Cones on Surfaces

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November 28, 2007

1 Cones

Let $\mathcal{V}$ be the Euclidean Plane (i.e. a real vector space of dimension 2 with an inner product $\langle \ , \ \rangle$). We can define a cone as the set of vectors whose angle to a given one-dimensional subspace is less than or equal to a given one.

If we take $c \in [0, 1]$ to be the cosine of the angle and $v \in \mathcal{V}$ to be a generator of the subspace, we obtain the following definition for the cone C:

$$C = \{w \in \mathcal{V} : |\langle v, w\rangle| \geq c \cdot \|v\| \cdot \|w\|\}$$

Notice that if $c = 1$ the cone C is a one-dimensional subspace, and if $c = 0$ then the cone is the entire plane.

Let $\mathcal{C}$ denote the set of all cones. The most natural paramatrization of the set of cones is probably the following: Take the closed disk $\mathcal{D} = \{v \in \mathcal{V} : \|v\| \leq 1\}$ and identify each point $v \in \mathcal{D}$ with a cone by setting $c = \|v\|$ in the above definition. More formally we have defined a function $C : \mathcal{D} \to \mathcal{C}$ by the following formula:

$$C(v) = \{w \in \mathcal{V} : |\langle v, w\rangle| \geq \|v\|^2 \cdot \|w\|\}$$

This paramatrization shows that we can think of $\mathcal{C}$ as the space we obtain by identifying all pairs of opposite vectors in $\mathcal{D}$. In other words the space of cones is itself a cone.

2 Elementary Geometry on the Space of Cones

We are now thinking of the space of cones as a cone with a 180 degree vertex angle. And we use $C : \mathcal{D} \to \mathcal{C}$ to paramatrize this space.

There is a nice geometric interpretation of this parametrization: Given a vector $v \in \mathcal{D}$ with $0 < \|v\| < 1$ take a line perpendicular to v at it's endpoint and note that it cuts the unit circle at two points v_1 and v_2. The two subspaces generated by v_1 and v_2 are the boundarys of $C(v)$.

Given the coordinate $v \in \partial\mathcal{D}$ of a one-dimensional subspace $C(v)$ the coordinates of the set of cones that contain this subspace as one of their boundarys is the union of two circles circle whose diameters are the line segments $[0, v]$ and $[0, -v]$. The interior of these two circles are the coordinates of cones that contain $C(v)$ in their interior (i.e. not as a boundary). All other coordinates map to cones that do not intersect $C(v)$ except at $0 \in V$.

It's also easy to find geometricaly the coordinates of cones that contain a given cone. Since a point $v \in \mathcal{D}$ with $0 < \|v\| < 1$ represents a cone that is formed by an interval of subspaces whose coordinates are an interval $[v_1, v_2]$ on the boundary of $\mathcal{D}$. We find that the coordinates of cones containing $C(v)$ is simply an intersection of circles whose diameter are radii of D.

The last thing we will mention in this section is that the concept of rotation makes sense on $\mathcal{C}$. Simply take a rotation on D and see how it projects to $\mathcal{C}$. The rotation of 180 degrees is the identity map since this is the angle at the vertex of $\mathcal{C}$.

3 Invertable Linear Maps

Any invertable linear map sends cones to cones. Therefore each such map $T : \mathcal{V} \to \mathcal{V}$ defines a transformation on $T_c : \mathcal{C} \to \mathcal{C}$. Trivially $(TS)_c = T_c S_c$. We will now attempt to describe the dynamics of this transformation.

If T is defined by the matrix:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

then by looking at the image of the cone generated by the upper right cuadrant of $\mathbb{R}^2$ we see that an invertable linear transformation doesn't always send the bisector of a cone to the bisector of it's image. This suggests that the easy way to calculate the image of a general cone might be to calculate the images of it's two boundary subspaces. This observation leads to what follows.

For $v \in \mathcal{D}$ with $0 < \|v\| < 1$ define v_1 and v_2 as in the previous section.

Given an invertable linear map $T : \mathcal{V} \to \mathcal{V}$ we define $T_d : \mathcal{D} \to \mathcal{D}$ as follows:

$$T_d v = \begin{cases} 0 & \text{if } v = 0 \\ \frac{1}{2}\left(\frac{Tv_1}{\|Tv_1\|} + \frac{Tv_2}{\|Tv_2\|}\right) & \text{if } 0 < \|v\| < 1 \\ \frac{Tv}{\|Tv\|} & \text{if } \|v\| = 1 \end{cases}$$

The map T_d is a lift of T_c which is now seen to be a homeomorphism.

What are the dynamics of T_c? That is, how does T transform cones under iteration?

If T is a multiple of the identity map it leaves all cones fixed and therefore T_c is the identity map on $\mathcal{C}$.

If T is a multiple of a axial symetry T_d is the same symetry and T_c is a kind of symmetry with respect to a radius of $\mathcal{C}$.

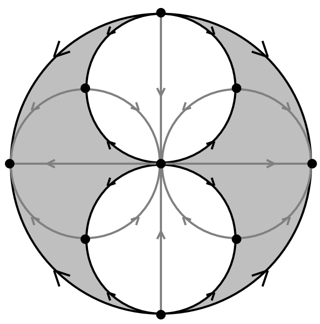
If T has two conjugate complex eigenvalues then it is a multiple of a rotation and we can see that T_c is the same rotation centered at the vertex of $\mathcal{C}$.

The other cases are a bit more difficult, we will illustrate a standard example in each case and then prove that all cases have been covered.

If we define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by the matrix

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

then the following picture illustrates the dynamics of T_d on $\mathcal{D}$.



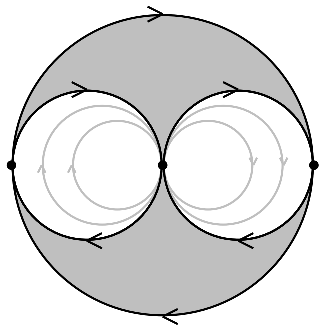
There are nine fixed points (represented by black dots) which are the coordinates of the five fixed cones under T. The four fixed points on the boundary map to the eigenspaces under the parametrization C. The coordinates of the eigenspace of eigenvalue 2 are atracting and their basin of attraction is shown in gray. There are four saddle fixed points which are coordinates for the two cones which have the eigenspaces as their boundary.

There is also a non-hyperbolic fixed point at 0 similar to a Leau Flower from complex dynamics, it attracts two disks which are the coordinates of the cones that contain the subspace of eigenvalue 1 in their interior. The gray circles indicate the boundary of the basin of 0 for the inverse transformation. All the lines and circles illustrated are invariant under T_d.

The picture on $\mathcal{D}$ for the original example defined by:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is the following:



The gray area is the set of coordinates of cones whose limit under iteration is the one-dimensional invariant subspace. The fixed point 0 attracts two disks once again in Leau Flower like fasion.

The last example we must consider is defined by:

$$\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$$

it's picture is the same as if we drop the minus sign. However the dynamics are different since subspaces aproach the attracting subspace alternating sides.

4 Classification of Cone Conjugacy Classes

We will call two invertable linear maps T and S Cone Conjugate if the maps T_c and S_c are topologicaly conjugate. That is, if there exists a homeomorphism $h : \mathcal{C} \to \mathcal{C}$ such that $hT_c = S_c h$.

If $T = A^{-1}SA$ then clearly T_c is Cone Conjugate to S_c by the homeomorphism A_c. Therefore the conjugation class depends only on the Jordan Normal Form of the transformation.

Also multiplication by any non-zero real number doesn't change the conjugation class.

The last observation is that since the dynamics on cones is determined by the dynamics on one-dimensional subspaces it is enough to find that T_c and S_c are conjugate when restricted to the boundary of $\mathcal{C}$ to assert that T and S are cone conjugate.

These actions on the boundary of $\mathcal{C}$ are extremely simple. We can easily see that all orientation preserving diffeomorphism on the circle with only one fixed point are in the same conjugacy class. And that all north-south diffeomorphisms of the circle are conjugate. Therefore we can deduce the following theorem which is simply a case by case discusion of each possible Jordan Form for T:

Theorem 4.1 *Every invertable linear map $T : \mathcal{V} \to \mathcal{V}$ is cone conjugate to the identity, an axial symmetry, a rotation, or one of the maps defined (in some fixed basis) by the matrices:*

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$$

So there are infinitely many conjugacy clases given by non trivial rotations, and five other classes.

Using the operator norm it makes sense to ask which conjugacy clases are open. A map in the interior of a conjugacy class is called Cone Structuraly Stable. Since only the last two clases are open, but maps in a non trivial rotation class form an open set we have the following:

Theorem 4.2 *Cone Structurally Stable systems are not dense.*