# Invariant Manifolds for Dominated Splitting on Surfaces

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### Abstract

In the context of the study of surface diffeomorphisms, we strengthen the known results on the existence of invariant manifolds asociated to non-hyperbolic compact invariant sets.

Let $M$ be a smooth Riemann surface without boundary, and let $f$ denote a $C^1$ generic difeomorphism of $M$.

Specifically assume that all periodic points of $f$ are hyperbolic, and that every chain-recurrent point of $f$ is approximated by periodic points (recall that a point is chain-recurrent if there exist periodic $\epsilon$-pseudo orbits through it for all positive $\epsilon$). The second property was proven to be generic only recently as a consecuence of an improved "Closing Lemma" (see [BC04]).

Suppose that $\Lambda$ is a compact invariant subset of $M$ such that the set of periodic points of saddle type is dense in $\Lambda$.

If $\Lambda$ is a hyperbolic set for $f$ then many dynamical properties are known thanks to the existence of stable and unstable manifolds through every $x \in \Lambda$. We will assume $\Lambda$ has a much weaker property, namely that it has a dominated splitting for $f$.

Dominated splitting is the property that there exist constants $C > 0$ and $\lambda \in (0, 1)$ such that for every periodic point of saddle type $x \in \Lambda$ the following condition on the tangent map holds (were $E^s$ and $E^u$ denote the stable and unstable subspaces as usual):

$$\left\| D_x f^k_{/E^s(x)} \right\| \cdot \left\| D_{f^k(x)} f^{-k}_{/E^u(f^k(x))} \right\| \leq C \cdot \lambda^k \text{ for all } k \geq 0$$

This domination condition states that the angle between any tangent vector at a periodic point of saddle type and $E^u$ contracts at a fixed exponential rate as we iterate $Df$ (the only exception being those vectors that belong to $E^s$). It also implies that stable and unstable subspaces vary continuously from periodic point to periodic point (regardless of the period of the points involved).

Furthermore it can be shown by restating this condition in terms of a continuous family of cones, that there is a open set $U$ containing $\Lambda$ such that if $\Lambda^+$ is set of points whose future orbit remains in the topological closure of $U$, and $\Lambda^-$ is defined analogously, then the sub bundle $E^s$ can be extended continuously to $\Lambda^+$, and the sub-bundle $E^u$ can be extended continuously to $\Lambda^-$. These extensions (we will call them $E$ and $F$ respectively) retain the property that $Df(E(x)) = E(f(x))$ and $Df^{-1}(F(x)) = F(f^{-1}(x))$ at all points where these expressions are defined. Also the domination condition above holds for all points $x \in \Lambda$ (periodic of not) replacing $E^s$ and $E^u$ by their extensions $E$ and $F$.

The fact that makes dominated splitting such a powerful tool in the study of surface maps, is that the proof of the stable manifold theorem can be adapted to provide center manifolds $W^{cs}(x)$ and $W^{cu}(x)$ for all $x \in \Lambda$. However the dynamical properties of these manifolds are much weaker then in the hyperbolic case. In particular dominated splitting doesn't rule out the possibility that there might even be exponential expansion in the $E$ direction, as long as this expansion is much weaker then the one in the $F$ direction. This means for example, that it is impossible to guarantee the invariance of the center stable manifolds under $f$. However it is clear by their construction that the manifolds are locally-invariant: that is, for each neighborhood of $f(x)$ in $W^{cs}(f(x))$ there exists a neighborhood of $x$ in $W^{cs}(x)$ such that it's image is contained in the former.

The precise theorem we will use is the following corollary of the work in [HPS77]:

**Theorem 0.1.** *Let $Emb^1((-1,1), M)$ denote the set of $C^1$ embeddings of the open interval $(-1,1)$ into $M$ endowed with the $C^1$ topology. There exist two continuous maps $\phi^{cs} : \Lambda^+ \to Emb^1((-1,1), M)$ and $\phi^{cu} : \Lambda^- \to Emb^1((-1,1), M)$ such that if we let $W^{cs}_\epsilon(x) = \phi^{cs}(x)((-\epsilon, \epsilon))$ and $W^{cu}_\epsilon(x) = \phi^{cu}(x)((-\epsilon, \epsilon))$ the following properties hold:*

1. *$\phi^{cs}(x)(0) = x$ for all $x \in \Lambda^+$ and $\phi^{cu}(x)(0) = x$ for all $x \in \Lambda^-$*
2. *$T_x W^{cs}_1(x) = E$ for all $x \in \Lambda^+$ and $T_x W^{cu}_1(x) = F$ for all $x \in \Lambda^-$*
3. *for every positive $\epsilon < 1$ there exists a positive $\delta$ such that:*
    (a) *$f(W^{cs}_\delta(x)) \subset W^{cs}_\epsilon(f(x))$ for all $x \in \Lambda^+$*

*(b)* $f^{-1}(W^{cu}_\delta(x)) \subset W^{cu}_\epsilon(f^{-1}(x))$ *for all* $x \in \Lambda^-$

Property 3 is too weak to obtain serious dynamical results. The purpose of this paper is to show that a stronger property actually holds if $\epsilon$ is small enough.

To be precise fix $\sqrt{\lambda} < \rho < 1$ and let $c = \frac{\rho}{\sqrt{\lambda}}$.

In first place we need to assume that the embeddings $\phi^{cs/cu}(x)$ all have the same derivative at 0, so that we can make the lengths of $W^{cs/cu}_\epsilon(x)$ very similar for all $x$ by taking $\epsilon$ small enough. In particular it should be true that for any $x, y \in \Lambda$ if $\beta_x$ is a connected component of $W^{cs/cu}_\epsilon(x) \setminus \{x\}$ and $\beta_y$ is analogously defined for $y$ then $\rho\,|\beta_y| < |\beta_x|$ (the absolute value symbols denote length of the respective curves).

Next we need to assume four more conditions on $\epsilon$ which are symmetric respect to time. Each condition is assumed to be valid for all $x \in \Lambda$. In the third condition $y \in W^{cs}_\epsilon$ and $E_1 = T_y W^{cs}_\epsilon(x)$ (analogously for the symmetric condition with $W^{cu}$ and $F_1$). In the forth condition it is assumed $y \in W^{cs}_\epsilon(x) \cap \Lambda^-$ so that $F$ is defined on $y$ (and once more the reader can supply the symmetric case). The conditions are as follows:

| | |
|---|---|
| $W^{cs}_\epsilon(x) \subset U$ | $W^{cu}_\epsilon(x) \subset U$ |
| $f(W^{cs}_\epsilon(x)) \subset W^{cs}_1(f(x))$ | $f^{-1}(W^{cu}_\epsilon(x)) \subset W^{cu}_1(f^{-1}(x))$ |
| $\Vert D_y f_{/E_1}\Vert \leq c \cdot \Vert D_x f_{/E}\Vert$ | $\Vert D_y f^{-1}_{/F_1}\Vert \leq c \cdot \Vert D_x f^{-1}_{/F}\Vert$ |
| $\Vert D_y f^{-1}_{/F}\Vert \leq c \cdot \Vert D_x f^{-1}_{/F}\Vert$ | $\Vert D_y f_{/E}\Vert \leq c \cdot \Vert D_x f_{/E}\Vert$ |

Our result is the following:

**Theorem 0.2.** *Under the conditions stated above, there exists* $\delta > 0$ *such that for all* $x \in \Lambda$ *the following holds:*

1. $f^k(W^{cs}_\delta(x)) \subset W^{cs}_\epsilon(f^k(x))$ *for all* $k \geq 0$
2. $f^{-k}(W^{cu}_\delta(x)) \subset W^{cu}_\epsilon(f^{-k}(x))$ *for all* $k \geq 0$

This result is non-trivial because property 3 in theorem 0.1 allows us to obtain $\delta$ for each fixed $k \geq 0$ but this $\delta$ goes to zero as $k$ increases.

Also we believe the result is interesting because we think it will have further consequences in the dynamics of generic diffeomorphisms of surfaces and 3-manifolds (since this proof is easily adapted to dominated splittings with a dimension 1 sub-bundle).

## 1 Proof of the Main Theorem

Since our result is symmetric with respect to $f$ and $f^{-1}$ we will prove only the statement about $W^{cu}$ (assuming only the right half of the conditions on $\epsilon$). Also since the truth of the statement for any iterate of $f$ implies the statement for $f$, we can assume that the constant $C$ of the dominated splitting is equal to 1 (by taking a large iterate $f^k$ so that $C\lambda^k < 1$).

For each periodic point of saddle type $x \in \Lambda$ let $I_x$ denote the largest open interval contained in $W^{cu}_\epsilon(x)$ such that:

1. $x \in I_x$
2. $f^{-k}(I_x) \subset W^{cu}_\epsilon(x)$ for all $k \geq 0$

We would like to show that $I_x$ contains a large interval for every $x$. More precisely we have to show that both connected components (from now on simply components) of $I_x \setminus \{x\}$ are large.

The fact that $I_x$ is a non-empty interval is guaranteed because a small portion of the unstable manifold of $x$ is contained in $W^{cu}_\epsilon(x)$. A trivial consequence of the maximality of $I_x$ is that $f^{-k}(I_x) \subset I_{f^{-k}(x)}$ for all $k \geq 0$.

Let $\alpha$ be a component of $I_x \setminus \{x\}$, a less trivial observation is the following:

**Remark 1.1.** *Either $\alpha$ is a component of $W^{cu}_\epsilon(x) \setminus \{x\}$ or $f^{-1}(\alpha)$ is a component of $I_{f^{-1}(x)} \setminus \{f^{-1}(x)\}$.*

We can obtain serious dynamical consequences out of the existence of small components $\alpha$. The first step is the following:

**Lemma 1.1.** *If $x \in \Lambda$ is a periodic point of saddle type, and $\alpha$ is a component of $I_x \setminus \{x\}$ such that:*

$$f^{-k}(\alpha) \textit{ is not a component of } W^{cu}_\epsilon(f^{-k}(x)) \setminus \{f^{-k}(x)\} \textit{ for any } k \geq 0$$

*then there exists $y$ in the orbit of $x$ and $\beta$ a component of $W^{cu}_\epsilon(y) \setminus \{y\}$ such that:*

$$f^k(\beta) \subset W^{cu}_\epsilon(f^k(y)) \textit{ for all } k \geq 0$$

*Proof.* Let $\pi(x)$ denote the period of $x$.

In the component of $W^{cu}_\epsilon(x) \setminus \{x\}$ that contains $\alpha$ take a strictly larger open interval $\alpha'$ such that:

$$f^{-k}(\alpha') \subset W^{cu}_\epsilon(f^{-k}(x)) \text{ for all } 0 \leq k \leq 2\pi(x)$$

Now let $K$ be the smallest positive integer such that the above inclusion fails. If we set $y = f^{-K}(x)$ and let $\beta$ be the component of $W^{cu}_\epsilon(y) \setminus \{y\}$ that intersects $f^{-K}(\alpha')$ then (by the second condition on $\epsilon$ in the above table) we have that $f^{-K}(\alpha') \supset \beta$. Therefore the following two conditions on $\beta$ hold:

1. $f^k(\beta) \subset W^{cu}_\epsilon(f^k(y))$ for all $0 \leq k \leq 2\pi(x)$
2. $f^{2\pi(x)}(\beta) \subset \beta$ (the 2 is necessary because $Df^{\pi(x)}$ might revert the $F$ direction)

the lemma follows easily. $\square$

The preceding lemma means that if small components of $I_x \setminus \{x\}$ exist, then we have periodic points of saddle type with very small expansion in their unstable direction (as is seen by the properties of $\beta$). In the following lemmas we will show that only a finite number of such "bad saddle points" can exist in $\Lambda$.

**Lemma 1.2.** *If $x \in \Lambda$ and $\beta_x$ is a component of $W^{cu}_\epsilon(x) \setminus \{x\}$ such that:*

$$f^k(\beta) \subset W^{cu}_\epsilon(f^k(x)) \text{ for all } k \geq 0$$

*then $\beta_x \subset \Lambda^+$ and*

$$\left\| D_y f^k_{/E} \right\| \leq \rho^k \text{ for all } k \geq 0 \text{ and all } y \in \beta$$

*Proof.* The fact that $\beta_x \subset \Lambda^+$ is a simple consequence of the first row of our table of assumptions on $\epsilon$.

We will now show that $\|D_x f^k_{/F}\| \leq \frac{1}{\sqrt{\lambda}^k}$ for all $k \geq 0$.

Suppose that this property fails for some $K$, and let $y = f^K(x)$ and $\beta_y$ be the component of $W^{cu}_\epsilon(y) \setminus \{y\}$ that contains $f^K(\beta_x)$. We have that $\|D_y f^{-K}_{/F}\| < \sqrt{\lambda}^K$ and therefore (by the third row of properties of $\epsilon$) that $\|D_z f^{-K}_{F_1}\| < \rho^k$ for all $z \in f^K(\beta_x)$. But this would imply that:

$$|\beta_x| = |f^{-K}(f^K(\beta_x))| < \rho^K |f^K(\beta_x)| \leq \rho |\beta_y|$$

contrary to our first assumption on $\epsilon$ (actually that assumption is more about making the parameterizations $\phi^{cs/cu}$ compatible with arc length and could be included in the statement of Theorem 0.1).

We have shown that $\|D_x f^k_{/F}\| \leq \frac{1}{\sqrt{\lambda}^k}$ or equivalently that:

$$\sqrt{\lambda}^k \leq \|D_{f^k(x)} f^{-k}_{/F}\| \text{ for all } k \geq 0$$

The domination condition now implies that $\|D_x f^k_{/E}\| \leq \sqrt{\lambda}^k$ for all $k \geq 0$ and this in turn implies (by the fourth row of conditions on $\epsilon$) that:

$$\left\| D_y f^k_{/E} \right\| \leq \rho^k \text{ for all } k \geq 0 \text{ and all } y \in \beta$$

as stated. $\square$

The following lemma asserts that exponential contraction in the $E$ direction provides uniform size stable manifolds.

**Lemma 1.3.** *There exists a positive number* $\epsilon_1$ *such that every* $x \in \Lambda^+$ *with* $\|D_x f^k_{/E}\| \leq \rho^k$ *for all* $k \geq 0$ *has the property that:*

$$W^{cs}_{\epsilon_1}(x) \subset W^s(x)$$

*Proof.* Fix a number $\rho_1$ in the open interval $(\rho, 1)$. And let $\epsilon_1$ be small enough so that if $x$ is any point in $\Lambda^+$, $y$ is in $W^{cs}_{\epsilon_1}(x)$ and $E_1$ is the tangent space at $y$ to $W^{cs}_{\epsilon_1}(x)$ the following two conditions hold:

1. $f(W^{cs}_{\epsilon_1}(x)) \subset W^{cs}_1(f(x))$
2. $\|D_y f_{/E_1}\| \leq \frac{\rho_1}{\rho} \cdot \|D_x f_{/E}\|$

the lemma easily follows and furthermore we can guarantee exponential contraction of $W^{cs}_{\epsilon_1}(x)$ of rate at least as small as $\rho_1$ under the future iterates of $f$. ☐

The content of these lemmas is the following situation: Let $x$ be a point (periodic or not) in $\Lambda$ and let $\beta$ be a component of $W^{cu}_\epsilon(x)$ such that $f^k(\beta) \subset W^{cu}_\epsilon(f^k(x))$ for all $k \geq 0$. For each $y \in \beta$ we have that $W^{cs}_{\epsilon_1}(y) \subset W^s(y)$. Let $B = \bigcup_{y \in \beta} W^{cs}_{\epsilon_1}(y)$, we will call $B$ an attracting box.

The following property of attracting boxes is central to the proof of the main theorem:

**Lemma 1.4.** *No attracting box can contain infinitely many periodic points.*

*Proof.* Let $x \in \Lambda$ and a component $\beta$ of $W^{cu}_\epsilon(x) \setminus \{x\}$ define an attracting box $B$ which contains infinitely many periodic points.

If $x$ is periodic, then by one-dimensional dynamics we know that all points in $\beta$ converge under $f^{2\pi(x)}$ to a fixed point of $f^{2\pi(x)}$. Since $B$ contains infinitely many periodic points we conclude that $\beta$ contains infinitely many fixed points of $f^{2\pi(x)}$ contradicting our hypothesis that all periodic points of $f$ are hyperbolic.

If $x$ is not periodic, take a sequence $n_k$ such that $f^{n_k}(x) \to y$ and $|f^{n_k}(\beta)| \to \delta$. Since $B$ contains many periodic points the lengths of the iterates of $\beta$ are bounded away from zero and therefore $\delta > 0$. We conclude that there is a component $\beta_y$ of $W^{cu}_\delta(y) \setminus \{y\}$ such that $f^{n_k}(\beta) \to \beta_y$ in the $C^1$ topology, and this component must contain infinitely many periodic points. By taking $g = f^k$ an iterate of $f$ such that $g(\beta_y) \cap \beta_y \neq \emptyset$ the arguments in [PS00, p.989] imply that $\bigcup_{n \in \mathbb{Z}} g^n(\beta_y)$ is a $g$-invariant one-dimensional manifold. This manifold would contain infinitely many periodic points of $f$ which would therefore have bounded periods contradicting once more the fact that all periodic points of $f$ are hyperbolic. $\square$

The previous two lemmas are strong enough to imply that only finitely many periodic points of saddle type can be in their hypothesis.

**Lemma 1.5.** *For all but a finite number of periodic points of saddle type $x \in \Lambda$ it holds that if $\alpha$ is component of $I_x \setminus \{x\}$ then there exists $k \geq 0$ such that $f^{-k}(\alpha)$ is a component of $W_\epsilon^{cu}(f^{-k}(x)) \setminus \{f^{-k}(x)\}$.*

*Proof.* If the lemma is false we have a sequence of attracting boxes $B_n$ defined by periodic points $x_n$ and components $\beta_n$ of $W_\epsilon^{cu}(x_n) \setminus \{x_n\}$.

We will assume that $x_n \to x$ as $n \to +\infty$, and that the curves $\beta_n$ all point toward the same side of $F$ in the following sense: there exists $\delta > 0$ such that if $x_n$ and $x_m$ are at a distance less than $\delta$ either $x_n \in B_m$ or $x_m \in B_n$. A equivalent condition is that $\beta_n \to \beta$ for some component $\beta$ of $W_\epsilon^{cu}(x) \setminus \{x\}$.

If there exists $n$ such that $x \in B_n$ then $B_n$ contains infinitely many periodic points. If not then $x$ and $\beta$ define an attracting box $B$ containing infinitely many periodic points. Both of these possibilities contradict the previous lemma. $\square$

Now we are in position to prove the main theorem for periodic points of saddle type, the general statement follows from the denseness of such points in $\Lambda$.

**Theorem 1.1.** *There exists a positive number $\delta$ such that for all periodic points $x \in \Lambda$ of saddle type $W_\delta^{cu}(x) \subset I_x$.*

*Proof.* If the theorem is false we have a sequence $\alpha_n$ of components of $I_{x_n} \setminus \{x_n\}$ for some sequence $x_n$ of periodic points of saddle type such that $|\alpha_n| \to 0$.

This in turn implies by the previous lemmas a sequence $y_n = f^{-k_n}(x_n)$ such that for some component $\beta_n$ of $W_\epsilon^{cu}(y_n) \setminus \{y_n\}$ we have that $\beta_n = f^{k_n}(\alpha_n)$. Also the boundedness of the derivative of $f^{-1}$ implies that $k_n \to +\infty$ as $n \to +\infty$.

For any point $z \in \beta_n$ we have that $f^{k_n}(z) \in \alpha_n$ and $f^{-2\pi(y_n)+k_n}(z) \in \alpha_n$. Therefore through $z$ there is a $|\alpha_n|$ pseudo-orbit.

Suppose without loss of generality that $y_n \to y$ and $\beta_n \to \beta$ where $\beta$ is some component of $W_\epsilon^{cu}(y) \setminus \{y\}$. Note that $y$ and $\beta$ define an attracting box $B$ since $f^k(\beta) \subset W_\epsilon^{cu}(f^k(y))$ for all $k \geq 0$. Also by the previous paragraph all points in $\beta$ are chain-recurrent (the pseudo orbits are constructed by taking a nearby point in $\beta_n$ and using it's pseudo-orbit). Since we have assumed that all chain recurrent points are approximated by periodic ones, this implies that $B$ contains infinitely many periodic points, contrary to the previous lemma. ☐

# References

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