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# Area-charge inequality for black holes

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**Abstract**

The inequality between area and charge $A \geqslant 4\pi Q^2$ for dynamical black holes is proved. No symmetry assumption is made and charged matter fields are included. Extensions of this inequality are also proved for regions in the spacetime which are not necessarily black hole boundaries.

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## 1. Introduction

In a recent series of articles [13, 1, 14, 26], the quasi-local inequality between area and angular momentum was proved for dynamical axially symmetric black holes (see also [5, 23, 22, 4] for a proof in the stationary case). In these articles, the assumption of axial symmetry is essential since it provides a canonical notion of quasi-local angular momentum. The natural question is whether similar kinds of inequalities hold without this symmetry assumption that certainly restricts their application in physically realistic scenarios. A natural first step to answer this question is to study the related inequality involving the electric charge, since the charge is always well defined as a quasi-local quantity.

In [17], the expected inequality for area and charge has been proved for stable minimal surfaces on time symmetric initial data. The main goal of this paper is to extend this result in several directions. First, we prove the inequality for generic dynamical black holes. Second, we also prove versions of this inequality for regions which are not necessarily black hole boundaries, that is, regions that can be interpreted as the boundaries of ordinary objects.

The plan of the paper is as follows. In section 2, we present our mains results which are given by theorems 2.1, 2.2 and 2.3. We also discuss in this section the physical meaning of these results. In section 3, we prove theorem 2.1 and in section 4 we prove theorems 2.2 and 2.3.

## 2. Main result

Consider Einstein equations with cosmological constant $\Lambda$

$$G_{ab} = 8\pi \left(T_{ab}^{EM} + T_{ab}\right) - \Lambda g_{ab}, \tag{1}$$

where $T_{ab}^{EM}$ is the electromagnetic energy–momentum tensor given by

$$T_{ab}^{EM} = \frac{1}{4\pi}\left(F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd}\right), \tag{2}$$

and $F_{ab}$ is the (antisymmetric) electromagnetic field tensor. The electric and magnetic charge of an arbitrary closed, oriented, two-surface $\mathcal{S}$ embedded in the spacetime are defined by

$$Q_E = \frac{1}{4\pi}\int_{\mathcal{S}} {}^*F_{ab}, \qquad Q_M = \frac{1}{4\pi}\int_{\mathcal{S}} F_{ab}, \tag{3}$$

where ${}^*F_{ab} = \frac{1}{2}\epsilon_{abcd}F^{cd}$ is the dual of $F_{ab}$ and $\epsilon_{abcd}$ is the volume element of the metric $g_{ab}$. It is important to emphasize that we do not assume that the matter is uncharged, namely we allow $\nabla_a F^{ab} = -4\pi j^b \neq 0$ (which is equivalent to $\nabla^a T_{ab}^{EM} \neq 0$). The only condition that we impose is that the non-electromagnetic matter field stress-energy tensor $T_{ab}$ satisfies the dominant energy condition.

The first main result of this paper is the following theorem.

**Theorem 2.1.** *Given an orientable closed marginally trapped surface $\mathcal{S}$ satisfying the spacetime stably outermost condition, in a spacetime which satisfies Einstein equations (1), with a non-negative cosmological constant $\Lambda$ and such that the non-electromagnetic matter fields $T_{ab}$ satisfy the dominant energy condition, it holds the inequality*

$$A \geqslant 4\pi \left(Q_E^2 + Q_M^2\right), \tag{4}$$

*where $A$, $Q_E$ and $Q_M$ are the area, electric and magnetic charges of $\mathcal{S}$ given by (3).*

For the definition of marginally trapped surfaces and the stably outermost condition, see definition 3.2 in section 3. This theorem represents a generalization of the result presented in [17] valid for stable minimal surfaces. In particular, it also incorporates the magnetic charge in the inequality. The query about the actual existence of magnetic charges is of experimental nature. From a theoretical perspective, magnetic monopoles can arise in standard electromagnetic theory as non-trivial topological configurations of the electromagnetic field[4] (see e.g. [30, 29]). In our present context, it is natural and straightforward to incorporate the magnetic charge, since the proof of theorem 2.1 only involves the flux of $F_{ab}$ through a minimal or marginally trapped surface, with no need of resorting to the singular magnetic monopole vector potential $A_a$.

Theorem 2.1 is the analog of the theorem proved in [26] for the angular momentum. The important difference is that in theorem 2.1, no symmetry assumption is made. Also the proof of this result is much simpler than the one in [26], and we explain this in detail in section 3.

Although the theorem proved in [17] (which we include as theorem 4.4 in this paper) for stable minimal surfaces embedded on maximal initial data is more restrictive than theorem 2.1, it is geometrically interesting and it has also relevant applications as the ones presented below. One important consequence of theorem 4.4 is that it allows for a suitable extension of inequality (4) to arbitrary surfaces, as it is proven in the following theorem.

[4] More specifically, although no local chart can be found where the corresponding vector potential $A_a$ is non-singular, the (curvature) electromagnetic field $F_{ab}$ is well defined globally on the non-trivial $U(1)$ principal fiber bundle. The magnetic charge is determined by the flux of $F_{ab}$ through $\mathcal{S}$, which is controlled by the first Chern class of the $U(1)$-bundle (in particular, this is the topological origin of the quantization of the product of electric and magnetic charges).

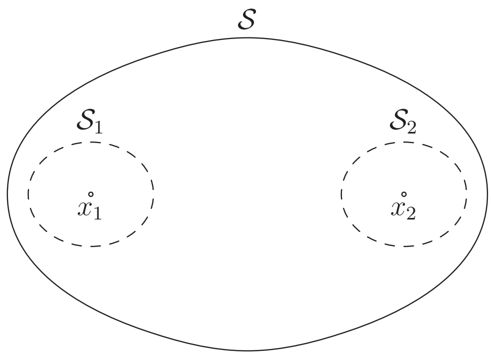


**Figure 1.** Brill–Lindquist data with large separation distance. The dashed surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ are minimal surfaces. The surface $\mathcal{S}$ is screening.

**Theorem 2.2** (Area, charge and global topology)**.** *Let $(\Sigma, (h, K), (E, B))$ be a complete, maximal and asymptotically flat (with possibly many asymptotic ends) initial data for Einstein–Maxwell equations. We assume that the non-electromagnetic matter fields are non-charged and that they satisfy the dominant energy condition. Then, for any oriented surface $\mathcal{S}$ screening an end $\Sigma_e$ we have*

$$A(S) \geqslant 4\pi \left(\bar{Q}_E^2 + \bar{Q}_M^2\right) \geqslant \frac{4\pi\,(Q_E^2 + Q_M^2)}{|H_2|}, \tag{5}$$

*where $Q_E$ and $Q_M$ are the electric and magnetic charges of $\mathcal{S}$, $\bar{Q}_E$ and $\bar{Q}_M$ are the absolute central charges of $\mathcal{S}$ and $H_2$ is the second Betti number of $\Sigma$.*

For the definitions of screening surface and absolute central charges, see section 4. It is important to note that all the charges in theorem 2.2 are produced by a non-trivial topology in the manifold (since by assumption the non-electromagnetic fields are uncharged in the whole initial surface $\Sigma$). That is, if the topology is trivial (i.e. $\Sigma = \mathbb{R}^3$) there is no charges and the theorem is also trivial. This is an important difference with theorem 2.1, where the charge can be produced by charged matter inside the trapped surfaces. Note also that this theorem has global requirements (namely, asymptotic flatness, completeness and the assumption that the matter is uncharged), in contrast with theorem 2.1 which is purely quasi-local in the sense that only conditions at the surface are used.

Let us discuss theorem 2.2 in some detail. In order to give an intuitive idea of the result and of the definitions involved, in the following we will analyze a particular class of examples.

Consider the well-known Brill–Lindquist initial data [10]. Brill–Lindquist data are time symmetric, conformally flat initial data with $N$ asymptotic ends. To simplify the discussion, we take $N = 3$ (in fact the discussion below applies to a much more general class of data which are not necessarily conformally flat). The manifold is $\Sigma := \mathbb{R}^3 \setminus \{x_1, x_2\}$, where $x_1$ and $x_2$ are arbitrary points in $\mathbb{R}^3$. Let $L = |x_1 - x_2|$ , where $|\cdot|$ denotes the Euclidean distance with respect to the flat conformal metric. The endpoints $x_1$ and $x_2$ have electric charges $Q_1$ and $Q_2$. The other end has charge $Q$ given by

$$Q = Q_1 + Q_2. \tag{6}$$

Consider families of initial data with fixed charges but different separation distance $L$. When $L$ is big enough, it can be proved that there exist only two stable minimal surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ surrounding each end point. See figure 1 (for a numerical picture of these surfaces see the original article [10]; the analytical proof that there exist only these two surfaces has been

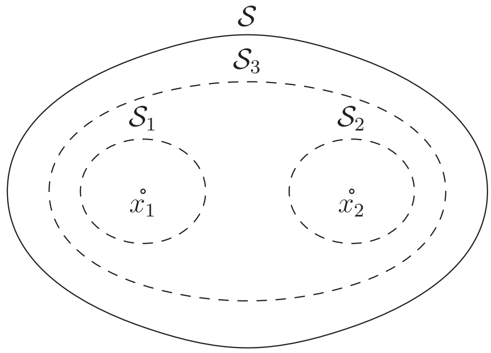


**Figure 2.** Brill–Lindquist data with small separation distance. A third minimal surface $\mathcal{S}_3$ appears enclosing the two ends $x_1$ and $x_2$ and the two minimal surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$. The surface $\mathcal{S}$ is screening but not necessarily minimal.

given in [12, 15]). Take a sphere $\mathcal{S}$ that encloses the two end points $x_1$ and $x_2$. This surface is screening (for a precise definition, see definition 4.1 in section 4). Since $\mathcal{S}_1$ and $\mathcal{S}_2$ are the only minimal surfaces, we have that

$$A \geqslant A_1 + A_2, \tag{7}$$

where $A$ is the area of $\mathcal{S}$ and $A_1$, $A_2$ are the areas of $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively. Applying theorem 4.4 for each minimal surfaces from (7), we obtain

$$A \geqslant 4\pi \left(Q_1^2 + Q_2^2\right). \tag{8}$$

Take now $L$ to be small enough. Then, a third minimal surface $\mathcal{S}_3$, with area $A_3$, which enclose the two ends appears. This surface is the outermost one and hence we have

$$A \geqslant A_3. \tag{9}$$

See figure 2. Then, using theorem 4.4 we get

$$A(\mathcal{S}) \geqslant 4\pi (Q_1 + Q_2)^2 = 4\pi Q^2 \tag{10}$$

where we have used that the charge of the surface $\mathcal{S}_3$ is equal to the charge of the end. If we combine inequality (10) with (8), we obtain the following:

$$A(\mathcal{S}) \geqslant 4\pi \inf\left\{Q_1^2 + Q_2^2, (Q_1 + Q_2)^2\right\}. \tag{11}$$

This inequality is valid for all screening surfaces $\mathcal{S}$ and it is independent of $L$. The right-hand side of this inequality is precisely the square of the absolute central charge defined in section 4, namely

$$\bar{Q}(\mathcal{S}) = \sqrt{\inf\left\{Q_1^2 + Q_2^2, (Q_1 + Q_2)^2\right\}}. \tag{12}$$

Note that if $Q_1$ and $Q_2$ have opposite signs, we get

$$\bar{Q}(\mathcal{S}) = |Q_1 + Q_2|, \tag{13}$$

and if they have the same signs we get

$$\bar{Q}(\mathcal{S}) = \sqrt{Q_1^2 + Q_2^2}. \tag{14}$$

The Betti number $H_2$ measures the number of holes of $\mathcal{S}$; in the present case we have $H_2 = 2$. It is clear that

$$\bar{Q}(\mathcal{S}) \geqslant \frac{|Q_1 + Q_2|}{2} = \frac{Q(\mathcal{S})}{H_2}. \tag{15}$$

This is precisely the second inequality in (5). Note that knowing the size of the parameter $L$ provides finer information. For example, take $Q_1 = -Q_2$. In that case, $Q(\mathcal{S}) = \bar{Q}(\mathcal{S}) = 0$ and theorem 2.2 is trivial. However, if $L$ is big, we have the non-trivial inequality (8).

Finally we present our third main result. As we discussed above, theorem 2.2 generalizes theorem 2.1 in the sense that it applies to surfaces that are not necessarily black hole horizons. However, in that theorem a strong restriction is made, namely that matter fields have no charges. The natural question is what happens for an ordinary charged object, is it possible to prove a similar kind of inequality? The answer is no. There exists an interesting and highly non-trivial counterexample. This counterexample was constructed by Bonnor in [6] and it can be summarized as follows: for any given positive number $k$, there exist static, isolated, non-singular bodies satisfying the energy conditions, whose surface area $A$ satisfies $A < kQ^2$. In [6], the inequality is written in terms of the mass; however, for this class of solution the mass is always equal to the charge of the body. The body is a highly prolated spheroid of electrically counterpoised dust. This suggests that for a body which is 'round' enough, a version of inequality (5) can still hold. From the physical point of view, we state that for an ordinary charged object we need to control another parameter (the 'roundness') in order to obtain an inequality between area and charge. Remarkably enough it is possible to encode this intuition in the geometrical concept of the isoperimetric surface: we say that a surface $\mathcal{S}$ is isoperimetric if among all surfaces that enclose the same volume as $\mathcal{S}$ does, $\mathcal{S}$ has the least area. Then, using the same technique as in the proof of theorem 4.4 and applying the results of [11], we obtain the following theorem for isoperimetric surfaces.

**Theorem 2.3.** *Consider an electro-vacuum, maximal initial data, with a non-negative cosmological constant. Assume that $\mathcal{S}$ is a stable isoperimetric sphere. Then,*

$$A(\mathcal{S}) \geqslant \frac{4\pi}{3}\left(Q_E^2 + Q_M^2\right), \tag{16}$$

*where $Q_E$ and $Q_M$ are the electric and magnetic charges of $\mathcal{S}$.*

We emphasize that this theorem is purely quasi-local (as theorem 2.1); it only involves conditions on the surface $\mathcal{S}$. In particular, it is assumed electro-vacuum only on $\mathcal{S}$, charged matter could exist inside or outside the surface.

## 3. Area–charge inequality for black holes

The aim of this section is to prove theorem 2.1. We follow the notation and definitions presented in [26]. Consider a closed orientable 2-surface $\mathcal{S}$ embedded in a spacetime $M$ with metric $g_{ab}$ and Lévi-Cività connection $\nabla_a$. We denote the induced metric on $\mathcal{S}$ as $q_{ab}$, with the Lévi-Cività connection $D_a$ and the Ricci scalar ${}^2R$. We will denote by $\mathrm{d}S$ the area measure on $\mathcal{S}$. Let us consider null vectors $\ell^a$ and $k^a$ spanning the normal plane to $\mathcal{S}$ and normalized as $\ell^a k_a = -1$, leaving a (boost) rescaling freedom $\ell'^a = f\ell^a$, $k'^a = f^{-1}k^a$. The expansion $\theta^{(\ell)}$ and the shear $\sigma^{(\ell)}_{ab}$ associated with the null normal $\ell^a$ are given by

$$\theta^{(\ell)} = q^{ab}\nabla_a \ell_b, \qquad \sigma^{(\ell)}_{ab} = q^c{}_a q^d{}_b \nabla_c \ell_d - \tfrac{1}{2}\theta^{(\ell)} q_{ab}, \tag{17}$$

whereas the normal fundamental form $\Omega^{(\ell)}_a$ is

$$\Omega^{(\ell)}_a = -k^c q^d{}_a \nabla_d \ell_c. \tag{18}$$

The spacetime metric $g_{ab}$ can be written in the following form:

$$g_{ab} = q_{ab} - \ell_a k_b - \ell_b k_a. \tag{19}$$

The surface $\mathcal{S}$ is a marginal outertrapped surface if $\theta^{(\ell)} = 0$. We will refer to $\ell^a$ as the *outgoing* null vector.

The following stability condition on marginally trapped surfaces introduced in [2, 3] plays a crucial role.

**Definition 3.1** (Andersson, Mars, Simon)**.** *Given a closed marginally trapped surface $\mathcal{S}$ and a vector $v^a$ orthogonal to it, we will refer to $\mathcal{S}$ as stably outermost with respect to the direction $v^a$ iff there exists a function $\psi > 0$ on $\mathcal{S}$ such that the variation of $\theta^{(\ell)}$ with respect to $\psi v^a$ fulfills the condition*

$$\delta_{\psi v}\theta^{(\ell)} \geqslant 0. \tag{20}$$

Here $\delta$ denotes the variation operator associated with a deformation of the surface $\mathcal{S}$ introduced in [2] (see also the treatment in [7]). Following [26] we will formulate this stability notion in a sense not referring to a particular stability direction, but just requiring stability along some outgoing non-timelike direction.

**Definition 3.2.** *A closed marginally trapped surface $\mathcal{S}$ is referred to as spacetime stably outermost if there exists an outgoing ($-k^a$-oriented) vector $x^a = \bar{\gamma}\ell^a - k^a$, with $\bar{\gamma} \geqslant 0$, with respect to which $\mathcal{S}$ is stably outermost.*

In the following, we denote by $X^a$ the vector $X^a = \psi x^a = \gamma\ell^a - \psi k^a$, with $\psi$ the function guaranteed by definition 3.1 and $\gamma \equiv \psi\bar{\gamma}$, so that $\delta_X\theta^{(\ell)} \geqslant 0$. Note that this *spacetime stability condition* includes, for an outgoing past null vector $x^a = -k^a$, the (outer trapping horizon) stability notions in [21, 28]. For a further discussion concerning this stability condition, see [26].

The following lemma provides the essential estimate for the matter fields on a stable marginally trapped surface $\mathcal{S}$. It is the analog of lemma 1 in [26]. Its proof essentially follows from setting the function $\alpha = 1$ used in that lemma. It is important to emphasize that no symmetry assumption is made. For completeness and since the final proof is much simpler, we present it here.

**Lemma 3.3.** *Given a closed marginally trapped surface $\mathcal{S}$ satisfying the spacetime stably outermost condition, the following inequality holds:*

$$\int_{\mathcal{S}} \left[ G_{ab}\ell^a \left( k^b + \frac{\gamma}{\psi}\ell^b \right) \right] \mathrm{d}S \leqslant 4\pi(1-g), \tag{21}$$

*where g is the genus of $\mathcal{S}$. If in addition we assume that the left-hand side in inequality (21) is non-negative and not identically zero, then it follows that $g = 0$ and hence $\mathcal{S}$ has the $\mathbb{S}^2$ topology.*

See also theorem 2.1 in [16], where a similar result with similar techniques has been proved (see especially the paragraph following inequality (2.8) in that paper).

**Proof.** First, we evaluate $\delta_X\theta^{(\ell)}/\psi$ for the vector $X^a = \gamma\ell^a - \psi k^a$ provided by definition 1 (use e.g. equations (2.23) and (2.24) in [7]) and impose $\theta^{(\ell)} = 0$. We obtain

$$\frac{1}{\psi}\delta_X\theta^{(\ell)} = D^a\Omega_a^{(\ell)} - {}^2\Delta\ln\psi - D_a\ln\psi D^a\ln\psi + 2\Omega_a^{(\ell)}D^a\ln\psi - \Omega_c^{(\ell)}\Omega^{(\ell)^c}$$
$$+\frac{1}{2}{}^2R - \frac{\gamma}{\psi}\left[\sigma_{ab}^{(\ell)}\sigma^{(\ell)ab} + G_{ab}\ell^a\ell^b\right] - G_{ab}k^a\ell^b.$$

We integrate this equation over the surface $\mathcal{S}$. On the left-hand side, we use the stability condition (20). The first two terms on the right-hand side integrate to zero. The next three terms can be arranged as a total square, namely

$$-\left(D_a\ln\psi - \Omega_a^{(\ell)}\right)\left(D^a\ln\psi - \Omega^{(\ell)^a}\right) = -D_a\ln\psi D^a\ln\psi + 2\Omega_a^{(\ell)}D^a\ln\psi - \Omega_c^{(\ell)}\Omega^{(\ell)^c}, \tag{22}$$

and hence the integral is non-positive. The integral of the scalar curvature is calculated using the Gauss–Bonnet theorem

$$\int_S \frac{1}{2} {}^2R \, \mathrm{d}S = 4\pi (1 - g). \tag{23}$$

Finally, the term with $\sigma^{(\ell)}_{ab} \sigma^{(\ell)ab}$ is non-positive. Collecting all these observations, inequality (21) follows. If the left-hand side of inequality (21) is non-negative it follows that $g$ can be 0 or 1. If it is not identically zero, then $g = 0$ and hence $\mathcal{S}$ has the $\mathbb{S}^2$ topology. □

The following lemma will allow us to write the relevant normal components of the electromagnetic field on the surface in terms of the charges. It is important to note that it is a pure algebraic result, Maxwell equations are not used. In particular, the generalization to Yang–Mills theories with a compact Lie group is direct and will be presented elsewhere.

**Lemma 3.4.** *Let $T^{EM}_{ab}$ be the electromagnetic energy–momentum tensor given by (2). Then, the following equality holds:*

$$T^{EM}_{ab} \ell^a k^b = \frac{1}{8\pi} \left[ \left( \ell^a k^b F_{ab} \right)^2 + \left( \ell^a k^b {}^*F_{ab} \right)^2 \right]. \tag{24}$$

**Proof.** The proof is a straightforward computation using the form of metric (19). We mention some useful intermediate steps. Using equation (19), we calculate

$$F_{ab} F^{ab} = -2(\ell^a k^b F_{ab})^2 - 4 q^{ab} k^c F_{ac} \ell^d F_{bd} + F_{ab} F_{cd} q^{ac} q^{bd} \tag{25}$$

and

$$\ell^a k^c F_{ab} F_c{}^b = (\ell^a k^b F_{ab})^2 + q^{ab} k^c F_{ac} \ell^d F_{bd}. \tag{26}$$

Noting that the pull-back of $F_{ab}$ on the surface $\mathcal{S}$ is proportional to the volume element $\epsilon_{ab}$ of the surface $\mathcal{S}$, we can evaluate $F_{ab} F_{cd} q^{ac} q^{bd}$ and $(\epsilon^{ab} F_{ab})^2$ to obtain

$$F_{ab} F_{cd} q^{ac} q^{bd} = \tfrac{1}{2} (\epsilon^{ab} F_{ab})^2 = 2 ({}^*F_{ab} \ell^a k^b)^2, \tag{27}$$

where the identity

$${}^*F_{ab} \ell^a k^b = \tfrac{1}{2} F_{ab} \epsilon^{ab} \tag{28}$$

has been used in the second equality. This identity follows from the relation $\epsilon_{ab} = \epsilon_{abcd} \ell^c k^d$. Inserting first (27) in equation (25) and then the resulting expression (together with (26)) into (2), we obtain (24). □

Note that the electric and magnetic charges (3) of $\mathcal{S}$ can be written as follows in terms of the null vector $\ell^a$ and $k^a$:

$$Q_E = \frac{1}{4\pi} \int_{\mathcal{S}} F_{ab} \ell^a k^b \mathrm{d}S, \qquad Q_M = \frac{1}{4\pi} \int_{\mathcal{S}} {}^*F_{ab} \ell^a k^b \mathrm{d}S. \tag{29}$$

Having proved these two lemmas, we have already the basic ingredients for the proof of our first main result.

**Proof of theorem 2.1.** We use inequality (21) and Einstein equations (1). Since the vector $k^a + \gamma/\psi \, \ell^a$ is timelike or null, using that the tensor $T_{ab}$ satisfies the dominant energy condition and that $\Lambda$ is non-negative, we get from (21) that

$$8\pi \int_{\mathcal{S}} T^{EM}_{ab} \ell^a k^b \, \mathrm{d}S \leqslant 8\pi \int_{\mathcal{S}} \left[ T^{EM}_{ab} \ell^a \left( k^b + \frac{\gamma}{\psi} \ell^b \right) \right] \mathrm{d}S \leqslant 4\pi (1 - g), \tag{30}$$

where in the last inequality we have used that $T_{ab}^{EM}\ell^a\ell^b \geqslant 0$ (this inequality follows directly from (2), i.e. the electromagnetic energy–momentum tensor satisfies the null energy condition). We use equality (24) to obtain from inequality (30) the following bound:

$$\int_{\mathcal{S}} [(\ell^a k^b F_{ab})^2 + (\ell^a k^{b*} F_{ab})^2]\, \mathrm{d}S \leqslant 4\pi(1-g). \tag{31}$$

If the left-hand side of inequality (31) is identically zero, then the charges are zero and inequality (4) is trivial. Then, we can assume that it is not zero at some point and hence we have that $g = 0$.

To bound the left-hand side of inequality (31), we use the Hölder inequality on $\mathcal{S}$ (following the spirit of the proof presented in [25] for the charged Penrose inequality) in the following form. For integrable functions $f$ and $h$, the Hölder inequality is given by

$$\int_{\mathcal{S}} fh\, \mathrm{d}S \leqslant \left(\int_{\mathcal{S}} f^2\, \mathrm{d}S\right)^{1/2} \left(\int_{\mathcal{S}} h^2\, \mathrm{d}S\right)^{1/2}. \tag{32}$$

If we take $h = 1$, then we obtain

$$\int_{\mathcal{S}} f\, \mathrm{d}S \leqslant \left(\int_{\mathcal{S}} f^2\, \mathrm{d}S\right)^{1/2} A^{1/2}, \tag{33}$$

where $A$ is the area of $\mathcal{S}$. Using this inequality in (31), we finally obtain

$$A^{-1}\left[\left(\int_{\mathcal{S}} \ell^a k^b F_{ab}\, \mathrm{d}S\right)^2 + \left(\int_{\mathcal{S}} \ell^a k^{b*} F_{ab}\, \mathrm{d}S\right)^2\right] \leqslant 4\pi. \tag{34}$$

Finally, we use equation (29) to express the left-hand side of (34) in terms of $Q_E$ and $Q_M$. Hence, inequality (4) follows. □

We note that up to the use of the Hölder inequality in equation (32), the line of reasoning in the proof above is also followed in [8]. Starting from the *outer* condition for trapping horizons in [21] (see also [28]), namely the stably outermost condition for a null $X^a$, a version of lemma 3.3 is derived there (their equation (20)). Then, the equality in lemma 3.4 is their equation (22). The last step completing the proof is though missing.

## 4. Area, charge and global topology

We consider maximal Einstein–Maxwell initial states $(\Sigma, (h, K), (E, B))$, with possibly many asymptotically flat (AF) ends. AF ends will be denoted by $\Sigma_e$. Our central object, the subject of our study, will be surfaces, $\mathcal{S}$, '*screening*' a given end $\Sigma_e$. Their definition is as follows.

**Definition 4.1** (Screening surfaces)**.** *Fix an AF end $\Sigma_e$ of $\Sigma$. A compact, oriented, but not necessarily connected surface $\mathcal{S}$ is said to screen the end $\Sigma_e$ if it is the boundary of an open and connected region $\Omega$ containing the given end but not any other. Such $\Omega$ is called a screened region.*

Every component of the surface $\mathcal{S}$ will always be given the orientation arising from the outgoing normal to $\Omega$.

Given an embedded oriented and compact surface $S$, and a divergenceless vector field $X^a$ we define the charge $Q(\mathcal{S})$ (relative to $X^a$) as

$$Q(\mathcal{S}) = \frac{1}{4\pi}\int_{\mathcal{S}} X_a n^a\, \mathrm{d}S, \tag{35}$$

where $n^a$ is the normal field to $\mathcal{S}$ in $\Sigma$, that together with the orientation of $\Sigma$ returns the orientation of $\mathcal{S}$. Note that because $X^a$ is divergenceless, the charge $Q(\mathcal{S})$ depends only on the homology class of $\mathcal{S}$, denoted by $[\mathcal{S}]$. When $X^a = E^a$ or $X = B$, that is, when $X^a$ is either the electric or the magnetic field, then the associated charges are the electric or the magnetic charges. To avoid excessive writing and to display certain generality, we will work most of the time with an arbitrary vector field $X^a$, instead of the specific vectors $E^a$ and $B^a$.

Note by the Gauss theorem that if $\mathcal{S}$ is screening then the electric or the magnetic charges of $\mathcal{S}$ are equal to the electric or the magnetic charges of the given end $\Sigma_e$.

In the following, we will discuss the notion of *absolute central charges* associated with an end which will play an important role in the proof of theorem 2.2. The relevant properties of charges and central charges are summarized in proposition 4.3. Then, we will explain in proposition 4.4 the basic inequality between area and charge for stable minimal surfaces. Using these elements we sketch then the idea of the proof of theorem 2.2. The rigorous proof is given immediately thereafter.

**Definition 4.2** (Absolute central charges)**.** *Fix an AF end $\Sigma_e$. Let $\mathcal{S} = \mathcal{S}_1 \cup \cdots \cup \mathcal{S}_{k(\Omega)} = \partial\Omega$ be a screening surface of the end $\Sigma_e$. Among the $\mathcal{S}_i$'s there are those that are part of the boundary of the unbounded connected components of $\Sigma \setminus \Omega$. Let us assume that $\{\mathcal{S}_1, \ldots, \mathcal{S}_{k(\Omega)}\}$ were ordered in such a way that $\{\mathcal{S}_1, \ldots, \mathcal{S}_{n(\Omega)}\}$, $n(\Omega) \leqslant k(\Omega)$, are such components. Then, define the absolute central electric or magnetic charges $\bar{Q}_E$ and $\bar{Q}_M$ associated with an end $\Sigma_e$ as*

$$\bar{Q} = \inf_{\Omega} \sqrt{\sum_{i=1}^{i=n(\Omega)} Q^2(\mathcal{S}_i)}, \tag{36}$$

*where for $\bar{Q}_E$, $Q$ is the electric charge and for $\bar{Q}_M$, $Q$ is the magnetic charge and where $\Omega$ ranges among the screened regions of $\Sigma_e$.*

We note now some basic facts about charges and absolute central charges.

**Proposition 4.3.** *Let $\Omega$ be a screened region of an end $\Sigma_e$, and let $\mathcal{S} = \partial\Omega$ be the screening surface. Then,*

*(1) $|Q(\Sigma_e)| = |\sum_{i=1}^{i=n(\Omega)} Q(\mathcal{S}_i)| \leqslant n(\Omega)^{\frac{1}{2}} (\sum_{i=1}^{i=n(\Omega)} Q^2(\mathcal{S}_i))^{\frac{1}{2}}$, where $Q$ is here either an electric or a magnetic charge.*
*(2) $n(\Omega) \leqslant |H_2|$, where $|H_2|$ is the second Betti number[5].*
*(3) $Q^2(\Sigma_e)/|H_2| \leqslant \bar{Q}^2(\mathcal{S})$.*

**Proof.**
*Item 1*. Let $\Sigma \setminus \Omega = \cup_{i=1}^{i=j(\Omega)} \Omega_i^c$, where the $\Omega_i^c$ are connected. Then, we have

$$Q(\Sigma_e) = \sum_{i=1}^{i=j(\Omega)} Q\big(\partial\Omega_i^c\big), \tag{37}$$

where $Q$ is either the electric or the magnetic charge. But we note that if $\Omega_i^c$ is a bounded component, then by the Gauss theorem $Q(\partial\Omega_i^c) = 0$. Using this and recalling then that the surfaces $\mathcal{S}_1, \ldots, \mathcal{S}_{n(\Omega)}$ are those that belong to the boundary of an unbounded connected component, $\Omega_i^c$, of $\Sigma \setminus \Omega$, we obtain

$$\bar{Q}(\Sigma_e) = \sum_{i=1}^{i=j(\Omega)} Q\big(\partial\Omega_i^c\big) = \sum_{i=1}^{i=n(\Omega)} Q(\mathcal{S}_i), \tag{38}$$

and the claim of *item 1* follows.

[5] Recall that the second homology group $H_2(\Sigma, \mathbb{Z})$ of a manifold $\Sigma$ (with finitely many AF ends) is always of the form $H_2 \sim \mathbb{Z}^{|H_2|} \oplus T$, where $T$ is a finite Abelian group called the *Torsion* and where $|H_2|$ is the second Betti number.

*Item 2*. We show now that the surfaces $\mathcal{S}_1, \ldots, \mathcal{S}_{n(\Omega)}$, which are orientable and oriented (from the outgoing normal to $\Omega$), are indeed linearly independent of $H_2(\Sigma, \mathbb{Z})$. Namely we show that if for some integer coefficients $a_i \in \mathbb{Z}$, $i = 1, \ldots, n(\Omega)$, we have

$$\sum_{i=1}^{i=n(\Omega)} a_i[\mathcal{S}_i] = 0 \tag{39}$$

in $H_2$, then $a_i = 0$ for $i = 1, \ldots, n(\Omega)$. Thus, $\mathbb{Z}^{n(\Omega)} \subset H_2$ and therefore $n(\Omega) \leqslant |H_2|$.

A simple and visual way to show this using triangulations of $\Sigma$ is as follows.

Suppose that a certain integer combination of $[\mathcal{S}_i]$ is zero in homology, namely suppose that $\sum_{i=1}^{i=n} a_i[\mathcal{S}_i] = \partial[\bar{C}_3]$ for some integer coefficients $a_i$ and a singular chain $[\bar{C}_3] = \sum b_i[\sigma_i]$, where $\sigma_i : \Delta^3 \to \Sigma$ is a singular three-simplex ([20], p 108). We consider now a closed region $\bar{\Sigma}$, with smooth boundary and containing in its interior the surfaces $\mathcal{S}_i$ and the singular simplices $\sigma_i(\Delta^3)$. It is clear that $\sum_{i=1}^{i=n} a_i[\mathcal{S}_i] = 0$ in $H_2(\bar{\Sigma}, \mathbb{Z})$.

Consider a triangulation of $\Sigma$ by embedded three-simplices (i.e. tetrahedrons) in such a way that every embedded two-simplex (i.e. triangle) of their boundaries is either disjoint from all the $\mathcal{S}_i$'s and $\partial\bar{\Sigma}$ or is inside and embedded in one of the $\mathcal{S}_i$'s or in $\partial\bar{\Sigma}$ (such triangulation always exists). In this way, we will assume $\bar{\Sigma}$ as a $\Delta$-complex ([20], p 104).

We recall that the homology groups of $\bar{\Sigma}$ as a $\Delta$-complex, denoted by $H_i^{\Delta}(\Sigma, \mathbb{Z}), i = 0, 1, 2, 3,$ and the homology groups of $\bar{\Sigma}$, denoted by $H_i(\bar{\Sigma}, \mathbb{Z})$, $i = 0, 1, 2, 3,$ are naturally isomorphic ([20], theorem 2.27).

For this reason, it is enough to argue in terms of chains of the $\Delta$-complex (triangulation) only. We will do that in the following.

Note that for the particular triangulation that we have chosen we can assume $[\mathcal{S}_i]$ as a two-chain of the $\Delta$-complex, namely a sum with coefficients in $\mathbb{Z}$ of oriented three-simplices of the $\Delta$-complex. The same happens with $\partial\bar{\Sigma}$. Suppose then that $\sum_{i=1}^{i=n(\Omega)} a_i[\mathcal{S}_i] = 0$ in $H_2^{\Delta}$, that is, suppose that

$$\sum_{i=1}^{i=n} a_i[\mathcal{S}_i] = \partial[C_3],$$

where $a_i \in \mathbb{Z}$, and $[C_3]$ is a three-chain of the $\Delta$-complex, namely a sum with coefficients in $\mathbb{Z}$ of oriented three-simplices of the $\Delta$-complex. We want to see that all the $a_i's$ must be zero. For this we will make use of smooth embedded, inextensible, oriented curves, denoted by $\xi$, such that

(1) $\xi$ ends along one direction at $\Sigma_e$ and ends along the other direction at another end $\Sigma'_e$ $(\Sigma'_e \neq \Sigma_e)$,
(2) if $\xi$ intersects a two-simplex of the $\Delta$-complex it does so in its interior and transversally to it. Thus, if $\xi$ intersects $\mathcal{S}_i$, then it does so transversally.

Thus, because $\xi$ and $\mathcal{S}_i$ are oriented, their intersection number ([19], chapter 3, section 3) denoted by $(\xi \cap \mathcal{S}_i)$ is well defined[6]. Moreover, we have

$$\sum_{i=1}^{i=n} a_i(\xi \cap [\mathcal{S}_i]) = (\xi \cap \partial[C_3]). \tag{40}$$

We note now that the boundary of any three-simplex of the $\Delta$-complex has signed the intersection number equal to zero to any such curve ($\xi$ gets out of the three-simplex the

[6] The authors of [19] use this notation for the intersection number *mod* 2.

same number of times it gets in). Therefore, the intersection number of any curve $\xi$ with $\partial[C_3]$ must be zero. Therefore, from (40) we get

$$\sum_{i=1}^{i=n} a_i(\xi \cap [\mathcal{S}_i]) = 0 \tag{41}$$

for any such curve $\xi$. Assume now that $a_j \neq 0$. Recalling the definition of the $\mathcal{S}_i'$s we can consider an inextendible curve $\xi$ as before, such that $(\xi \cap S_j) = 1$ and $(\xi \cap \mathcal{S}_i) = 0$ for $i \neq j$. Indeed the curve $\xi$ can be chosen to intersect $S_j$ only once and avoiding intersecting $\mathcal{S}_i,\ i \neq j$. Then, the intersection number of $\xi$ to $\sum a_i[\mathcal{S}_i]$ must be equal to

$$\sum_{i=1}^{i=n(\Omega)} a_i(\xi \cap [\mathcal{S}_i]) = a_j \neq 0, \tag{42}$$

which is a contradiction. This finishes the proof of the second *item*.

*Item 3*. This *item* follows directly from *items 1* and *2*. □

We discuss now the basic relation between charge and area for stable minimal surfaces. We recall first the setup. Let $(\Sigma, h)$ be an oriented Riemannian three-manifold, with possibly many asymptotically flat ends. Suppose that its scalar curvature $R$ satisfies $R \geqslant 2|X|^2$, where the vector field $X^a$ is divergenceless. Then, for any oriented surface $\mathcal{S}$, the charge $Q([\mathcal{S}])$ is given by (35). Then, in this setup, we have the following result proved in [17]. For completeness, we repeat its proof.

**Theorem 4.4** (Gibbons)**.** *Let $\mathcal{S}$ be a stable minimal surface. Then,*

$$A \geqslant 4\pi Q^2, \tag{43}$$

*where A is the area of $\mathcal{S}$ and Q is its charge.*

**Proof.** The stability inequality (where $D$ is the covariant derivative with respect to the Riemannian metric $h$)

$$\int_S |D\alpha|^2 + \frac{1}{2}{}^2\!R\alpha^2 \,\mathrm{d}S \geqslant \frac{1}{2}\int_S R \,\mathrm{d}S \tag{44}$$

with $\alpha = 1$ gives

$$4\pi \geqslant \frac{1}{2}\int_S R\,\mathrm{d}S \geqslant \int_S |X|^2\,\mathrm{d}S \geqslant \frac{(\int_S X_a n^a \,\mathrm{d}S)^2}{A} = \frac{(4\pi Q)^2}{A}, \tag{45}$$

where the last inequality follows from the Cauchy–Schwarz inequality. □

Note that part of the argument above shows that

$$A \leqslant \frac{4\pi}{\overline{|X|^2}}, \tag{46}$$

where $\overline{|X|^2}$ is the average of $|X|^2$ over $S$. Combining this and (43) in the case of the electromagnetic field (Einstein–Maxwell), we get

$$\overline{|E|^2 + |B|^2} \leqslant \frac{1}{Q_E^2 + Q_M^2}. \tag{47}$$

*In other words, the average of the electromagnetic energy over $\mathcal{S}$ is bounded above by the sum of the squares of the electric and magnetic charges.* In a mean sense, the electromagnetic energy cannot be arbitrarily large over $\mathcal{S}$ if $S$ is minimal and stable.

We are ready to discuss and give the proof of theorem 2.2. As said before and to simplify the writing, we will work with a system of the form

$$R \geqslant 2|X|^2,$$
$$D_a X^a = 0,$$

instead of the system

$$R \geqslant 2(|E|^2 + |B|^2), \tag{48}$$

$$D_a E^a = 0, \tag{49}$$

$$D_a B^a = 0 \tag{50}$$

but the argumentation is exactly parallel in this last case.

In this setup, the proof of theorem 2.2 follows from propositions (4.3) and (4.4) and an application of a result of Meeks–Simon–Yau [27]. Indeed, we start by choosing an end $\Sigma_e$ and a screening surface $\mathcal{S}$. We apply then theorem 1 in [27] to obtain a smooth measure-theoretical limit of isotopic variations of $\mathcal{S}$, whose area realizes the infimum of the areas of all the isotopic variations of $\mathcal{S}$. The important fact is that because $\mathcal{S}$ is screening and the limit surfaces (possibly repeated) are a measure-theoretical limit of isotopic variations of $S$, then there is a subset of connected limit surfaces whose union is a screening surface. The inequality (5) follows then applying (43) to any one of these stable components of the limit and using *Item 3* in proposition 4.3.

**Proof of theorem 2.2.** Let $\mathcal{S}$ be an oriented surface embedded in $\Sigma$ and screening the end $\Sigma_e$. Following [27], theorem 1, there exist embedded minimal surfaces, $S_1, \ldots, S_k$, and natural numbers $n_1, \ldots, n_k$ ($n_i \geqslant 0$) such that

(1) $A(\mathcal{S}) \geqslant \inf_{\tilde{\mathcal{S}}\sim\mathcal{S}} A(\tilde{\mathcal{S}}) = n_1 A(\mathcal{S}_1) + \cdots + n_k A(\mathcal{S}_k)$, where $\tilde{\mathcal{S}} \sim \mathcal{S}$ signifies that the infimum is taken over surfaces $\tilde{S}$ isotopic to $S$, and

(2) there is a sequence of surfaces $\{\tilde{\mathcal{S}}\}$ isotopic to $\mathcal{S}$ such that for any continuous function $h$ we have

$$\lim \int_{\tilde{\mathcal{S}}} h \, \mathrm{d}S = \sum_{i=1}^{i=k} n_i \int_{\mathcal{S}_i} h \, \mathrm{d}S \tag{51}$$

which implies, choosing $h = 1$, that $\lim A(\tilde{\mathcal{S}}) = n_1 A(\mathcal{S}_1) + \cdots + n_k A(\mathcal{S}_k)$.

We claim that because $\mathcal{S}$ screens the end $\Sigma_e$, there is a subset of surfaces $\mathcal{S}_1, \ldots, \mathcal{S}_k$ screening $\Sigma_e$. Namely, we claim that there is a screened region $\bar{\Omega}$, such that $\partial\bar{\Omega}$ is a union of some or all of the surfaces $\mathcal{S}_1, \ldots, \mathcal{S}_k$. Let us postpone this technical point to the end and assume for the moment that the surfaces $\mathcal{S}_i$'s were ordered in such a way that $\mathcal{S}_1, \ldots \mathcal{S}_l$, $l \leqslant k$, is such a set of oriented surfaces, or in other words that $\partial\bar{\Omega} = \mathcal{S}_1 \cup \cdots \cup \mathcal{S}_l$.

We therefore calculate

$$A(S) \geqslant \sum_{i=1}^{i=k} n_i A(\mathcal{S}_i) \geqslant 4\pi \sum_{i=1}^{i=l} n_j Q^2(\mathcal{S}_i) \tag{52}$$

$$\geqslant 4\pi \sum_{i=1}^{i=l} Q^2(\mathcal{S}_i) \geqslant 4\pi Q^2 \geqslant \frac{4\pi Q^2}{|H_2|}. \tag{53}$$

The claim of theorem 2.2 follows.

We prove now that there is a subset of the $\mathcal{S}_1, \ldots, \mathcal{S}_k$ screening $\Sigma_e$. For this we will show that every embedded inextensible curve $\xi$ starting at $\Sigma_e$ and ending at $\Sigma_e \neq \Sigma_e$ has to intersect one of the $\mathcal{S}_1, \ldots, \mathcal{S}_k$. If that is the case, define $\Omega$ as the set of points $p$ in $\Sigma \setminus (\mathcal{S}_1 \cup \cdots \cup \mathcal{S}_k)$, such that there is an inextensible embedded curve $\beta$ starting at $\Sigma_e$ and ending at $p$ and not touching any of the surfaces $\mathcal{S}_1, \ldots, \mathcal{S}_k$. Such an open set would not contain any end different from $\Sigma_e$ and its boundary would be a subset of $\mathcal{S}_1, \ldots, \mathcal{S}_k$. Then, the closure $\bar{\Omega}$ of $\Omega$ must be a screened region and its boundary $\partial\bar{\Omega}$ must be a subset of the $\mathcal{S}_1, \ldots, \mathcal{S}_j$. Note that $\partial\bar{\Omega}$ is not necessarily equal to $\partial\Omega$.

Suppose now that there is an inextensible embedded curve $\xi$ starting at $\Sigma_e$ and ending at $\Sigma_e' \neq \Sigma_e$.

Let now $T(r)$, for $r$ small, be a tubular neighborhood of $\xi$ of radius $r$ such that $T(r) \cap (S_1 \cup \cdots \cup S_k) = \emptyset$. Let $\varphi$ be a non-negative function such that $\varphi = 1$ on $T(r/2)$ and 0 on $T(r/2)^c$ ($T^c(r/2)$ is the complement of $T(r/2)$ in $\Sigma$) and let $f$ be a function of support in $T(r)^c$. Then, we have

$$\lim \int_{\tilde{S}} f + \varphi \, \mathrm{d}S = \sum_{i=1}^{i=k} \int_{\mathcal{S}_i} f + \varphi \, \mathrm{d}S = \sum_{i=1}^{i=k} \int_{\mathcal{S}_i} f \, \mathrm{d}S. \tag{54}$$

On the other hand, we have

$$\lim \int_{\tilde{S}} f \, \mathrm{d}S = \sum_{i=1}^{i=k} \int_{\mathcal{S}_i} f \, \mathrm{d}S \tag{55}$$

and

$$\lim \int_{\tilde{S}} \varphi \, \mathrm{d}S \geqslant c > 0 \tag{56}$$

for some fixed constant $c > 0$ and for every element of the sequence $\tilde{S}$. This last inequality follows easily from the fact that every element $\tilde{S}$ must intersect every curve at a distance $d < r/2$ from $\xi$ (otherwise the intersection number between $\xi$ and $\tilde{S}$ would be zero, which would imply that the intersection number between $\xi$ and $S$ would be zero). Inequalities (55) and (56) contradict (54). □

Finally we give the proof of theorem 2.3.

**Proof of theorem 2.3.** In [11], it has been shown that an isoperimetric stable sphere $\mathcal{S}$ satisfies the following inequality[7]:

$$12\pi \geqslant \frac{1}{2} \int_{\mathcal{S}} R \, \mathrm{d}S. \tag{57}$$

Note the extra factor 3 in comparison with (44). The left-hand side of (57) is bounded in the same way as in the proof of theorem 4.4. □

We would like to point out that inequalities of type (5) are precursors of further inequalities between mass and charge squared. Indeed, using the Riemannian–Penrose inequality [9] and theorem 4.4, one can easily prove for instance the following.

**Theorem 4.5** (Mass, charge and global topology)**.** *Let $(\Sigma, (g, K), (E, B))$ be a maximal initial state for the Einstein–Maxwell equations, with asymptotically flat ends. Then, for a given end $\Sigma_e$ we have*

$$4m^2 \geqslant \frac{Q_E^2 + Q_M^2}{|H_2|}, \tag{58}$$

*where m is the mass of $\Sigma_e$ and $Q_E$ and $Q_B$ are its electric and magnetic charges.*

For a different treatment of these types of inequalities, see for instance [24, 18].

[7] There is a missing 1/2 factor in front of $R$ in equation (2) of [11]. This factor translates into equation (5) in that reference.

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