{f}(k)\hat{g}(k) dk. \quad (42)$$

That is, we have the following two identical inner products

$$\langle f, g \rangle = \int_0^\infty f(\rho)g(\rho) d\rho, \quad \langle \hat{f}, \hat{g} \rangle = \int_0^\infty \hat{f}(k)\hat{g}(k) dk, \quad (43)$$

defined naturally in the physical and in the momentum space, respectively.

The crucial property of the Hankel transformation  $\mathfrak{H}$  for our purposes is its behavior with respect to the differential operator  $P$ , namely

$$\mathfrak{H}(P(v)) = -k^2 \mathfrak{H}(v). \tag{44}$$

This property follows after integration by parts and use of Eq. (36). Note also that the operator  $P$  is self adjoint with respect to  $\langle , \rangle$ , namely

$$\langle w, P(v) \rangle = \langle P(w), v \rangle. \tag{45}$$

**2.4. The Hankel Transform Acting on the Differential Operators**

To solve Eqs. (34)–(35) we will apply the Hankel transform (40) to obtain simpler equations in momentum space. By Eq. (44), the Hankel transform  $\mathfrak{H}$  acts naturally on the operator  $P$ . However, this is not the case for the operator  $T$ . This operator is the main source of the difficulties. The next lemma characterize the action of  $\mathfrak{H}$  on  $T$ .

**Lemma 1.** *Let  $\hat{\gamma}(k) = \mathfrak{H}(\gamma)$  be of compact support. Then, the following relation holds*

$$\mathfrak{H}(T(\gamma))(k) = -k^{\frac{3}{2}} E(k), \tag{46}$$

where we have defined

$$E(k) = \int_k^\infty \frac{\hat{\gamma}(\bar{k})}{\bar{k}^{\frac{1}{2}}} d\bar{k}. \tag{47}$$

which is a function of compact support.

Note that the function  $E(k)$  satisfies the following equation

$$\frac{dE(k)}{dk} = -\frac{\hat{\gamma}(k)}{k^{\frac{1}{2}}}. \tag{48}$$

This relation will be useful later on.

*Proof.* From the definition we have that

$$\mathfrak{H}(T(\gamma))(k) = \int_0^\infty \sqrt{\rho} \partial_\rho \left( \gamma \rho^{-3/2} \right) (\rho k)^{\frac{1}{2}} J_1(k\rho) d\rho, \tag{49}$$

and

$$\gamma(\rho) = \int_0^\infty \hat{\gamma}(\bar{k}) (\rho \bar{k})^{\frac{1}{2}} J_1(\bar{k}\rho) d\bar{k}. \tag{50}$$

Multiplying by  $\rho^{-3/2}$  and taking a  $\rho$  derivative in Eq. (50) under the integral, we obtain

$$\partial_\rho \left( \gamma \rho^{-\frac{3}{2}} \right) = \int_0^\infty \hat{\gamma}(\bar{k}) \bar{k}^{\frac{1}{2}} \partial_\rho \left( \frac{J_1(\bar{k}\rho)}{\rho} \right) d\bar{k}. \tag{51}$$

We use the relation (77) to compute the derivative with respect to  $\rho$  of  $J_1$ . We have

$$\left(\frac{J_1(k\rho)}{\rho}\right)' = k^2 \frac{d}{dx} \left(\frac{J_1(x)}{x}\right) = -\frac{k}{\rho} J_2(k\rho). \quad (52)$$

where we have defined  $x = k\rho$ . Hence we obtain

$$\partial_\rho \left(\gamma \rho^{-\frac{3}{2}}\right) = - \int_0^\infty \hat{\gamma}(\bar{k}) \bar{k}^{\frac{3}{2}} \frac{J_2(\bar{k}\rho)}{\rho} d\bar{k}. \quad (53)$$

Inserting the expression (53) for  $\partial_\rho (\gamma \rho^{-3/2})$  into Eq. (49) we get

$$\mathfrak{H}(T(\gamma))(k) = - \int_0^\infty k^{\frac{1}{2}} \bar{k}^{\frac{3}{2}} \hat{\gamma}(\bar{k}) d\bar{k} \int_0^\infty J_2(\bar{k}\rho) J_1(k\rho) d\rho. \quad (54)$$

From Eq. (75) we know that the integral

$$\int_0^\infty J_2(\bar{k}\rho) J_1(k\rho) d\rho, \quad (55)$$

is equal to  $k/\bar{k}^2$  if  $k < \bar{k}$  and it is equal to zero if  $k > \bar{k}$ . Then, the conclusion of the lemma follows.  $\square$

The next step is to analyze Eq. (35). In this equation there is no time derivative. We want to solve this equation for an arbitrary given function  $w$ , which is not necessarily a solution of the other Eq. (34). In the next lemma, we construct such solution using the Hankel transform.

**Lemma 2.** *Let  $w$  be a given function, with  $\hat{w}(k) = \mathfrak{H}(w)$  of compact support. Then, the solution  $\hat{\gamma} = \mathfrak{H}(\gamma)$  of compact support of the Eq. (35) in momentum space is given by*

$$\hat{\gamma}(k) = 2\hat{w}(k) + \frac{2k^{\frac{3}{2}}}{(1+k^2)^{\frac{3}{2}}} \int_k^\infty \frac{(1+\bar{k}^2)^{\frac{1}{2}}}{\bar{k}^{\frac{1}{2}}} \hat{w}(\bar{k}) d\bar{k}. \quad (56)$$

We also have that  $E(k)$ , defined by (47), is given by

$$E(k) = \frac{2}{(1+k^2)^{\frac{1}{2}}} \int_k^\infty \frac{(1+\bar{k}^2)^{\frac{1}{2}}}{\bar{k}^{\frac{1}{2}}} \hat{w}(\bar{k}) d\bar{k}. \quad (57)$$

*Proof.* We apply the transform  $\mathfrak{H}$  to Eq. (35). Using (44) we get

$$-(1+k^2)\hat{\gamma}(k) - \mathfrak{H}(T(\gamma)) = -2(1+k^2)\hat{w}(k). \quad (58)$$

We can use Lemma 1 to express  $\mathfrak{H}(T(\gamma))$  in terms of  $\hat{\gamma}$ . However, it is convenient to express everything in terms of  $E(k)$  (defined by (47)) instead of  $\hat{\gamma}(k)$ . Using (46), (47), and (48) we obtain

$$\frac{dE(k)}{dk} + \frac{k}{1+k^2} E(k) = -\frac{2}{k^{1/2}} \hat{w}(k). \quad (59)$$

We multiply by the integrating factor  $(1 + k^2)^{\frac{1}{2}}$  to get

$$\frac{d}{dk} \left( (1 + k^2)^{\frac{1}{2}} E \right) = -2 \frac{(1 + k^2)^{\frac{1}{2}}}{k^{\frac{1}{2}}} \hat{w}(k). \tag{60}$$

Integrating this equation and forgetting the integrating constant to have a solution of compact support, we obtain Eq. (57). Equation (56) follows directly using (48).  $\square$

Finally, in the next lemma we solve the whole system (34)–(35).

**Lemma 3.** *Let  $F(k)$  and  $G(k)$  be arbitrary functions of compact support. Then  $w = \mathfrak{H}^{-1}(\hat{w})$  and  $\gamma = \mathfrak{H}^{-1}(\hat{\gamma})$  define a solution of the system of Eqs. (34)–(35), where*

$$\hat{w}(t, k) = \frac{k^{\frac{3}{2}}}{(1 + k^2)^{\frac{1}{2}}} \int_k^\infty \left( F(\bar{k}) \cos((1 + \bar{k}^2)^{\frac{1}{2}} t) + G(\bar{k}) \sin((1 + \bar{k}^2)^{\frac{1}{2}} t) \right) d\bar{k}, \tag{61}$$

$$\hat{\gamma}(t, k) = 2\hat{w}(t, k) + \frac{2k^{\frac{3}{2}}}{(1 + k^2)^{\frac{3}{2}}} \int_k^\infty \frac{(1 + \bar{k}^2)^{\frac{1}{2}}}{\bar{k}^{\frac{1}{2}}} \hat{w}(t, \bar{k}) d\bar{k}. \tag{62}$$

*Proof.* First, note the Eq. (62) is the solution of Eq. (35) given by Lemma 2 if we consider  $\hat{w}(t, k)$  as a given function. The only part we have to prove is that Eq. (61) is also a solution of (34) in which  $\gamma$  is given by (62).

We apply the Hankel transform to Eq. (34) and we obtain the following equation in momentum space

$$\ddot{\hat{w}}(t, k) = -(1 + k^2)\hat{w}(t, k) + \mathfrak{H}(T(\gamma)). \tag{63}$$

Using the expression for  $\mathfrak{H}(T(\gamma))$  obtained in Lemma 2 we obtain

$$\ddot{\hat{w}}(t, k) = -(1 + k^2)\hat{w}(t, k) - \frac{2k^{\frac{3}{2}}}{(1 + k^2)^{\frac{1}{2}}} \int_k^\infty \frac{(1 + \bar{k}^2)^{\frac{1}{2}}}{\bar{k}^{\frac{1}{2}}} \hat{w}(t, \bar{k}) d\bar{k}. \tag{64}$$

This equation involves only  $\hat{w}$ , and hence we have reduced the system (34)–(35) to only one equation for one unknown. But Eq. (64) is not a differential equation, it is an integro-differential equation. We can get a simpler expression if we define  $a(t, k)$  by

$$a(t, k) = \frac{(1 + k^2)^{\frac{1}{2}}}{k^{\frac{3}{2}}} \hat{w}(t, k). \tag{65}$$

In terms of  $a(t, k)$ , Eq. (64) is written as

$$\ddot{a}(t, k) = -(1 + k^2)a(t, k) - 2 \int_k^\infty \bar{k} a(t, \bar{k}) d\bar{k}. \tag{66}$$

Although Eq. (66) looks certainly simpler than Eq. (65), the essential difficulty remains the same. However, a remarkable cancellation occurs if we take

a derivative with respect to  $k$  allowing us to convert this equation into a pure differential equation. Namely, let us define

$$b(t, k) = \frac{\partial}{\partial k} a(t, k). \quad (67)$$

Then, taking a derivative with respect to  $k$  of Eq. (66) we obtain the following remarkable simple differential equation for  $b(k)$

$$\ddot{b}(t, k) = -(1 + k^2)b(t, k). \quad (68)$$

The important fact is that the term  $-2ka(t, k)$  that appears when the  $k$  derivative is applied to  $-k^2a(t, k)$  in Eq. (66) cancels out by the derivative of the integral (the factor 2 in front of the integral is crucial). And hence in the final expression only appears  $b(t, k)$  and not  $a(t, k)$ .

The solution of Eq. (68) is given by

$$b(t, k) = -F(k) \cos((1 + k^2)^{\frac{1}{2}}t) - G(k) \sin((1 + k^2)^{\frac{1}{2}}t), \quad (69)$$

for arbitrary functions  $F(k)$  and  $G(k)$ . We have written Eq. (69) with a minus sign just for convenience.

The function  $a(t, k)$  is calculated integrating (67), namely

$$a(t, k) = - \int_k^{\infty} b(t, \bar{k}) d\bar{k}. \quad (70)$$

There is a constant of integration that we have set to zero in (70), otherwise the function  $\hat{w}$  defined by (65) will not be of compact support.

Using (69), (70), and (65), expression (61) follows.  $\square$

### 3. Proof of Theorem 1

*Proof.* The theorem is a straightforward consequence of Lemma 3, and the scaling invariance of the equations. Namely, consider the solution of Eqs. (34)–(35) founded in Lemma 3. To be consistent with the notation used in Sect. 2.2, this solution should be denoted by  $\tilde{w}_1(t, \rho)$  and  $\tilde{\gamma}_1(t, \rho)$ . Using the scaling (33), out of  $\tilde{w}_1(t, \rho)$ ,  $\tilde{\gamma}_1(t, \rho)$  we construct

$$\tilde{w}(t, \rho, \lambda) = \tilde{w}_1(\lambda t, |\lambda|\rho), \quad \tilde{\gamma}(t, \rho, \lambda) = \tilde{\gamma}_1(\lambda t, |\lambda|\rho). \quad (71)$$

These are solutions of (27)–(28). We apply the inverse Fourier transform (26 to  $\tilde{w}(t, \rho, \lambda)$  and  $\tilde{\gamma}(t, \rho, \lambda)$  to obtain the desired result. In (6) we have redefined the function  $G(k, \lambda)$  to make simpler the connection with the initial data.  $\square$

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### Appendix A. Bessel Functions and Hankel Transform

We collect in this appendix properties on Bessel functions and Hankel transform that we use in this article. A general reference is the classical book [5] and also the book [7] for the Hankel transform.

We will consider Bessel functions of the first kind, denoted by  $J_\nu(x)$ . We will restrict ourselves to the case where  $\nu$  is a positive integer.

Bessel functions are solutions of the Bessel equation

$$\frac{d^2 J_\nu}{dx^2} + \frac{1}{x} \frac{dJ_\nu}{dx} + \left(1 - \frac{\nu^2}{x^2}\right) J_\nu = 0. \tag{72}$$

For  $\nu \geq 0$ , a series expansion of these function is given by

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{j=0}^{\infty} \frac{(-1)^j}{j! \Gamma(j + \nu + 1)} \left(\frac{x}{2}\right)^{2j}, \tag{73}$$

where  $\Gamma$  denotes the standard Gamma function. Since in our case  $\nu$  is a positive integer we have  $\Gamma(j + \nu + 1) = (j + \nu)!$ . From (73) we deduce that  $J_\nu(x)$  is an even function of  $x$  for  $\nu$  even and an odd function of  $x$  for  $\nu$  odd.

We have the following orthogonality property for Bessel functions

$$\int_0^\infty \rho J_\nu(k\rho) J_\nu(\bar{k}\rho) d\rho = \frac{1}{k} \delta(\bar{k} - k). \tag{74}$$

The following integral is used in the proof of Lemma 1 (see [5], p. 406)

$$\int_0^\infty J_\nu(at) J_{\nu-1}(bt) dt = \begin{cases} b^{\nu-1}/a^\nu, \\ 1/(2b), \\ 0, \end{cases} \tag{75}$$

where the first value corresponds to  $b < a$ , the second to  $b = a$  and the third to  $b > a$ .

We make use also of the following relation ([5], p. 45)

$$x \frac{dJ_\nu(x)}{dx} - \nu J_\nu(x) = -x J_{\nu+1}(x). \tag{76}$$

For the case  $\nu = 1$  this relation can be written in the form

$$\frac{d}{dx} \left( \frac{J_1(x)}{x} \right) = -\frac{J_2(x)}{x}. \tag{77}$$

The Hankel transform of order  $\nu$  of the function  $f$  is defined as

$$\mathfrak{H}_\nu(f) = \int_0^\infty f(\rho) \sqrt{k\rho} J_\nu(k\rho) d\rho, \tag{78}$$

and the inverse is given by

$$\mathfrak{H}_\nu^{-1}(F) = \int_0^\infty F(k) \sqrt{k\rho} J_\nu(k\rho) dk. \quad (79)$$

The formula (74) guarantee that  $\mathfrak{H}_\nu^{-1}(\mathfrak{H}_\nu(f)) = f$  with  $f$  in an appropriate functional space. More precisely the Hankel transform (of order one) is an automorphism on the space  $\mathcal{H}_1$  of  $C^\infty$  functions  $f : \mathbb{R}^+ \rightarrow \mathbb{R}$  provided with the family of seminorms  $\gamma_{m,k}$  which for each pair of non-negative integers  $m$  and  $k$  are defined as

$$\gamma_{m,k}(f) = \sup_{0 < \rho < \infty} |\rho^m (\rho^{-1} \partial_\rho)^k [\rho^{-3/2} f(\rho)]|. \quad (80)$$

A function  $f$  in  $\mathcal{H}_1$  has an expansion of the form

$$f(\rho) = \rho^{\frac{3}{2}} (a_0 + a_2 \rho^2 + \dots + a_{2l} \rho^{2l} + R_{2l}(\rho)), \quad (81)$$

and is of fast decay, namely, decays faster than any power of  $1/\rho$  as  $\rho \rightarrow \infty$  (Lemma 5.2.1 in p. 130 of [7]).

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