

Energy and Volume: A Proof of the Positivity of ADM Energy Using the Yamabe Invariant of Three-Manifolds

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Abstract: We give a new proof of the positivity (non-negativity) of ADM energy¹ using the Yamabe invariant of three-manifolds. From a physical point of view, the new proof is motivated by a formula (explicitly non-negative) for the total ADM energy of emerging (asymptotically flat) stationary solutions on maximally expanding compact cosmologies. Mathematically, the proof is an application of the Thurston Geometrization of three-manifolds.

Introduction

Remark 1. The *Yamabe invariant* of a compact three-manifold as it is used in the present article, means the supremum over all conformal classes of three-metrics, of the infimum of the *Yamabe functional* (see below) at each conformal class. In the literature, the Yamabe invariant is also known under the name of *sigma constant* [1]. Also, it is common to call the *Yamabe invariant* to the infimum of the Yamabe functional on a given conformal class [6]. Our terminology shouldn't be confused with it.

The proof of the positivity of the ADM mass on asymptotically flat Riemannian three-manifolds² of non-negative scalar curvature has a rich and long history. The first proof, given by Schoen and Yau [11] in 1979 (see also [12]), was partially motivated to conclude the proof of the *Yamabe problem*. Later, in 1981, Witten [13] gave another proof using four-spinors which explicitly displayed the non-negativity of mass through a Bochner-type formula. In 1997, Lohkamp [7] gave a different geometric proof, studying deformations of the scalar curvature on localized regions. More recently, the positivity of mass has been proved through the sharper lower estimates for the mass provided by

¹ Properly speaking, we give a new proof of the *Riemannian positive energy Theorem*. Namely, we prove that an asymptotically flat Riemannian three-manifold with non-negative scalar curvature cannot have negative mass.

² We will always restrict to dimension three.

the *Penrose inequality* [5]. Here, we will provide a proof of the non-negativity of mass which in a vague sense closes a circle of concepts opened since the original proof by Schoen and Yau motivated by the Yamabe problem. Being still a little imprecise, the present proof shows that the non-negativity of mass is implied by the solvability of the Yamabe problem on manifolds with non-positive Yamabe invariant and the computation [2] (after the proof of the Thurston Geometrization Conjecture) of the Yamabe invariant of three-manifolds, for manifolds whose Yamabe invariant is also non-positive.³

We will give thus a new proof of the following well known statement.

Theorem 1. *Say (M, g) is an asymptotically flat Riemannian three-manifold of non-negative scalar curvature. Then, if $m \neq 0$ it is $m > 0$.*

Remark 2. The theorem doesn't tell (and we won't prove) that the manifold is flat and topologically \mathbb{R}^3 when $m = 0$. By asymptotically flat we mean $|\partial^{(i)}(g - g_S)| = \mathcal{O}(r^{-2-i})$ with $i = 0, 1, 2$ and g_S the Schwarzschild metric $g_S = (1 + \frac{m}{2r})^4(dr^2 + r^2d\Omega^2)$.

Let us give below a brief introduction of the main elements involved in the proof. We will end up explaining the ideas behind the main argument.

Say (M, g) is a compact Riemannian three-manifold, define the *Yamabe functional* on the conformal class $[g]$ of g as

$$Y(\tilde{g}) = \frac{\int_M \tilde{R} dv_{\tilde{g}}}{V_{\tilde{g}}^{\frac{1}{3}}},$$

where \tilde{g} is a metric in the conformal class of g , i.e. $\tilde{g} = e^{2f}g$, $V_{\tilde{g}}$ is the volume of M under the volume form of \tilde{g} and \tilde{R} the scalar curvature of \tilde{g} . Denote by $\lambda([g])$ the infimum of Y in $[g]$ and define the *Yamabe invariant* $Y(M)$ of M as the supremum of $\lambda([g])$ over all conformal classes $[g]$. A landmark in geometric analysis is the resolution of the *Yamabe problem* (see [6] for a survey).

Theorem 2. *(Yamabe, Aubin, Trudinger, Schoen). Say g is not conformal to the standard sphere. Then, $\lambda([g]) < \lambda(S^3)$ and there is a metric in $[g]$ of constant scalar curvature reaching $\lambda([g])$.*

Observe that the Yamabe functional is scale invariant therefore if \tilde{g} realizes $\lambda([g])$ so does any scaling of it. Observe too that if $Y(M) < 0$ the Yamabe invariant is equal to minus the infimum of the two-third power of the volumes of Yamabe metrics of scalar curvature negative one. Therefore maximizing the scalar curvature among unit volume Yamabe metrics is equivalent to minimizing the volume among Yamabe metrics of constant scalar curvature minus one. What is the relation between the signature of $Y(M)$ and the topology of M , and how much is its value? This question was partially answered after the proof of the Thurston Geometrization Conjecture [3, 9] via the Ricci flow. Let us briefly review how to obtain Thurston's geometrization on three-manifolds (see [1] for a summary) as it is relevant to the article and to give a partial answer to the question before. Given a three-manifold M , to obtain the geometric decomposition one first performs the prime decomposition, i.e. factors M into a unique (up to reordering) connected sum of prime three-manifolds P_i . A prime three-manifold is one which is not the three

³ It is worth to remark that (in dimension three) the Positivity Energy Theorem is needed to settle the Yamabe problem in the case that the infimum of the Yamabe functional on the given conformal class is positive. Thus it is not needed to solve the Yamabe problem on manifolds with non-positive Yamabe invariant.

sphere and it is either $S^2 \times S^1$ or it is irreducible (i.e. every two sphere bounds a disc). On each one of the resulting (prime) three-manifolds one performs the torus decomposition (JSJ) by excising incompressible tori (those whose fundamental group injects). In this way we obtain a set of manifolds with toric boundaries. Thurston’s geometrization asserts that each one of the resulting pieces admits a geometric structure among eight possible [1]. In particular, after this decomposition is carried out, there is a possibly empty set of manifolds with or without boundary admitting a complete hyperbolic metric of finite volume. We will denote such summands as H_i . An important property [8] of the prime decomposition is that if $M = P_1 \natural \dots \natural P_m$ and $N = \tilde{P}_1 \natural \dots \natural \tilde{P}_n$, then $M \natural N = P_1 \natural \dots \natural P_m \natural \tilde{P}_1 \natural \dots \natural \tilde{P}_n$, where each \natural denotes a connected sum. This property will be used in *Step 3* inside the proof of Theorem 1. The following partial answer to the question above was proved in [2].

Theorem 3. *Say M is a three-manifold whose Thurston decomposition has at least one component with hyperbolic geometry. Then, $Y(M) < 0$ and*

$$Y(M) = -6 \left(\sum V(H_i) \right)^{\frac{2}{3}}.$$

In this case the Yamabe invariant is only sensitive to the hyperbolic sector of the Thurston decomposition. We will exploit this fact to give a proof of the positivity of mass proceeding by contradiction and showing that if not the volume of a certain Yamabe metric of scalar curvature negative one is below $(-Y(M))^{\frac{3}{2}}$ with M a three-manifold of negative Yamabe invariant. Let us explain the main steps to carry out this program. Roughly speaking a metric is asymptotically flat at spatial infinity if asymptotically it looks as the Schwarzschild metric

$$g = \frac{1}{\left(1 - \frac{2m}{r}\right)} dr^2 + r^2 d\Omega, \tag{1}$$

where a priori the mass m can have arbitrary signature.⁴ Let $S(r)$ be the sphere of constant coordinate radius r , and let n be the outgoing normal. Being spherically symmetric the Schwarzschild metric is well known to be conformally flat. In fact defining

$$v = \frac{m}{2} \frac{1 + \sqrt{1 - \frac{2m}{r}}}{1 - \sqrt{1 - \frac{2m}{r}}},$$

we get explicitly

$$g = \left(1 + \frac{m}{2v}\right)^4 (dv^2 + v^2 d\Omega^2),$$

which after the change of variables $u = \ln v$ transforms into

$$g = 4m^2 \cosh^4 \frac{(u - \ln m/2)}{2} (du^2 + d\Omega^2), \tag{2}$$

if $m > 0$ and

$$g = 4m^2 \sinh^4 \frac{(u - \ln -m/2)}{2} (du^2 + d\Omega^2), \tag{3}$$

⁴ We will assume throughout that the mass m is different from zero.

if $m < 0$. The conformal presentation above (Eqs. (2) and (3)) clearly display the dependence of the normal component of the Ricci curvature with respect to the signature of mass. In fact recalling that $Ric(n, n) = -2e^{-2f} \partial_u^2 f$ with f half the logarithm of the conformal factor in Eqs. (2) and (3) respectively, we get

$$Ric(n, n) = \frac{-1}{8m^2 \cosh^6 \frac{u - \ln m/2}{2}},$$

if $m > 0$ and

$$Ric(n, n) = \frac{1}{8m^2 \sinh^6 \frac{u - \ln -m/2}{2}},$$

if $m < 0$. Thus if the mass is positive the normal component of the curvature is negative while it is positive if the mass is negative.⁵ We will use this fact to make a volume comparison. Observe that the mean curvature of a sphere $S(r)$ (as a two-spheres embedded in Schwarzschild space) is $\theta = \frac{2\sqrt{1 - \frac{2m}{r}}}{r}$. As a consequence, the surrounding geometry of the spheres $S(r)$ inside the Schwarzschild space gets closer and closer to the surrounding geometry of the spheres $S_E(r)$ of radius r inside Euclidean space. Thus, large spheres provide a starting point from which to compare volumes between the Schwarzschild and the Euclidean spaces. Let us describe this in more detail. Fix a center o in Euclidean three-space and denote by $S_E(r)$ the two-spheres with center o and radius r . In Schwarzschild space let $d(r_1, r_0)$ ($r_1 < r_0$) be the Riemannian distance between the spheres $S(r_1)$ and $S(r_0)$. Finally denote by $V_E(r, r_0)$ the Euclidean volume lying between the spheres $S_E(r)$ and $S_E(r_0)$ and denote by $V_S(r_1, r_0)$ the volume lying between $S(r_1)$ and $S(r_0)$ inside the Schwarzschild space. In the sense of the classical volume comparison, one would like to compare the volumes $V_S(r_1, r_0)$ and $V_E(r_0 - d(r_1, r_0), r_0)$ as r_1 decreases, starting from $r_1 = r_0$, with r_0 a large radius. According to the Bishop-Gromov volume comparison, as r_1 decreases the volume $V_S(r_1, r_0)$ increases faster than the volume $V_E(r_0 - d(r_1, r_0), r_0)$ when $m > 0$ (because $Ric(n, n) < 0$) and slower if $m < 0$ (because $Ric(n, n) > 0$). Let us quantify this volume comparison. From the expansion of $\frac{1}{\sqrt{1-x}}$ we get

$$\frac{1}{\sqrt{1 - \frac{2m}{r}}} = 1 + \frac{m}{r} + \frac{3}{2} \frac{m^2}{r^2} + \frac{5}{2} \frac{m^3}{r^3} + \frac{35}{8} \frac{m^4}{r^4} + \mathcal{O}(r^{-5}). \tag{4}$$

The distance $d(r_1, r_0)$ is estimated as

$$d(r_1, r_0) = r_0 - r_1 + m \ln \frac{r_0}{r_1} + \mathcal{O}(r_1, r_0), \tag{5}$$

where $\mathcal{O}(r_1, r_0) \rightarrow 0$ as $r_1, r_0 \rightarrow \infty$ and the volume $V_S(r_1, r_0)$ as

$$V_S(r_1, r_0) = \Omega_1 \left(\frac{1}{3}(r_0^3 - r_1^3) + \frac{m}{2}(r_0^2 - r_1^2) + \frac{3m^2}{2}(r_0 - r_1) + \frac{5m^3}{2} \ln \frac{r_0}{r_1} + \mathcal{O}(r_0, r_1) \right), \tag{6}$$

⁵ One may interpret that by saying that if $m < 0$ the Riemannian slice focuses into a naked singularity (at $u = \ln - m/2$) while if $m > 0$ the slice gets thickened (forming a horizon at $u = \ln m/2$) preventing any singularity.

with $\mathcal{O}(r_0, r_1)$ as above and Ω_1 the volume of the unit two-dimensional sphere. Formulas (5) and (6) show that given r_1 the difference in volume between the Schwarzschild and Euclidean space increases to infinity if $m > 0$ as r_0 goes to infinity (and comparing them from r_0) and to negative infinity if $m < 0$. As said before, we will use this fact, proceeding by contradiction and assuming there exists an asymptotically flat metric with negative mass, to find a Yamabe metric of constant scalar curvature negative one whose volume is below $(-Y(M))^{3/2}$ for M certain compact three-manifold with negative Yamabe invariant. In other words we increase the Yamabe invariant of M . The idea is the following. Pick a (any) compact hyperbolic manifold H . Say g_H is the (unique) hyperbolic metric of sectional curvature -1 . Scale g_H to get a metric g_K of sectional curvature $-K$ (and scalar curvature $-6K$) in such a way that the local geometry is almost flat. Pick the (hypothetical) asymptotically flat metric of negative mass and, loosely speaking, place it inside H in replacement of a big ball (in the metric g_K) whose geometry is not far from flat. In this way, as has been argued above, one obtains a new metric g_{\sharp} (and a new manifold M) whose volume is substantially below the volume of (H, g_K) . The crucial point is to show that the gluing can be done with enough care to guarantee that the Yamabe metric of scalar curvature $-6K$ (in the conformal class of g_{\sharp} in M) has a volume still below the volume of (H, g_K) . It is then shown that $-Y(M) > -Y(H) > 0$ which gives the mentioned contradiction.

A cosmological motivation. The reasoning above aroused in the study of the long time evolution of constant mean curvature cosmological solutions having maximal rate of expansion, i.e. expanding as the $\mathcal{K} = -1$ Robertson-Walker cosmological model does. We mention here (sketchily) the main lines of the motivation⁶. For a detailed analysis of the presentation below see [10]. Say $H = M$ is a hyperbolic manifold. For any CMC (constant mean curvature) state (g, K) , where g is a three-metric and K the second fundamental form, define the *reduced volume*⁷ $\mathcal{V}(g, K) = \mathcal{H}^3 V_g$, where $\mathcal{H} = \frac{-k}{3}$ is the Hubble parameter. It is known that the infimum of \mathcal{V} in the set of all CMC states is given by $(-\frac{1}{6}Y(M))^{3/2}$, and therefore $\mathcal{V}_{inf} = V_{g_H}$. Define the *CMC energy* as

$$E = \frac{1}{4\pi\mathcal{H}}(\mathcal{V} - \mathcal{V}_{inf}). \tag{7}$$

Say N is the lapse when we take the mean curvature k as time. Define the Newtonian potential $\phi = \frac{Nk^2}{3} - 1$. Then ϕ satisfies the Poisson like equation

$$\Delta\phi + |K|^2\phi = |\hat{K}|^2,$$

where \hat{K} is the traceless part of K . Consider for simplicity an empty-matter universe which is expanding and suppose that some gravitational energy collapsed into a set of black holes that, asymptotically in time, become asymptotically flat and away from each other. Using the Einstein equations it is seen that the CMC energy evolves as

$$\frac{dE}{d\sigma} = -\frac{1}{4\pi} \int_H \tilde{N} |\hat{K}|^2 dv_g + E,$$

⁶ Although the mathematics can be made rigorous assuming suitable hypotheses, the hypotheses themselves are, mathematically speaking speculative [10].

⁷ Under the name of *Reduced Hamiltonian* the quantity (up to a constant) \mathcal{V} was first introduced and studied in the context of long time evolution by Fischer and Moncrief [4].

where $\sigma = -\ln -k$ is the logarithmic time. Making the assumption that the volumes enclosed by the black holes grow slower than $\frac{1}{r^2}$ and that at big balls surrounding them (centers of mass) the potential is approximately $\phi \sim \frac{-m_i}{r}$ we get after integration by parts that after a long time $\frac{dE}{d\sigma} \sim 0$ and

$$E \sim \sum m_i + \int_{(\cup B_i)^c} |\hat{K}|^2 (1 + \phi) dv_g. \tag{8}$$

The formula above can be interpreted as $E \sim M + R$, where M is the total mass of the black holes and R the total energy of radiation. Observe that the expression R coincides with the kinetic term in the linearized ADM energy (see [10] for a more complete description). A remarkable fact about the formula (8) is that the energy E on the left-hand side is explicitly positive (because of its definition in (7)). This is the fact that inspired the present proof of the positivity of mass.

Proof of Theorem 1. As it was explained in the introduction, we will proceed by contradiction and assume we have an asymptotically flat Riemannian manifold (M, g) with non-negative scalar curvature and negative mass. We will use fresh notation in this section. Fix any compact hyperbolic three-manifold H , with hyperbolic metric g_H of sectional curvature minus one having volume $V(H)$. The proof of Theorem 1 is made in four steps. In the first three steps we will assume that outside a compact set in M the metric g is exactly Schwarzschild. We will explain in the fourth step the necessary modifications to account for the general case. The first step consists in gluing the metric g to a hyperbolic metric g_K of sectional curvature $-K$ in the hyperbolic disc model, carefully controlling the quotient $-R/(6K)$ with R the scalar curvature of the resulting Riemannian metric. Once this is done we place it inside the hyperbolic manifold H with metric $\frac{1}{K}g_H$ of sectional curvature $-K$. Call the resulting Riemannian manifold $(M\sharp H, g_\sharp)$. The second step proceeds to construct a barrier for the solution ϕ of the Yamabe problem (with scalar curvature $-6K$) with base metric g_\sharp . In the third step we show, using the barrier found in the second step, that the volume of $(M\sharp H, \phi^4 g_\sharp)$ is below the minimum possible provided by the Yamabe invariant of $M\sharp H$.

Step 1. As said we assume that outside a compact set in M the metric g is of the form

$$g = \left(1 + \frac{m}{2r}\right)^4 (dr^2 + r^2 d\Omega^2). \tag{9}$$

Recall that the disk representation of a hyperbolic space of sectional curvature $-K$ (and therefore scalar curvature $-6K$) is

$$g_K = \frac{1}{\left(1 - \frac{Kr^2}{4}\right)^2} (dr^2 + r^2 d\Omega^2). \tag{10}$$

We represent a conformally flat metric by $e^{2f} g_F$ with $g_F = dr^2 + r^2 d\Omega^2$ a flat metric. We start the gluing of the metrics (9) and (10) by linearly deforming the exponent f from some radius r_0 to some radius r_1 . A schematic representation can be seen in Fig. 1.

Denote by f_S and f_K half the logarithm of the conformal factors in the metrics (9) and (10) respectively, namely

$$f_K = -\ln \left(1 - \frac{Kr^2}{4}\right),$$

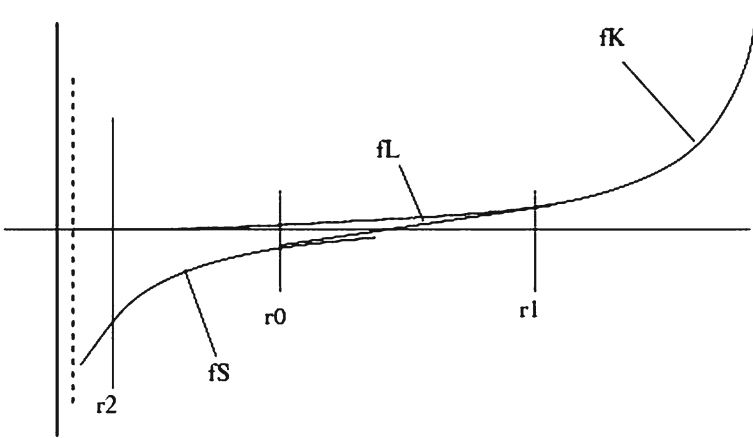


Fig. 1. A schematic representation of the gluing

and

$$f_S = 2 \ln \left(1 + \frac{m}{2r} \right).$$

Denote by f_L the linear function tangent to f_S at r_0 . There are unique K and r_1 such that f_L is tangent to f_K at r_1 . We get them solving the system of equations

$$\frac{-m}{r_0^2 \left(1 + \frac{m}{2r_0} \right)} = \frac{K r_1}{2 \left(1 - \frac{K r_1^2}{4} \right)}, \tag{11}$$

$$-\ln \left(1 - \frac{K r_1^2}{4} \right) = -\frac{m}{r_0^2 \left(1 + \frac{m}{2r_0} \right)} (r_1 - r_0) + 2 \ln \left(1 + \frac{m}{2r_0} \right). \tag{12}$$

We display now the dependence of r_1 and K with respect to r_0 and m . Make $\delta = r_1/r_0$. From Eq. (11) we get

$$\frac{1}{\left(1 - \frac{K r_1^2}{4} \right)} = -\delta \frac{m}{2r_0} \frac{1}{\left(1 + \frac{m}{2r_0} \right)} + 1.$$

Putting this into Eq. (12) and making $U = 1 + \frac{m}{2r_0}$ we get

$$\ln \left(\delta \frac{1-U}{U} + 1 \right) - \ln U^2 = 2 \frac{1-U}{U} (\delta - 1),$$

and rearranging terms

$$\frac{\ln \left(\delta \frac{1-U}{U} + 1 \right) \frac{1}{U^2}}{2 \frac{1-U}{U}} = \delta - 1.$$

When $r_0 \rightarrow \infty$, $U \rightarrow 1$ and $\delta \rightarrow 4$ (use the equivalent $\ln x \sim x - 1$ as U tends to one). Note that $K \sim -\frac{2m}{\delta r_0^3}$ as $r_0 \rightarrow \infty$.

Let R be the scalar curvature of $e^{2f_L} g_F$ considered in the domain where r lies between r_0 and r_1 . As will be explained in *Step 2* we need to estimate the maximum value of $-R/(6K)$ in the interval $[r_0, r_1]$, we do that below. Being spherically symmetric, the scalar curvature R is expressed as

$$R = -4e^{-2f} \left(f'' + \frac{2f'}{r} + \frac{f's^2}{2} \right), \tag{13}$$

where $' = \frac{d}{dr}$. Thus, as $f = f_L$ is linear and increasing, $-R$ is a decreasing function of r in the interval $[r_0, r_1]$. Therefore $-R/6K$ is maximum at r_0 . We have

$$\begin{aligned} -R(r_0) &= 4e^{2f(r_0)} \left(-\frac{2m}{r_0^3 \left(1 + \frac{m}{2r_0}\right)} + \frac{m^2}{2r_0^4 \left(1 + \frac{m}{2r_0}\right)^2} \right) \\ &= 16e^{2f(r_0)} \frac{(1-U)}{r_0^2 U} \left(1 + \frac{(1-U)}{2U} \right). \end{aligned}$$

From Eq. (11) we get

$$K = 4\delta \frac{(1-U)}{U} \frac{1}{r_1^2} \frac{1}{\left(1 + \frac{\delta}{U}(1-U)\right)}.$$

Thus $-R/(6K) \rightarrow (2/3)\delta \rightarrow 8/3$ as $r_0 \rightarrow \infty$. So far we constructed a C^1 exponent, to get a smooth exponent (that we will denote \bar{f}) in the conformal factor ($e^{2\bar{f}}$) we need to deform (slightly) the functions f_L and f_S at r_0 and the functions f_L and f_K at r_1 . As will be explained later we want to do so without changing much the maximum of $8/3$ for the quotient $-R/(6K)$. This is a delicate operation as the scalar curvature Equation involves the second derivative of f . Let us explain how the deformation is performed at r_0 . The deformation at r_1 proceeds along similar lines. Pick a function ξ of one variable, positive and symmetric around the origin with support in $[-1, 1]$ and total integral one. Define the kernel $\xi_\epsilon = \frac{1}{\epsilon} \xi(\frac{x}{\epsilon})$. Say f is the function equal to f_S before r_0 and the function f_L after r_0 . We smooth it out by convolving it with the kernel ξ_ϵ , i.e. we consider the function

$$\bar{f}(r) = \int f(t)\xi_\epsilon(r-t)dt.$$

Integrating by parts we get $\bar{f}'(r) = \int f'(t)\xi_\epsilon(r-t)dt$ and $\bar{f}''(r) = \int^{r_0} f''(t)\xi_\epsilon(r-t)dt$ (note that $f''(r) = 0$ if $r > r_0$). This shows that given β there is γ such that we can modify f on $[r_0 - \gamma, r_0 + \gamma]$ to get $|\bar{f} - f(r_0)| \leq \beta$, $|\bar{f}' - f'(r_0)| \leq \beta$ on $[r_0 - \gamma, r_0 + \gamma]$ while making the second derivative \bar{f}'' increasing (observe that f_S'' is increasing). In particular we can see from Eq. (13) that, up to β , $-R/(6K)$ passes from zero to $8/3$ in the interval $[r_0 - \gamma, r_0 + \gamma]$. This finishes the construction.

Summarizing, we have constructed a metric on M equal to g until $r_0 - \gamma$, hyperbolic of sectional curvature $-K$ after $r_1 + \gamma$ and conformally flat with linear exponent in the conformal factor in the interval $[r_0 + \gamma, r_1 - \gamma]$ in such a way that the quotient $-R/(6K)$ has a maximum in M approaching $8/3$ as r_0 tends to infinity if we choose $\gamma(r_0) \rightarrow 0$ as $r_0 \rightarrow \infty$ conveniently.

We place now the metric constructed above inside the scaled hyperbolic manifold $(H, \frac{1}{K}g_H)$. Observe that the annulus $[r_1 + \gamma, r_1 + 1]$ is isometric to an annulus

$B(p, s_1)/B(p, s_0)$ in $(H, \frac{1}{K}g_H)$, where $B(p, s)$ denotes a ball centered at p and of radius s in $(H, \frac{1}{K}g_H)$. After excising the ball $B(p, s_0)$ we identify both annulus by the given isometry, thus constructing a new manifold denoted as $M_{\sharp}H$ and a Riemannian metric on it denoted as g_{\sharp} .

Step 2. We study now the constant scalar curvature equation

$$6K\phi^5 = 8\Delta\phi - R_{\sharp}\phi, \tag{14}$$

for a metric $g_Y = \phi^4 g_{\sharp}$. We will construct an upper barrier ϕ_T for the solution ϕ allowing us to estimate the total volume of $(M_{\sharp}H, g_Y)$. Let ϕ_K be the conformal factor such that $\phi_K^4 g_{\sharp} = g_K$ on $[r_0 - 2\gamma, r_1 + 1]$. Define the upper barrier ϕ_T to the solution ϕ of Eq. (14) as (we will prove soon this is actually a barrier)

$$\phi_T = \begin{cases} \text{i) } 1 \text{ on the } H - \text{ side of the two - sphere } \{r = r_1\}, \\ \text{ii) } \phi_K(r) \text{ for } r \text{ in } [r_0 - 2\gamma, r_1 + 1], \\ \text{iii) in such a way that } F = 6K\phi_T^5 + R_{\sharp}\phi_T - 8\Delta\phi_T \geq 0, \\ \qquad \qquad \qquad \text{for } r \text{ in } [r_0 - 3\gamma, r_0 - 2\gamma], \\ \text{iv) } \phi_T(r_0 - 2\gamma) \leq \phi_T \text{ (constant)} \leq \phi_T(r_0 - 2\gamma) + \gamma, \\ \qquad \qquad \qquad \text{on the } M - \text{ side of } \{r = r_0 - 3\gamma\}. \end{cases} \tag{15}$$

The construction in the third step in the definition (15) above can be done following the next argument. For a function ϕ_T depending only on the radius, write⁸ $\Delta\phi = \frac{1}{A(s)}(A(s)\phi'_T(s))'$, where s is the radial distance, i.e. $\frac{ds}{dr} = (1 + \frac{m}{2r})^2$, $A(s)$ is the area of the two sphere with s constant and $' = \frac{d}{ds}$. For $r > r_0 - 2\gamma$ we have defined ϕ_T as $\phi_T = \phi_K$ therefore

$$\phi_T = \frac{1}{\left(1 - \frac{Kr^2}{4}\right)^{\frac{1}{2}} \left(1 + \frac{m}{2r}\right)}, \tag{16}$$

for r greater but close to $r_0 - 2\gamma$. A straightforward computation gives $\phi'_T r_0 - 2\gamma \sim m(\frac{1}{2} - \frac{1}{2\delta})\frac{1}{r_0^2} < 0$ as $r_0 \rightarrow \infty$. Pick a function of one variable ξ , being zero for $x < -1$ and one for $x > 0$ with a graph as is represented in Fig. 2. Define ϕ_T on $[r_0 - 3\gamma, r_0 - 2\gamma]$ by running the ODE

$$(A(s)\phi'_T(s))' = \xi\left(\frac{r - (r_0 - 2\gamma)}{\gamma}\right) \frac{6K\phi_K^5(s)A(s)}{8}$$

backwards and starting from $r_0 - 2\gamma$. Say that at $r = r_0 - 2\gamma$ and at $r = r_0 - 3\gamma$ it is $s = s_0$ and $s = s_1$ respectively. We have

$$A(s)\phi'_T(s) = A(s_0)\phi'_T(s_0) - \int_s^{s_0} \frac{1}{8}A(s)\xi\left(\frac{r - (r_0 - 2\gamma)}{\gamma}\right) 6K\phi_K^5(s)ds. \tag{17}$$

Fix Ξ (see Fig. 2) in such a way that $A(s_1)\phi'_T(s_1) = 0$. Then it is $\phi'_T = 0$ and ϕ_T constant for $s < s_1$. Observe that from Eq. (17) it is $\phi'_T \leq 0$ and consequently ϕ_T is increasing in the decreasing direction of s . Therefore

$$F = 6K\phi_T^5 + R_{\sharp}\phi_T - 8\Delta\phi_T = 6K\phi_T^5 - \frac{8}{A}(A\phi'_T)' \geq 6K\phi_K^5(s_0)(1 - \xi) \geq 0.$$

⁸ This formula can be seen easily by integration by parts in a region between two spheres (say $S(s_1)$ and $S(s)$) and then differentiating with respect to s .

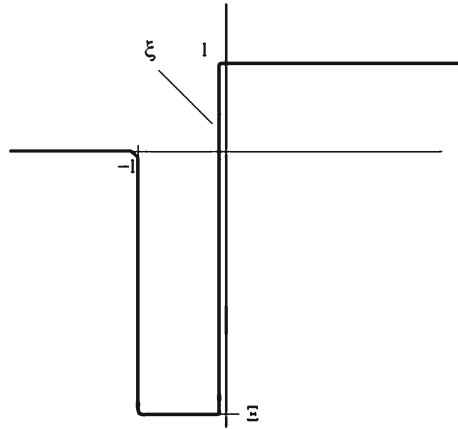


Fig. 2. A schematic representation of ξ

This finishes the construction. Note that $\phi_T \geq 1$ everywhere. To show that ϕ_T is an upper barrier for ϕ , we proceed by contradiction and assume that the infimum of $\phi_T - \phi$ is less than zero. Then at the point q where it takes place it is $\phi(q) > \phi_T(q) \geq 1$. From the equation

$$8\Delta(\phi_T - \phi) = 6K(\phi_T^5 - \phi^5) + R_{\sharp}(\phi_T - \phi) - F,$$

we get

$$6K(\phi_T^5(q) - \phi^5(q) + \frac{R_{\sharp}}{6K}(\phi_T(q) - \phi(q))) \geq F(q) \geq 0. \tag{18}$$

Express $\phi_T^5 - \phi^5$ as $(\phi_T - \phi)(\phi_T^4 + \phi_T^3\phi + \phi_T^2\phi^2 + \phi_T\phi^3 + \phi^4)$. Plugging it into Eq. (18) gives

$$6K(\phi_T(q) - \phi(q)) \left((\phi_T^4 + \phi_T^3\phi + \phi_T^2\phi^2 + \phi_T\phi^3 + \phi^4)(q) + \frac{R_{\sharp}}{6K} \right) \geq 0.$$

The factor $\left((\phi_T^4 + \phi_T^3\phi + \phi_T^2\phi^2 + \phi_T\phi^3 + \phi^4)(q) + \frac{R_{\sharp}}{6K} \right)$ is greater or equal to $5 - 8/3$ as r_0 goes to infinity, thus if r_0 is chosen sufficiently large $\phi_T(q) \geq \phi(q)$ which gives a contradiction.

Step 3. We compare now the volumes of $V(M_{\sharp}H, g_Y)$ and $V(H, \frac{1}{K}g_H)$. Fix a coordinate radius $r = r_2$ with $s = s_2$. We observe that the ball $B(p, s_2)$ in $(H, \frac{1}{K}g_H)$ and the compact side of the two sphere $r = r_2$ in (M, g) have bounded volume as $r_0 \rightarrow \infty$. Also observe that because $\phi_T \rightarrow 1$ as $r_0 \rightarrow \infty$, the volume of the annulus $[r_2, r_1]$ in $(M_{\sharp}H, g_Y)$ minus the volume of annulus $B(p, s_1)/B(p, s_2)$ in $(H, \frac{1}{K}g_H)$ tends to $-\infty$ as r_0 tends to ∞ . Thus for r_0 sufficiently large it is $V(M_{\sharp}H, g_Y) < V(H, \frac{1}{K}g_H)$.

Now the manifold $M_{\sharp}H$ has (independently of the nature of M) the hyperbolic piece H in its Thurston decomposition. Therefore by Theorem 3 it must be $V(M_{\sharp}H, g_Y) \geq V(H, \frac{1}{K}g_H)$, which is absurd. This finishes the proof of Theorem 1 in the case the metric g is exactly Schwarzschild outside a compact set in M .

Step 4. We treat now the case when the metric g is not exactly Schwarzschild outside a compact set. We start by introducing a notation. We say that a function h is an $\mathcal{O}(r^{-\alpha})$

as $r_0 \rightarrow \infty$ if $\sup\{h/r^{-\alpha}\}$ is bounded as $r_0 \rightarrow \infty$. Now we observe that the argument carried out in steps one, two and three above can be repeated if we can (slightly) deform the metric g to a metric \tilde{g} inside the annulus $[r_0 - r_0^{\frac{1}{2}+\epsilon}, r_0]$ to get an exact Schwarzschild metric of mass m after r_0 in such a way that $R_{\tilde{g}}$ is an $\mathcal{O}(r^{-3-2\epsilon})$ as $r_0 \rightarrow \infty$ and in particular with $\sup\{-R_{\tilde{g}}/r^3\}$ tending to zero as r_0 tends to ∞ . Let φ be a smooth non-negative function of one variable x with range in $[0, 1]$, being one for $x < 0$ and zero for $x > 1$. Consider the metric $\tilde{g} = g_S + \varphi\left(\frac{r-r_0+r_0^{\frac{1}{2}+\epsilon}}{r_0^{\frac{1}{2}+\epsilon}}\right)(g - g_S)$. Write $\tilde{\nabla} = \nabla^S + \Gamma$.

A computation gives

$$R_{\tilde{g}} = Ric^{(S)\alpha\beta}\tilde{g}_{\alpha\beta} - \frac{1}{2}\tilde{g}^{\alpha\beta}\nabla_{\alpha}^S\nabla_{\beta}^S \ln|\tilde{g}| + \Gamma_{\mu\rho}^{\nu}\Gamma_{\nu\bar{\rho}}^{\rho}\tilde{g}^{\mu\bar{\rho}}, \tag{19}$$

where $\sqrt{|\tilde{g}|}$ is the quotient of the volume forms of \tilde{g} and g_S , i.e. $dv_{\tilde{g}} = \sqrt{|\tilde{g}|}dv_{g_S}$. The Christoffel symbols Γ are computed as

$$\Gamma_{\alpha\beta}^{\mu} = \frac{1}{2}(\nabla_{\alpha}^S(\tilde{g}_{\beta m} - g_{(S)\beta m}) + \nabla_{\beta}^S(\tilde{g}_{\alpha m} - g_{(S)\alpha m}) - \nabla_m^S(\tilde{g}_{\alpha\beta} - g_{(S)\alpha\beta}))g_S^{m\mu}. \tag{20}$$

Substituting $\tilde{g} - g_S$ for $(\varphi)(g - g_S)$ in Eq. (20) it is seen that Γ and $\nabla^S\Gamma$ are $\mathcal{O}(r^{-\frac{5}{2}-\epsilon})$ and $\mathcal{O}(r^{-3-2\epsilon})$ as $r_0 \rightarrow \infty$ respectively. This implies that the last term of Eq. (19) is an $\mathcal{O}(r^{-5-2\epsilon})$ as $r_0 \rightarrow \infty$. To analyze the second term in Eq. (19) we recall that $\Gamma_{\nu\rho}^{\nu} = \partial_{\rho} \ln \sqrt{|\tilde{g}|}$, which makes it an $\mathcal{O}(r^{-3-2\epsilon})$ as $r_0 \rightarrow \infty$. The first term in Eq. (19) is seen to be an $\mathcal{O}(r^{-5})$ as $r_0 \rightarrow \infty$ by noting that it can be written as $Ric^{(S)\alpha\beta}(\tilde{g}_{\alpha\beta} - g_{(S)\alpha\beta})$. \square

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