

VERY FLAT REDUCTIVE MONOIDS

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ABSTRACT. This is a preliminary version, to be published elsewhere. Given a semisimple algebraic group G_0 over an algebraically closed field k of arbitrary characteristic, we construct a reductive monoid, the *universal semigroup*, which gives a flat deformation of G_0 to an affine semigroup, the *asymptotic semigroup*, whose algebra of regular functions is obtained by changing the product in $k[G_0]$. This construction, of geometric nature, generalises the construction of the universal and asymptotic semigroups done by Vinberg in characteristic zero.

1. INTRODUCTION

Let k be an algebraically closed field of arbitrary characteristic, and M be an algebraic normal irreducible monoid over k of unit group G . Then G acts by right and left multiplication over M , in such a way that the orbit of 1 is an open dense subset isomorphic to $(G \times G)/\Delta(G)$, $\Delta(G)$ the diagonal. Suppose that G is reductive, with commutator G_0 ; then the quotient $\pi : M \rightarrow A_M = M//_{G_0 \times G_0}$ exists and is a commutative monoid, with unit group the torus G/G_0 . We call this quotient the *abelianisation* of M . We say that M is *very flat* if the abelianisation is a flat morphism with reduced (as schemes) and irreducible fibres. In this paper we study the geometry of very flat monoids. In particular, given a semisimple group G_0 , we find a minimal element of the family $\mathcal{FM}(G_0)$ of very flat monoids with unit group such that its commutator is G_0 . This monoid, the *universal semigroup*, is such that its abelianisation is isomorphic to the affine space \mathbb{A}^n , n the semisimple rank of G_0 , and such that any very flat monoid $M \in \mathcal{FM}(G_0)$ is a fibred product over \mathbb{A}^n of A_M and the universal semigroup. Moreover, let S be the universal semigroup associated to G_0 ; then the abelianisation $\pi : S \rightarrow A_S \cong \mathbb{A}^n$ gives a flat deformation of G_0 to the algebraic semigroup $\pi^{-1}(0)$. We call $\pi^{-1}(0)$ the *asymptotic semigroup* associated to G_0 . The algebra of regular functions of the asymptotic semigroup is obtained from $k[G_0]$ by changing the product (and leaving the coproduct as is).

These constructions are valid in arbitrary characteristic, and thus generalise those done by Vinberg for characteristic zero in [15] and [16].

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2. PRELIMINARIES

In this section we recall the basic facts about the classification of *reductive embeddings* – i.e. embeddings of a reductive group – to be used in this work.

Definition 1. Let G be a reductive group. An homogeneous space G/H is *spherical* if there exists a Borel subgroup B , of G such that BH is open in G . If X is a normal irreducible G -variety with an open orbit isomorphic to G/H , then X is called a *G/H -spherical variety*. We say that X is *simple* if the action of G over X has only one closed orbit.

If G is a reductive group, then $G \cong (G \times G)/\Delta(G)$, is a spherical homogeneous space. A *reductive embedding* is a G -spherical variety.

A reductive monoid is an irreducible algebraic monoid with unit group a reductive group. We have the following:

Theorem 1. ([12]) *The normal reductive monoids are exactly the affine embeddings of reductive groups. The normal commutative reductive monoids are exactly the affine embeddings of tori.* \square

Notation. From now on reductive monoids are supposed to be normal (and irreducible), unless stated otherwise. All reductive groups are supposed connected.

In [15] Vinberg classified all reductive monoids in characteristic zero in terms of the decomposition of their algebra of regular functions for the action of the unit group. This classification is dual of their classification as spherical varieties. We refer the reader to [12] for a complete description of the classification of reductive embeddings, and to [7] for the general case of spherical varieties. We summarize this classification in order to fix notations. Let G/H be a spherical homogeneous space. We denote $k(G/H)^{(B)}$ the set of B -eigenvectors of $k(G/H)$, the field of rational functions of G/H , and $\Lambda_{G/H}$ the set of weights of $k(G/H)^{(B)}$. We consider the space $\text{Hom}_{\mathbb{Z}}(\Lambda_{G/H}, \mathbb{Z}) \otimes \mathbb{Q} = \mathcal{Q}(G/H)$. The restriction to $k(G/H)^{(B)}$ induces an injection of the set of G -invariant valuations of the field $k(G/H)$, into $\mathcal{Q}(G/H)$. Its image is a rational polyhedral cone, the *valuation cone*, denoted $\mathcal{V}(G/H)$. On the other hand, if we denote $\mathcal{D}(G/H)$ the set of the irreducible B -stable divisors of G/H , the *colors*, then there exists a map (not necessarily injective) $\rho : \mathcal{D}(G/H) \rightarrow \mathcal{Q}(G/H)$.

To each simple spherical variety X with open orbit G/H corresponds an unique *colored cone*, namely a pair $(\mathcal{C}(X), \mathcal{F}(X))$ constructed as follows: let Y be the unique closed G -orbit of X , and put $\mathcal{F}(X) = \{D \in \mathcal{D}(G/H) \mid \overline{D} \supset Y\}$. Let $\mathcal{B}(X)$ be the subset of $\mathcal{V}(G/H)$ consisting of the valuations associated to the irreducible G -stable divisors of X . Then $\mathcal{C}(X)$ is the strictly convex polyhedral cone generated by $\mathcal{B}(X)$ and $\rho(\mathcal{F}(X))$. This cone verifies that $\mathcal{C}(X)^\circ$, the relative interior of $\mathcal{C}(X)$, intersects the valuation cone.

Conversely, a pair $(\mathcal{C}, \mathcal{F})$ verifying the properties described above determines an unique G/H -spherical variety such that $(\mathcal{C}(X), \mathcal{F}(X)) = (\mathcal{C}, \mathcal{F})$. Non-simple

spherical varieties can be classified by means of *colored fans*, which are collections of colored cones with some compatibility restrictions.

Finally, we recall that a dominant G -equivariant morphism $\varphi : X \rightarrow X'$ between spherical varieties with open orbits G/H and G/H' respectively, induces a morphism $\varphi_* : \mathcal{Q}(G/H) \rightarrow \mathcal{Q}(G/H')$ such that $\varphi_*(\mathcal{E}_X) \subset \mathcal{E}_{X'}$, where \mathcal{E}_X denotes the colored fan associated to X .

Let us fix some notations. Let G be a reductive group, and let T be a maximal torus of G , B a Borel subgroup containing T , and B^- its opposite Borel subgroup. We denote $\Xi(T)$ the set of weights and $\Xi_+(T)$ the semigroup of dominant weights with respect to B . We denote W the Weyl group associated to T , and $C = C(G)$ the Weyl chamber associated to (B, T) . We denote $\alpha_1, \dots, \alpha_l$ and $\omega_1, \dots, \omega_l$ the simple roots and fundamental weights associated to (B, T) respectively.

Finally, we denote $\Xi_*(T)$ the set of one parameter subgroups (1-PS) of T . We identify $\Xi_*(T)$ with $\Xi(T)$ by means of a W -invariant form $\langle \cdot, \cdot \rangle$, in such a way that $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$.

The subvarieties $\overline{Bs_{\alpha_i}B^-}$, $s_{\alpha_i} \in W$ the reflection associated to the simple root α_i , $i = 1, \dots, l$, are exactly the irreducible $(B \times B^-)$ -stable divisors of G (the colors).

The combinatorial data associated to the spherical homogeneous space $G = (G \times G)/\Delta(G)$ is the space $\mathcal{Q}(G) = \Xi(T) \otimes \mathbb{Q}$, with valuation cone $\mathcal{V}(G) = -C$ and colors α_i^\vee , $i = 1, \dots, l$ ([12]).

Let S be a reductive monoid with unit group G . We denote $\mathcal{V}(S) = \mathcal{C}(S) \cap \mathcal{V}(G)$.

3. ABELIANISATION OF REDUCTIVE MONOIDS

We keep the notations of the preceding section.

Let G be a reductive group and let Z be the *connected* center of G . We denote as before G_0 the commutator of G . Let S be a reductive monoid of unit group $G = (G_0 \times Z)/Z_0$, where $Z_0 = Z \cap G_0$. It seems natural to try to describe the geometry of S in terms of the commutative monoid $\overline{T} \subset S$. However, for the very flat monoids, Vinberg ([15],[16]) has shown that it suffices to consider \overline{Z} to give a geometrical picture of S , in characteristic zero. In the rest of this work we show how to extend Vinberg results to arbitrary characteristic, giving a geometric construction of the *enveloping* and *asymptotic semigroups*.

Definition 2. Let S be a (not necessarily normal) reductive monoid of unit group G . The abelianisation of S is the geometric quotient $\pi : S \rightarrow A_S = S // (G_0 \times G_0)$ for the action by left and right multiplication, *i.e.* A_S is the affine algebraic variety such that $k[A_S] = k[S]^{G_0 \times G_0}$.

If S is normal, A_S is so. Moreover, $k[A_S] \subset k[S]$ is $(G \times G)$ -stable, and $\pi : S \rightarrow A_S$ is a $(G \times G)$ -morphism.

Instead of using \overline{Z} , we will describe the geometry of very flat monoids by means of its abelianisation (wich in fact is closely related to \overline{Z}):

Theorem 2. ([15, thm. 3],[11]) *Let S be a reductive monoid (not necessarily normal) with unit group G , and let $\pi : S \rightarrow A_S$ be the abelianisation of S . Then the following properties are verified:*

- i) $\pi^{-1}(\pi(1)) = G_0$.
- ii) A_S is a commutative monoid, with unit group $G/G_0 \cong Z/Z_0$.
- iii) $\pi^{-1}(G/G_0) = G$.
- iv) $\pi(\overline{Z}) = A_S$, and $A_S \cong \overline{Z}/Z_0$.
- v) *If moreover S is normal then the induced map*

$$\pi_* : \mathcal{Q}(G) \cong \mathcal{Q}(G_0) \times \mathcal{Q}(Z) \rightarrow \mathcal{Q}(G/G_0) \cong \mathcal{Q}(Z/Z_0)$$

is given by the projection over the second coordinate $(\lambda, \mu) \mapsto \mu$, and the following properties of π_ are verified:*

- $\pi_*(\mathcal{C}(S)) = \mathcal{C}(A_S)$. In particular, $\pi_*^{-1}(\mathcal{C}(A_S)) = \mathcal{C}(S) + \mathcal{Q}(G_0)$, and $\pi_*(\mathcal{C}(S) \cap \mathcal{Q}(Z)) = \mathcal{C}(A_S)$.
- *If we denote $\pi_Z : \overline{Z} \rightarrow A_S$ the restriction of π to \overline{Z} , then $\pi_{Z*} : \mathcal{Q}(Z) \rightarrow \mathcal{Q}(Z/Z_0) \cong \mathcal{Q}(Z)$ is the identity. In particular,*

$$\mathcal{C}(A_S) = \mathcal{C}(S) \cap \mathcal{Q}(Z) .$$

The algebra $k[A_S]$ decomposes for the action of $(Z/Z_0) \times (Z/Z_0)$ as follows:

$$k[A_S] = \bigoplus_{\mu \in \mathcal{L}(A_S)} ke^\mu ,$$

where $\mathcal{L}(A_S) = \{\mu \in \Xi(Z/Z_0) = \Xi(Z)^{Z_0} : \mu \in (\mathcal{C}(S) \cap \mathcal{Q}(Z))^\vee\}$ (the dual is taken in $\mathcal{Q}(Z)$), and ke^μ is the simple representation of $(Z/Z_0) \times (Z/Z_0)$ of weight $(\mu, -\mu)$.

PROOF. Assertions i) – iv) are proved in [15, §3] in the characteristic zero case. These proofs remain valid in arbitrary characteristic with slight modifications (see for exemple [11]). Assertion v) is proved straightforward from i) – iv) by “dualizing”. For a complete proof, we refer the reader to [11]. \square

Given a very flat monoid, we want to see it as the fibred product of its abelianisation with a “minimal” monoid (over the abelianisation of this one). For this, we must first understand the relationship between monoids and their abelianisation at the level of morphisms:

Let $\varphi : S' \rightarrow S$ be a morphism of algebraic monoids. Then φ induces a morphism (of algebraic monoids) between the abelianisations $\varphi_{ab} : A_{S'} \rightarrow A_S$ in such a way that the diagram :

$$\begin{array}{ccc}
 S' & \xrightarrow{\varphi} & S \\
 \pi' \downarrow & & \downarrow \pi \\
 A_{S'} & \xrightarrow{\varphi_{ab}} & A_S
 \end{array}$$

is commutative.

Consider the fibred product

$$\widehat{S} = A_{S'} \times_{A_S} S = \{(a, s) \in A_{S'} \times S : \varphi_{ab}(a) = \pi(s)\}.$$

It is clear that \widehat{S} is a submonoid of $A_{S'} \times S$. The canonical projections $\widehat{\pi} : \widehat{S} \rightarrow A_{S'}$ and $\widehat{\varphi} : \widehat{S} \rightarrow S$ are morphisms of algebraic monoids. By the universal property of the fibred product, there exists a unique morphism (of algebraic monoids) $\gamma : S' \rightarrow \widehat{S}$, such that the following diagram is commutative:

$$\begin{array}{ccccc}
 S' & \xrightarrow{\varphi} & S & & \\
 \pi' \downarrow & \searrow \gamma & \nearrow \widehat{\varphi} & & \downarrow \pi \\
 & & \widehat{S} & & \\
 & \searrow \widehat{\pi} & & & \\
 A_{S'} & \xrightarrow{\varphi_{ab}} & A_S & &
 \end{array}$$

The morphism φ is called *excellent* if γ is an isomorphism. In this case, φ sends the commutator of $G(S')$ isomorphically over the commutator of $G(S)$.

Let us study the fibred product \widehat{S} in more detail.

Proposition 1. *Let S be a (not necessarily normal) reductive monoid of unit group G and let $\pi : S \rightarrow A_S$ be the abelianisation of S . Let A be a commutative monoid of unit group a torus R , such that there exists a morphism of monoids $\varphi : A \rightarrow A_S$. Then:*

- i) $\widehat{S} = A \times_{A_S} S = \{(a, s) \in A \times S \mid \varphi(a) = \pi(s)\}$ is a monoid of unit group $\widehat{G} = G(\widehat{S}) = R \times_{Z/Z_0} G$. In particular, the commutator of \widehat{G} is $\widehat{G}_0 = \{1\} \times G_0 \cong G_0$.
- ii) The connected center \widehat{Z} of \widehat{G} is isomorphic to $R \times_{Z/Z_0} Z$, and

$$\widehat{Z}_0 = \widehat{Z} \cap \widehat{G}_0 = \{1\} \times Z_0 \cong Z_0.$$

In particular, $\widehat{G} = (G_0 \times \widehat{Z})/\widehat{Z}_0 \cong (G_0 \times (R \times_{Z/Z_0} Z))/\widehat{Z}_0$.

- iii) The abelianisation of \widehat{S} is A . Moreover, consider the cartesian diagram:

$$\begin{array}{ccc}
 \widehat{S} & \xrightarrow{\widehat{\varphi}} & S \\
 \widehat{\pi} \downarrow & & \downarrow \pi \\
 A & \xrightarrow{\varphi} & A_S
 \end{array}$$

Then $\varphi = \widehat{\varphi}_{ab}$.

- iv) Let $x \in A$ be such that the fibre $\pi^{-1}(\varphi(x))$ is reducible as scheme (resp. irreducible, resp. normal). Then $\widehat{\pi}^{-1}(x)$ is reducible as scheme (resp. irreducible, resp. normal).
- v) If $\pi : S \rightarrow A_S$ is flat, then $\widehat{\pi} : \widehat{S} \rightarrow A$ is flat.

PROOF. It is clear that \widehat{S} is a submonoid of $A \times S$. An element $(a, s) \in \widehat{S}$ is invertible if and only if $a \in R$ and $s \in G$, so $\widehat{G} = R \times_{A_S} G$. As $\varphi(R) \subset Z/Z_0$ and $\pi(Z) = Z/Z_0$, we deduce that $\widehat{G} = R \times_{Z/Z_0} G = R \times_{Z/Z_0} ((Z \times G_0)/Z_0)$.

On the other hand, the commutator of two elements $(r_1, g_1), (r_2, g_2) \in \widehat{G}$ is

$$(r_1 r_2 r_1^{-1} r_2^{-1}, g_1 g_2 g_1^{-1} g_2^{-1}) = (1, g_1 g_2 g_1^{-1} g_2^{-1}),$$

so the commutator of \widehat{G} is $\widehat{G}_0 = \{1\} \times G_0 \cong G_0$.

Analogously, we prove that $\widehat{Z} = R \times_{Z/Z_0} Z$. It follows that $\widehat{Z}_0 = \{1\} \times Z_0$.

It is easy to see that $(A \times_{A_S} S) // (G_0 \times G_0) \cong A \times_{A_S} A_S = A$, and $\varphi = \widehat{\varphi}_{ab}$.

In order to prove iv), let $x \in A$ and consider $\mathcal{M} \subset k[A]$, the maximal ideal associated to x . Let $\mathcal{M}' \subset k[A_S]$ be the preimage of \mathcal{M} by φ^* . Then \mathcal{M}' is the maximal ideal of $\varphi(x)$, and

$$k[\widehat{S}]/\mathcal{M}k[\widehat{S}] \cong (k[A]/\mathcal{M}) \otimes_{k[A_S]/\mathcal{M}'} (k[S]/\mathcal{M}'k[S]) = k[S]/\mathcal{M}'k[S].$$

It follows that the fibre $\widehat{\pi}^{-1}(x)$ is reducible as scheme (resp. irreducible, resp. normal) if and only if $\pi^{-1}(\varphi(x))$ is.

Finally, we recall that flatness is stable under base extension. \square

Proposition 2. *We keep the notations of Proposition 1. Suppose that S and A are normal varieties. Then*

- i) *The normalisation \widehat{S}_{norm} of \widehat{S} – which is a reductive monoid of unit group \widehat{G} ([12]) – verifies:*

$$\mathcal{V}(\widehat{S}_{norm}) = \mathbb{Q}^+ \left\{ \begin{array}{l} (\lambda, (\mu_1, \mu_2)) \in (\Xi_*(T_0) \times \Xi_*(\widehat{Z}))^{\widehat{Z}_0} : \\ (\lambda, \mu_2) \in \mathcal{V}(S), \mu_1 \in \mathcal{C}(A), \varphi(\mu_1(t)) = \pi(\mu_2(t)) \end{array} \right\},$$

where λ is an 1-PS of G_0 and (μ_1, μ_2) is an 1-PS of $R \times Z$.

In particular, $A_{\widehat{S}_{norm}} \cong A$.

- ii) *If moreover $\pi : S \rightarrow A_S$ is flat with normal fibres, then \widehat{S} is normal.*

PROOF. The 1-PS of \widehat{G} are the triples $(\lambda, \mu_1, \mu_2) \in \Xi_*(G_0) \times \Xi_*(R) \times \Xi_*(Z)$ such that $\pi(\mu_2(t)) = \varphi(\mu_1(t))$. It suffices to show that the 1-PS $(\lambda, \mu_1, \mu_2) \in \widehat{G}$ belonging to $-C(\widehat{G})$ such that the limit $\lim_{t \rightarrow 0} (\lambda, \mu_1, \mu_2)(t)$ exists in \widehat{S} , are exactly those such that $(\lambda, \mu_2) \in \mathcal{V}(S)$, and $\mu_1 \in \mathcal{C}(A)$ ([12, §4.2]). Recall that $C(\widehat{G})$ is the Weyl chamber of \widehat{G} .

Let $\nu = (\lambda, \mu_1, \mu_2) \in \Xi_*(\widehat{G})$ be a 1-PS. Then $\nu \in -C(\widehat{G})$, if and only if $\lambda \in -C(G_0)$. Moreover, the limit $\lim_{t \rightarrow 0} \nu(t) = \lim_{t \rightarrow 0} (\lambda(t), \mu_1(t), \mu_2(t))$ exists in \widehat{S} if and only if the limits

$$\lim_{t \rightarrow 0} \widehat{\varphi}(\lambda(t), \mu_1(t), \mu_2(t)) = \lim_{t \rightarrow 0} \lambda(t) \mu_2(t)$$

and

$$\lim_{t \rightarrow 0} \widehat{\pi}(\lambda(t), \mu_1(t), \mu_2(t)) = \lim_{t \rightarrow 0} \mu_1(t)$$

exist in S and A respectively. It follows that the limit $\lim_{t \rightarrow 0} \nu(t)$ exists in \widehat{S} if and only if $(\lambda, \mu_2) \in \mathcal{V}(S)$, and $\mu_1 \in \mathcal{C}(A)$. As an 1-PS of \widehat{G} has a limit in \widehat{S} if and only if it has a limit in \widehat{S}_{norm} , assertion i) is verified.

If $\pi : S \rightarrow A_S$ is a flat morphism, then $\widehat{\pi} : \widehat{S} \rightarrow A$ is flat by base extension. By Proposition 1-iv), if the fibres of π are normal, then the fibres of $\widehat{\pi}$ are normal. It follows from [8, §21.E] that \widehat{S} is normal. \square

Remark. We can identify $\mathcal{Q}(\widehat{G})$ with $\ker((\varphi_*, 0) - (0, \pi_*)) \subset \mathcal{Q}(R) \times \mathcal{Q}(G)$. Under this identification, $\mathcal{V}(\widehat{S}_{norm}) = (\mathcal{V}(A) \times \mathcal{V}(S)) \cap \mathcal{Q}(\widehat{G})$.

4. ENVELOPING AND ASYMPTOTIC SEMIGROUPS OF A SEMISIMPLE GROUP

Let G_0 be a semisimple group. We keep the notations of the precedings paragraphs, and denote $\mathcal{FM}(G_0)$ the class of very flat monoids whose unit group has commutator isomorphic to G_0 . This family of monoids has been studied by Vinberg ([15]) in characteristic 0. He has shown that it has a minimal element, the *enveloping semigroup*, that allows to describe the monoids of the family as fibred products. Moreover, the abelianisation of the enveloping semigroup gives us a flat deformation of G_0 over a semigroup, the *asymptotic semigroup*, associated to a certain graded algebra ([16]). In this section, we give a geometric construction of the enveloping and asymptotic semigroups, valid in arbitrary characteristic, and study their geometry.

Definition 3. Let G_0 be a semisimple group. We denote $\text{Env}(G_0)$ the monoid of unit group $G = (G_0 \times T_0)/Z_0$ associated to the colored cone:

$$(\{\mathbb{Q}^+\{(-\omega_1, \omega_1), \dots, (-\omega_l, \omega_l)\} + \mathbb{Q}^+\mathcal{D}(G), \mathcal{D}(G)\}.$$

We call this monoid the *enveloping semigroup associated to G_0* .

Remarks. By Theorem 2, the abelianisation of the enveloping semigroup is a (T_0/Z_0) -toric variety with associated cone $\mathbb{Q}^+(\alpha_1, \dots, \alpha_l)$, thus $A_{\text{Env}(G_0)} \cong \mathbb{A}^l$.

An easy calculation shows that in characteristic zero, the preceding definition coincides with the one given in [15].

Notation. From now on, we consider a semisimple simply connected group G_0 , and its adjoint group $G_{ad} = G_0/Z_0$, where Z_0 is the center of G_0 , unless otherwise

stated. This is done without loss of generality, by the fact that reductive monoids “behave well” under quotient by finite central subgroups ([11], [12]).

Proposition-Definition ([3],[14]). Let G be a reductive group and Z be its center. There exists an unique simple projective embedding, without colors, of $G_{ad} = G/Z \cong (G_{ad} \times G_{ad})/\Delta(G_{ad})$. We call this embedding the *wonderful embedding of G_{ad}* (or G).

Let us summarize the properties of the wonderful embedding to be used in our work:

The wonderful embedding X is a smooth variety.

If $C = C(G)$ denotes the Weyl chamber of G associated to a Borel subgroup B , the colored cone associated to the wonderful embedding is:

$$(\mathcal{C}(X), \mathcal{F}(X)) = (\mathcal{V}(G_{ad}), \emptyset) = (-C(G_{ad}), \emptyset).$$

We denote X_i the irreducible component (divisor) of $X \setminus G_{ad}$ that has associated valuation $-\omega_i$. The closed irreducible $(G \times G)$ -stable subsets of X are exactly the sets $X_I = \bigcap_{i \in I} X_i$, $I \subset \{1, \dots, l\}$, with the convention $X_\emptyset = X$.

The unique closed orbit of X is $Y = X_{\{1, \dots, l\}} \cong (G/B) \times (G/B^-)$.

Moreover, there exists a morphism $\gamma : \mathbb{A}^l \rightarrow X$, where \mathbb{A}^l is the affine space of dimension l , and an open neighborhood $\mathcal{U} \subset X$ of $1 \in G_{ad}$ such that the map

$$U \times U^- \times \mathbb{A}^l \ni (u, u', a) \mapsto (u, u') \cdot \gamma(a) \in \mathcal{U}$$

is an isomorphism. The morphism γ verifies that for all $(t, s) \in T_0 \times T_0$, $\gamma^{-1}((t, s) \cdot 1) = (t^{\alpha_1} s^{-\alpha_1}, \dots, t^{\alpha_l} s^{-\alpha_l})$ (recall that α_i , $i = 1, \dots, l$, are the simple roots of G_0).

Finally, a G -embedding has no colors if and only if it dominates equivariantly X ([7]).

Notation. Let $\lambda \in \Xi_+(T)$ be a dominant weight of G . We denote $L(\lambda) \rightarrow G/B$ the line bundle associated to the weight λ ([6]). The external tensor product $L(\lambda) \boxtimes L(-\lambda) \rightarrow (G/B) \times (G/B^-)$ is denoted $\mathcal{L}(\lambda)$.

We recall some properties about line bundles over the wonderful embedding of G_0 needed for our work.

Let X be the wonderful embedding of G_0 , and Y its unique closed $(G_0 \times G_0)$ -orbit. The Picard group of X is isomorphic to the subgroup of the group of weights of $G_0 \times G_0$ generated by $(\omega_i, -\omega_i)$, $i = 1, \dots, l$ ([14]), in such a way that if L_λ is the line bundle associated to the weight $(\lambda, -\lambda)$, then the restriction of L_λ to the closed orbit $G/B \times G/B^-$ coincides with $\mathcal{L}(\lambda)$. Moreover, L_λ is $(G_0 \times G_0)$ -linearized (see [9] for a definition of linearisation), in such a way that over the $(B_0^- \times B_0)$ -fixed point of Y , $(B_0^- \times B_0)$ acts by multiplication by $(-\lambda, \lambda)$.

We denote L_i the line bundle L_{ω_i} , which is associated to the divisor $\overline{D_i} = \overline{B_0 s_\alpha B_0^-} \subset X$. If $\lambda = \sum n_i \omega_i$, then $L_\lambda \cong L_1^{\otimes n_1} \otimes \dots \otimes L_l^{\otimes n_l}$. On the other

hand, the line bundles L_{α_i} are associated to the divisors X_i ([14]). Moreover, $H^0(X, L_{\alpha_i}) \neq 0$.

Finally, we recall that the bundles L_i are generated by their global sections and that $L_{\sum_{i=1}^l \omega_i} = L_1 \otimes \cdots \otimes L_l$ is an ample line bundle ([14]). We consider the action of $G_0 \times G_0$ over the sections of this $(G_0 \times G_0)$ -linearized bundle and its induced action on the bundle $\varrho : E = \bigoplus_{i=1}^l L_i^\vee \rightarrow X$.

In [1], Brion and Polo have described some properties of the algebra

$$R(X) = \bigoplus_{n_1, \dots, n_l \in \mathbb{Z}} H^0(X, L_1^{\otimes n_1} \otimes \cdots \otimes L_l^{\otimes n_l}),$$

in arbitrary characteristic. This algebra turns out to be the algebra of regular functions of the envelopping semigroup:

Theorem 3. *Let G_0 be a semisimple simply connected group, and X its wonderful embedding. Then the algebra of regular functions of the envelopping semigroup of G_0 verifies:*

$$(1) \quad k[\text{Env}(G_0)] \cong R(X) = \bigoplus_{n_1, \dots, n_l \in \mathbb{Z}} H^0(X, L_1^{\otimes n_1} \otimes \cdots \otimes L_l^{\otimes n_l}).$$

Moreover, $k[\text{Env}(G_0)]$ is generated by its subspaces $H^0(X, L_i)$, $i = 1, \dots, l$, and by $\sigma_1, \dots, \sigma_l$, where $\sigma_i \in H^0(X, L_{\alpha_i})$ is a global $(B \times B^-)$ -equivariant section of L_{α_i} .

The sections $\sigma_1, \dots, \sigma_l$ form a regular sequence in $k[\text{Env}(G_0)]$, and the quotient $k[\text{Env}(G_0)]/(\sigma_1, \dots, \sigma_l)$ is isomorphic to $\bigoplus_{n_i \in \mathbb{Z}} H^0(Y, L_1^{n_1} \otimes \cdots \otimes L_l^{n_l})$, where Y is the closed orbit of X .

PROOF. Equation (1) will be a consequence of the construction developed below (cf. Corollary 1). The stated properties of $k[\text{Env}(G_0)] = R(X)$ are proved in [1, Cor. 9 and 10]. \square

4.1. A geometrical construction of the envelopping semigroup.

In this paragraph, we give a construction of $\text{Env}(G_0)$ using bundles over the wonderful embedding of G_0 .

If G_0 is not simply connected, it is known that there exists a simply connected group \widetilde{G}_0 such that $G_0 \cong \widetilde{G}_0/\Gamma$, where Γ is a finite central subgroup. In this case, by general properties of reductive embeddings, (see [15] and [12]) we have:

$$\text{Env}(G_0) = \text{Env}(\widetilde{G}_0/\Gamma) = \text{Env}(\widetilde{G}_0)/\Gamma,$$

so we can suppose without loss of generality that G_0 is simply connected. We denote Z_0 its center, and $G_{ad} = G_0/Z_0$ its adjoint group.

The following Proposition, easy to prove, generalizes some results of Fujita and Demazure ([5], [2]). For a proof, we refer the reader to [11].

Proposition 3. *We keep the notations of the preceding paragraphs. Consider the bundle $\varrho : E = L_1^\vee \oplus \cdots \oplus L_l^\vee \rightarrow X$. Then the algebra of regular functions of $E_I = \varrho^{-1}(X_I)$ is of finite type and normal, and satisfies:*

$$H^0(E_I, \mathcal{O}_{E_I}) \cong \bigoplus_{\lambda \in \Xi_+(T_0)} H^0(X_I, L_\lambda) \cong \bigoplus_{n_1, \dots, n_l \geq 0} H^0(X_I, L_1^{\otimes n_1} \otimes \cdots \otimes L_l^{\otimes n_l}),$$

In particular,

$$(2) \quad S_0 = \text{Spec } H^0(E, \mathcal{O}_E) = \text{Spec } \bigoplus_{\lambda \in \Xi_+(T_0)} H^0(X, L_\lambda)$$

is a normal affine variety, and the canonical morphism $\varphi : E \rightarrow S_0$ is a $(G_0 \times G_0)$ -equivariant proper birational morphism, with connected fibres (obvious actions).

Let E_i be the zero section of L_i^\vee , i.e. the image of the canonical closed immersion $\bigoplus_{j \neq i} L_j^\vee \hookrightarrow E$. Consider $E' = E \setminus (\cup_{1 \leq i \leq l} E_i)$, and set $E'_I = \varrho^{-1}(X_I) \cap E'$. Then

$$H^0(E'_I, \mathcal{O}_{E'_I}) \cong \bigoplus_{\lambda \in \Xi(T_0)} H^0(X_I, L_\lambda)$$

is an algebra of finite type and normal.

Moreover, the canonical morphism

$$\varphi' : E' \rightarrow \text{Spec } \bigoplus_{\lambda \in \Xi_+(T_0)} H^0(X, L_\lambda)$$

is birational. □

We will show that the variety S_0 defined in equation (2) is a reductive monoid with unit group $(G_0 \times T_0)/Z_0$. Moreover, $\varphi : E \rightarrow S_0$ is the *decoloration* of S_0 (cf. Definition 4 below). This fact will allow us to determine the colored cone associated to S_0 .

In general, S_0 is not very flat: the enveloping semigroup associated to G_0 will be constructed as the normalization of the fibred product between S_0 and \mathbb{A}^l , over the abelianisation of S_0 (cf. Theorem 5 below).

Theorem 4. *We keep the preceding notations and let $G = (G_0 \times T_0)/Z_0$. Then E is a simple G -embedding, without colors, and*

$$G \cong \text{Spec } \bigoplus_{\lambda \in \Xi(T_0)} H^0(G_{ad}, L_\lambda|_{G_{ad}}) = \text{Spec } \bigoplus_{n_1, \dots, n_l \in \mathbb{Z}} H^0(G_{ad}, L_1^{\otimes n_1} \otimes \cdots \otimes L_l^{\otimes n_l}|_{G_{ad}}) \hookrightarrow E' \subset E.$$

Moreover, there exists a bijection between the $(G \times G)$ -stable irreducible closed subsets of E (and thus the orbits) and the couples (I, J) of subsets of $\{1, \dots, l\}$, given by $(I, J) \leftrightarrow E_{I,J} = \bigoplus_{i \in I} L_i^\vee|_{X_J}$.

In particular, the colored cone associated to E in $\mathcal{Q}(G)$ is:

$$(\mathbb{Q}^+ \{(-\omega_1, \omega_1), \dots, (-\omega_l, \omega_l), (0, \alpha_1), \dots, (0, \alpha_l)\}, \emptyset) .$$

PROOF. First we observe that the total space E is a smooth variety. The torus T_0 acts by multiplication by ω_i on the fibres of L_i^\vee , in such a way that we have an action of T_0 on E . On the other hand, the action of $G_0 \times G_0$ over the sections of the bundles L_i^\vee induces an action of $G_0 \times G_0$ on E .

Thus, we have an action $G_0 \times G_0 \times T_0$ on E . We identify T_0 with $(T_0 \times T_0) / \Delta(T_0)$ by $t \mapsto (1, t)$, and we get an action of $(G_0 \times T_0) \times (G_0 \times T_0)$ on E . We claim that this action is the one needed in order to prove the theorem.

Let us explicit the action of $G_0 \times G_0 \times T_0$ on E :

Consider an affine open subset $V \subset X$ such that there exist trivialisations over V for the bundles L_i^\vee . Consider a section s_i of L_i^\vee that does not vanish on V , $i = 1, \dots, l$. We have an isomorphism $V \times \mathbb{A}^l \cong \varrho^{-1}(V)$, given by $(x, t_1, \dots, t_l) \mapsto \sum_i t_i s_i(x)$. If $(a, b, z) \in G_0 \times G_0 \times T_0$, then $(a, b) \cdot V$ is an open subset of X , isomorphic to V as a variety. the action of (a, b, z) over $\sum_i t_i s_i(x) \in V$ is given by:

$$(3) \quad (a, b, z) \cdot \sum_{i=1}^l t_i s_i(x) = \sum_{i=1}^l t_i \omega_i(z) ((a, b) \cdot s_i) ((a, b) \cdot x) .$$

In order to prove that the action of $G_0 \times G_0 \times T_0$ on E induces an action of $G \times G$ such that E is a G -embedding, we must first show that if $z \in Z_0$, then $(1, z^{-1}, z)$ acts trivially on E (i.e. the action found factorizes through the quotient $(G_0 \times T_0) / Z_0 \times (G_0 \times T_0) / Z_0$). Next, we must show that there exists an element such that its isotropy group is $\Delta(G)$. The orbit passing by this element is necessarily open by dimension considerations.

If we substitute (a, b, z) in equation (3) by $(1, z^{-1}, z)$, $z \in Z_0$, we obtain:

$$(1, z^{-1}, z) \cdot \sum_{i=1}^l t_i s_i(x) = \sum_{i=1}^l t_i \omega_i(z) \omega_i(z^{-1}) s_i(x) = \sum_{i=1}^l t_i s_i(x) ,$$

so the action of $G_0 \times G_0 \times T_0$ of E induces one of $G \times G$.

Next, we calculate the $(G \times G)$ -orbits of E or, equivalently, we calculate the $(G_0 \times G_0 \times T_0)$ -orbits. As T_0 acts on the fibres, it is clear that every orbit is contained in the preimage by ϱ of one $(G_0 \times G_0)$ -orbit of X .

On the other hand, for every couple $I, J \subset \{1, \dots, l\}$, there exists a $(G_0 \times G_0 \times T_0)$ -equivariant closed immersion:

$$E_{I,J} = \bigoplus_{i \in I} L_i^\vee|_{X_J} \hookrightarrow \bigoplus_{i=1}^n L_i^\vee = E .$$

It is clear that the $E_{I,J}$ are the only irreducible $(G \times G)$ -stable closed subsets of E . Every $E_{I,J}$ contains an unique open orbit $\mathcal{O}_{I,J}$, obtained by discarding the zero sections over \mathcal{O}_J .

In particular, $\text{Spec} \bigoplus_{\lambda \in \Xi(T_0)} \Gamma(G_{ad}, L_\lambda|_{G_{ad}}) \cong \mathcal{O}_{\{1, \dots, l\}, \emptyset} \subset \varrho^{-1}(G_{ad})$ is an open orbit, and if $Y \subset X$ is the unique closed orbit of the wonderful embedding, then the unique closed orbit of E is $\mathcal{O}_{\emptyset, \{1, \dots, l\}} = \theta(Y)$, the zero section. It follows that E is a simple $(G \times G)$ -spherical variety.

Let us show that $\mathcal{O}_{\{1, \dots, l\}, \emptyset}$ is isomorphic to G as $(G \times G)$ -variety. It follows that E is a simple G -embedding. Consider an open neighborhood $\mathcal{U} \subset X$ around $1 \in G_{ad} = G_0/Z_0$, isomorphic to $U \times U^- \times \mathbb{A}^l$. The bundles L_i^\vee can be trivialized over \mathcal{U} ([14]), so $\varrho^{-1}(\mathcal{U}) \cong U \times U^- \times \mathbb{A}^l \times \mathbb{A}^l$. Consider sections s_i of the bundles L_i^\vee , $i = 1, \dots, l$, over a neighbourhood of 1, such that s_i trivializes the bundle L_i^\vee . It suffices to show that $\Delta(G)$ is the isotropy group of $\sum_{i=1}^l s_i(1)$, that is:

$$(G_0 \times G_0 \times T_0)_{\sum_{i=1}^l s_i(1)} = (\Delta(G_0) \times \{1\}) (\{(1, z^{-1}, z) \in G_0 \times G_0 \times T_0 : z \in Z_0\}) .$$

From equation (3), we deduce that an element (a, b, z) belongs to $(G_0 \times G_0 \times T_0)_{\sum_{i=1}^l s_i(1)}$ if and only if $ab^{-1} \in Z_0$ and $\omega_i(z)((a, b) \cdot s_i)(1) = s_i(1)$, for all $i = 1, \dots, l$. In this case, there exists $z_0 \in Z_0$ such that $b = az_0$, so $s_i(1) = \omega_i(z)((a, az_0) \cdot s_i)(1) = \omega_i(z)\omega_i(z_0)s_i(1)$. It follows that $\omega_i(z z_0) = 1$ for all $i = 1, \dots, l$; i.e. $z = z_0^{-1}$, so E is a G -embedding.

Finally, we calculate the colored cone $(\mathcal{C}(E), \mathcal{F}(E))$ associated to E .

First, we find the set of colors $\mathcal{F}(E)$. For this we observe that $\varrho(\overline{Bs_{(\alpha, 0)}B^-}) = \overline{B_0 s_\alpha B_0^-} \subset X$, which does not intersect Y . It follows that E is without colors.

Next, we calculate $\mathcal{C}(E) = \mathcal{V}(E)$. For this, it suffices to find the 1-PS of G which belong to $-C(G) = -C(G_0) \times \mathcal{Q}(T_0)$, such that their limit as t goes to zero exists in E ([12, §4.2]).

An 1-PS $(\lambda, \mu) \in \Xi_*(T_0 \times T_0)^{Z_0}$ is given by $(\lambda(t), \mu(t)) \cdot \sum s_i(1) = \sum \omega_i(\mu(t) - \lambda(t))s_i(\lambda(t)^{-1})$. Considering the isomorphism $\varrho(\mathcal{U}) \cong U \times U^- \times \mathbb{A}^l \times \mathbb{A}^l$ we have

$$\begin{aligned} \mathcal{U} \supset BB^- &\leftrightarrow U \times U^- \times (k^*)^l \times (k^*)^l \subset U \times U^- \times \mathbb{A}^l \times \mathbb{A}^l \\ (\lambda(t), \mu(t)) &\leftrightarrow (1, 1, \lambda(t)^{-\alpha_1}, \dots, \lambda(t)^{-\alpha_l}, \omega_1(\mu(t) - \lambda(t)), \dots, \omega_l(\mu(t) - \lambda(t))) . \end{aligned}$$

Then, $(\lambda(t), \mu(t)) \leftrightarrow (1, 1, t^{-\langle \lambda, \alpha_1 \rangle}, \dots, t^{-\langle \lambda, \alpha_l \rangle}, t^{\langle \mu - \lambda, \omega_1 \rangle}, \dots, t^{\langle \mu - \lambda, \omega_l \rangle})$. We deduce that the limit $\lim_{t \rightarrow 0} (\lambda(t), \mu(t))$ exists if and only if $\lambda \in -C(G_0)$, et $\mu - \lambda \in \mathbb{Z}^+ \{\alpha_1, \dots, \alpha_l\}$.

It follows that

$$\mathcal{C}(E) = \mathcal{V}(E) = \mathbb{Q}^+ \{(-\omega_1, \omega_1), \dots, (-\omega_l, \omega_l), (0, \alpha_1), \dots, (0, \alpha_l)\} .$$

□

We recall an useful result about the sections of bundles over the wonderful embedding.

Proposition 4. ([3, prop. 8.2],[14]) *We keep the notations of the preceding paragraphs. Consider $I \subset \{1, \dots, l\}$ and $\lambda = \sum_{i=1}^l n_i \omega_i \in \Xi(T_0)$. We define*

$$\mathcal{L}_{I,\lambda} = \left\{ \gamma \in \Xi_+(T_0) : \gamma = \lambda - \sum_{j \in I^c} t_j \alpha_j, t_j \in \mathbb{Z}^+ \right\}.$$

There exists a morphism $V_\gamma \times V_\gamma^ \hookrightarrow H^0(X_I, L_\lambda)$ if and only if $\gamma \in \mathcal{L}_{I,\lambda}$. \square*

Corollary 1. *Equation (1) is verified:*

$$k[\text{Env}(G_0)] \cong R(X) = \bigoplus_{n_1, \dots, n_l \in \mathbb{Z}} H^0(X, L_1^{\otimes n_1} \otimes \dots \otimes L_l^{\otimes n_l}).$$

PROOF. Indeed, observe as before that the action of G_0 over the sections $H^0(X, L_1^{\otimes n_1} \otimes \dots \otimes L_l^{\otimes n_l})$ induces an action of $G \times G$ over $R(X)$, such that the weights of the vectors in $R(X)^{U \times U^-}$ are exactly those appearing in $\mathcal{C}(\text{Env}(G_0))^\vee \cap C(G)$, where $G = G(\text{Env}(G_0))$. Moreover, by Proposition 3 the open orbit in $\text{Spec } R(X)$ of the induced action is isomorphic to G . It follows that $\text{Spec } R(X) \cong \text{Env}(G_0)$ ([7]). \square

Definition 4. Let X be a normal G -embedding, with associated colored fan \mathcal{E} . Consider the normal G -embedding \tilde{X} associated to the fan \mathcal{E}' obtained from \mathcal{E} by intersecting with $\mathcal{V}(G)$. Then the morphism $\varphi : \tilde{X} \rightarrow X$ is proper and birational, and is minimal for these properties; we call φ (or \tilde{X}) the *decoloration* of X ([13]).

It follows immediately from the classification of reductive monoids ([12, Prop. 12]) that two reductive monoids are isomorphic if and only if their decolorations are isomorphic.

Theorem 5. *The variety S_0 defined by equation (2) is a reductive monoid with zero, with unit group $G = (G_0 \times T_0)/Z_0$. The colored cone in $\mathcal{Q}(G)$ associated to S_0 is*

$$(4) \quad (\mathcal{C}(S_0), \mathcal{F}(S_0)) = \left(\mathbb{Q}^+ \left\{ (-\omega_i, \omega_i), (0, \alpha_j^\vee), (\alpha_k^\vee, 0) ; i, j, k = 1 \dots, l \right\}, \mathcal{D}(G) \right).$$

The morphism $\varphi : E \rightarrow S_0$ is the decoloration of S_0 . In particular, φ is a proper birational morphism, so it is surjective.

Moreover, if $\pi : S \rightarrow \mathbb{A}^l$ and $\pi_0 : S_0 \rightarrow A_0$ are the abelianisations of $\text{Env}(G_0)$ and S_0 respectively, then there exists a morphism $\gamma : \mathbb{A}^l \rightarrow A_0$ such that $\text{Env}(G_0) = (\mathbb{A}^l \times_{A_0} S_0)_{\text{norm}}$.

PROOF. We have seen that S_0 is an affine normal variety. The action of $G \times G$ on E induces an action on $H^0(E, \mathcal{O}_E) = k[S_0]$. We consider the induced action of $G \times G$ over S_0 . By Proposition 3, $\varphi : E \rightarrow S_0$ is a birational proper $(G \times G)$ -morphism. It follows that $\varphi_0(G)$ is an open $(G \times G)$ -orbit of S_0 , so S_0 is an

affine G -embedding, i.e. a monoid of unit group G . Moreover, the maximal ideal $\mathcal{M} = \bigoplus_{\lambda \in \Xi_+(T_0) \setminus \{0\}} H^0(X, L_\lambda)$ is $(G \times G)$ -stable, so the unique closed orbit of S_0 is the closed point corresponding to \mathcal{M} . In particular, S_0 has a zero.

The colored cone associated to S_0 is the dual of the cone generated by the weights of the regular functions belonging to $k[S_0]^{U \times U^-}$ (recall that if a monoid has a zero, then all the colors are present). By Proposition 4 this semigroup is:

$$\mathcal{L}_{S_0} = \{(\lambda, \mu) \in \Xi_+(T_0) \times \Xi_+(T_0) : \mu - \lambda \in \mathbb{Z}^+ \{\alpha_1, \dots, \alpha_l\}\} .$$

Dualizing, we obtain equation (4).

Finally, the morphism $\varphi : E \rightarrow S_0$ is the decoloration of S_0 because E is without colors and that $\mathcal{C}(E) = \mathcal{V}(S_0)$.

Let $G = (G_0 \times T_0)/Z_0$; in order to avoid confusion, we denote $Z \cong \{1\} \times T_0$ the connected center of G .

The associated cones of \mathbb{A}^l and A_0 as a (Z/Z_0) -toric varieties are $\mathcal{C}(\mathbb{A}^l) = \mathbb{Q}^+ \{\omega_1, \dots, \omega_l\} \subset \mathcal{Q}(Z/Z_0) = \mathcal{Q}(Z)$, and $\mathbb{Q}^+ \{\alpha_1^\vee, \dots, \alpha_l^\vee\}$ respectively. It follows that the identity $\text{id} : \mathcal{Q}(Z/Z_0) \rightarrow \mathcal{Q}(Z/Z_0)$ induces a morphism $\gamma : \mathbb{A}^l \rightarrow A_0$.

Consider the fibred product $S = \mathbb{A}^l \times_{A_0} S_0$.

By Proposition 2, S_{norm} is a reductive monoid with unit group $G(S) = G \times_{Z/Z_0} (Z/Z_0) = G$, such that the associated colored cone verifies:

$$\mathcal{V}(S_{norm}) = \mathbb{Q}^+ \left\{ \begin{array}{l} (\lambda, (\mu_1, \mu_2)) \in (\Xi(T_0) \times \Xi(\widehat{Z}))^{\widehat{Z}_0} : \\ (\lambda, \mu_2) \in \mathcal{V}(S), \mu_1 \in \mathcal{C}(\mathbb{A}^l), \gamma(\mu_1(t)) = \pi_0(\mu_2(t)) \end{array} \right\} ,$$

or equivalently:

$$\mathcal{V}(S_{norm}) = \mathbb{Q}^+ \left\{ \begin{array}{l} (\lambda, \mu) \in (\Xi(T_0) \times \Xi(Z))^{Z_0} : \\ \mu - \lambda \in \mathbb{Z}^+ \{\alpha_1, \dots, \alpha_l\}, \mu \in \mathbb{Z}^+ \{\omega_1, \dots, \omega_l\} \end{array} \right\} ,$$

which is equal to $\mathcal{V}(\text{Env}(G_0))$, so $S \cong \text{Env}(G_0)$ ([12, Prop. 12]). \square

The $(G \times G)$ -orbits of $\text{Env}(G_0)$, where $G = (G_0 \times T_0)/Z_0$, are in bijection with the colored faces of $\mathcal{C}(\text{Env}(G_0))$. So it is possible to give a combinatorial characterization of the orbits:

Theorem 6 ([15], [11]). *There exists a bijection between the couples (I, J) of subsets of $\{1, \dots, l\}$ such that every connected component of J (in the Dynkin diagram with indices $1, \dots, l$) intersects I , and the $(G \times G)$ -orbits of $\text{Env}(G_0)$. If $\mathcal{O}_{I,J}$ is the orbit associated to (I, J) by the bijection constructed, then*

$$\dim \mathcal{O}_{I,J} = \dim(G \times G) / P_{J^c} ,$$

where P_{J^c} is the parabolic subgroup generated by the parabolics $P_{\alpha_i} \times Q_{\alpha_i}$, $i \in J^c$, P_α being the minimal parabolic subgroup associated to the simple root α , and Q_α its opposite. \square

We study now the geometry of the abelianisation of the envelopping semigroup. We keep the notations of the preceding paragraphs. Consider $J \subset \{1, \dots, l\}$ and let I be its complement. Consider the closed subvariety X_J of the wonderful embedding X .

If $\lambda \in \Xi(T)$ and global sections $\sigma_i \in H^0(X, L_{\alpha_i})$, $i = 1, \dots, l$, as in Theorem 3, we consider the injection

$$\prod_{i \in I} \sigma_i^{t_i} : H^0 \left(X_J, L_{\lambda - \sum_{i \in I} t_i \alpha_i} \right) \hookrightarrow H^0(X_J, L_\lambda)$$

$$\sigma \mapsto \sigma \prod_{i \in I} \sigma_i^{t_i}$$

We denote its image $F_{\sum_{i \in I} t_i \alpha_i} H^0(X_J, L_\lambda)$, and consider the filtration of $H^0(X_J, L_\lambda)$ by $F_n H^0(X_J, L_\lambda) = H^0(X_J, \mathcal{I}_Y^n \otimes L_\lambda)$. We recall that $Y = X_{\{1, \dots, l\}}$ is the unique closed orbit, and that the sheaf of ideals \mathcal{I}_Y of Y is generated in \mathcal{O}_{X_J} by the regular sequence $\{\sigma_i, i \in I\}$ ([1]).

Proposition 5. ([1]) *The graduation $F_n H^0(X_J, L_\lambda)$ is finite, and such that*

$$F_n H^0(X_J, L_\lambda) = \sum_{\sum_{i \in I} t_i = n} F_{\sum_{i \in I} t_i \alpha_i} H^0(X_J, L_\lambda) .$$

We denote $n_0(J, \lambda) \in \mathbb{Z}^+$ the integer such that $F_n H^0(X_J, L_\lambda) = 0$ if $n > n_0$, and such that $F_{n_0} H^0(X_J, L_\lambda) \neq 0$.

The n th layer of the associated graded module verifies:

$$\mathrm{gr}_n H^0(X_J, L_\lambda) = F_n H^0(X_J, L_\lambda) / F_{n+1} H^0(X_J, L_\lambda) \cong \bigoplus_{\substack{\gamma = \lambda - \sum_{i \in I} t_i \alpha_i \\ \sum_{i \in I} t_i = n, \gamma \in \Xi_+(T_0)}} H^0(Y, L_\gamma)$$

Moreover, consider the restriction $r : H^0(X, L_\lambda) \rightarrow H^0(X_J, L_\lambda)$. Then

$$r(F_n H^0(X, L_\lambda)) \subset F_n H^0(X_J, L_\lambda) .$$

PROOF. We prove the last assertion. For the other ones, it suffices to translate the proof of [1, Thm. 7] for the case $J = \emptyset$, to this more general case.

It suffices to prove that if $\sum t_i = n$, then $r(F_{\sum t_i \alpha_i} H^0(X, L_\lambda)) \subset F_n H^0(X_J, L_\lambda)$. We recall that $F_{\sum t_i \alpha_i} H^0(X, L_\lambda) = \prod \sigma_i^{t_i} H^0(X, L_{\lambda - \sum t_i \alpha_i})$. Taking restriction to X_J we get:

$$r(F_{\sum t_i \alpha_i} H^0(X, L_\lambda)) = \frac{\prod \sigma_i^{t_i}|_{X_J} r(H^0(X, L_{\lambda - \sum t_i \alpha_i}))}{\prod \sigma_i^{t_i}|_{X_J} H^0(X_J, L_{\lambda - \sum t_i \alpha_i})} \subset$$

It follows that if there exists an $i \in J$ such that $t_i \neq 0$, then $r(F_{\sum t_i \alpha_i} H^0(X, L_\lambda)) = 0$, so

$$r(F_n H^0(X, L_\lambda)) \subset \sum_{\sum_{i \in I} t_i = n} \prod_{i \in I} \sigma_i^{t_i} H^0(X_J, L_{\lambda - \sum t_i \alpha_i}) =$$

$$\sum_{\sum_{i \in I} t_i = n} F_{\sum_{i \in I} t_i \alpha_i} H^0(X_J, L_\lambda) = F_n H^0(X_J, L_\lambda) .$$

□

In [1] it is proved that if $\lambda \in \Xi_+(T_0)$, then the restriction $r : H^0(X, L_\lambda) \rightarrow H^0(X_J, L_\lambda)$ is surjective. The following Lemma shows that in fact this is true for all $\lambda \in \Xi(T_0)$.

Lemma 1. *Let $\lambda \in \Xi(T_0)$. Then the restriction $r : H^0(X, L_\lambda) \rightarrow H^0(X_J, L_\lambda)$ is surjective.*

PROOF. It suffices to prove that $r_n = r|_{F_n H^0(X, L_\lambda)} : F_n H^0(X, L_\lambda) \rightarrow F_n H^0(X_J, L_\lambda)$ is surjective for all n . We prove the assertion by recursion.

First, we observe that

$$0 \neq F_{n_0(J, \lambda)} H^0(X_J, L_\lambda) \cong \text{gr}_{n_0(J, \lambda)} H^0(X_J, L_\lambda) = \bigoplus_{\substack{\gamma = \lambda - \sum_{i \in I} t_i \alpha_i \\ \sum_{i \in I} t_i = n_0(J, \lambda), \gamma \in \Xi_+(T_0)}} H^0(Y, L_\gamma).$$

$$\text{In particular, } \text{gr}_{n_0(J, \lambda)} H^0(X, L_\lambda) = \bigoplus_{\substack{\gamma = \lambda - \sum_{i \in I} t_i \alpha_i \\ \sum_{i \in I} t_i = n_0(J, \lambda), \gamma \in \Xi_+(T_0)}} H^0(Y, L_\gamma) \neq 0.$$

It follows that $F_{n_0(J, \lambda)} H^0(X, L_\lambda) \neq 0$ and r_{n_0} is surjective. In particular, $n_0(J, \lambda) \leq n_0(\emptyset, \lambda) = n_0$.

Suppose now that r_{n+1} is surjective. Take $s \in F_n H^0(X_J, L_\lambda)$, and consider $\pi_n(s)$, where $\pi_n : F_n H^0(X_J, L_\lambda) \rightarrow \text{gr}_n H^0(X_J, L_\lambda)$ is the canonical projection. There exists an element $\tilde{s} \in F_n H^0(X, L_\lambda)$ such that $\tilde{r}_n(\pi(\tilde{s})) = \pi(s)$, where $\tilde{r}_n : \text{gr}_n H^0(X, L_\lambda) \rightarrow \text{gr}_n H^0(X_J, L_\lambda)$ denotes the (surjective) morphism induced by r . By the recursion hypothesis, it follows that $r(\tilde{s} + F_{n+1} H^0(X, L_\lambda)) = s + F_{n+1} H^0(X_J, L_\lambda)$, and the proposition is proved. □

Consider the envelopping semigroup and its abelianisation $\pi : \text{Env}(G) \rightarrow \mathbb{A}^l$. The associated cone of \mathbb{A}^l is generated by the fundamental weights $\omega_1, \dots, \omega_l$. If $I \subset \{1, \dots, l\}$, we denote \mathcal{O}_I the orbit associated to the face generated by $\{\omega_i \mid i \in I\}$. Then the ideal associated to $\overline{\mathcal{O}_I}$ is $\bigoplus_{\gamma \notin \mathcal{M}_I} k e^\gamma$, where \mathcal{M}_I denotes the semigroup generated by the simple roots $\alpha_i, i \in I$.

Theorem 7. *We keep the precedings notations. The preimage of $\overline{\mathcal{O}_I}$ is isomorphic to $\text{Spec} \bigoplus_{\lambda \in \Xi(T_0)} H^0(X_J, L_\lambda)$. In particular, $\pi^{-1}(\overline{\mathcal{O}_I})$ is an irreducible normal affine variety. Moreover, if $x \in \mathbb{A}^l$ then $\pi^{-1}(x)$ is a irreducible normal affine variety.*

PROOF. Consider the restriction

$$r : k[\text{Env}(G_0)] = \bigoplus_{\lambda \in \Xi(T_0)} H^0(X, L_\lambda) \twoheadrightarrow \bigoplus_{\lambda \in \Xi(T_0)} H^0(X_J, L_\lambda).$$

It suffices to show that $\ker r = \mathcal{I}_{\overline{\mathcal{O}_I}} k[\text{Env}(G)]$. Observe that $\mathcal{I}_{\overline{\mathcal{O}_I}} k[\text{Env}(G_0)]$ is the ideal generated by $\{\sigma_i : i \in J\}$, σ_i as in Theorem 3.

Consider the morphism between the graded modules induced by r :

$$\tilde{r} : \bigoplus_{n=1}^{n_0} \text{gr}_n H^0(X, L_\lambda) \rightarrow \bigoplus_{n=1}^{n_0} \text{gr}_n H^0(X_J, L_\lambda) ,$$

where $\text{gr}_n H^0(X_J, L_\lambda) = 0$ if $n_0(J, \lambda) < n \leq n_0$. In order to calculate $\ker(\tilde{r})$, we consider the restriction to the n th summand:

$$\tilde{r}_n : \bigoplus_{\substack{\gamma = \lambda - \sum t_i \alpha_i \\ \sum t_i = n, \gamma \in \Xi_+(T_0)}} H^0(Y, L_\gamma) \rightarrow \bigoplus_{\substack{\gamma = \lambda - \sum_{i \in I} t_i \alpha_i \\ \sum_{i \in I} t_i = n, \gamma \in \Xi_+(T_0)}} H^0(Y, L_\gamma)$$

It is clear that $\ker \tilde{r}_n = \bigoplus_{\sum_{i \in I} t_i = n, t_i \alpha_i \in \Xi_+(T_0) \setminus \mathcal{M}_I} H^0(Y, L_{\lambda - \sum t_i \alpha_i})$. A standard argument shows that $\ker(r)$ is then generated by $\{\sigma_i : i \in J\}$, and the first assertion of the theorem is proved.

By Proposition 3, $\pi^{-1}(\overline{\mathcal{O}_I})$ is a irreducible normal variety. As \mathcal{O}_I is an homogeneous space the restriction $\pi|_{\pi^{-1}(\mathcal{O}_I)}$ is a fibration; it follows that if $x \in \mathcal{O}_I$, then $\pi^{-1}(x)$ is a normal variety. In order to prove that $\pi^{-1}(x)$ is irreducible, it suffices to prove that the isotropy group of a point in \mathcal{O}_I (for the action of $G(\text{Env}(G_0)) \times G(\text{Env}(G_0))$) is connected. We recall that $G = G(\text{Env}(G_0)) \cong (G_0 \times Z)/Z_0$, where $Z \cong T_0$ is the connected center of G , and $Z_0 = G_0 \cap Z$.

Consider an idempotent $e_I \in \mathcal{O}_I$, and let $Z_I \subset Z/Z_0$ be its isotropy group for the action of Z/Z_0 on \mathbb{A}^l . An element $(g, g') = ((a, z), (b, w)) \in (G \times Z)/Z_0 \times (G \times Z)/Z_0$ acts in \mathbb{A}^l by

$$(\overline{(a, z)}, \overline{(b, w)}) \cdot x = zw^{-1} \cdot x = \pi(zw^{-1})x \quad , \quad x \in \mathbb{A}^l ,$$

where $\pi : Z \rightarrow Z/Z_0$ denotes the quotient map. It follows that $(g, g') \in (G \times G)_{e_I}$ if and only if $zw^{-1} \in \pi^{-1}(Z_I)$, that is

$$(G \times G)_{e_I} = ((G_0 \times \pi^{-1}(Z_I))/Z_0 \times (G_0 \times \pi^{-1}(Z_I))/Z_0) \Delta Z \subset G \times G .$$

If we prove that $H = (G_0 \times \pi^{-1}(Z_I))/Z_0$ is connected, we are done. Consider Z'_I , the connected component of the identity of $\pi^{-1}(Z_I)$. We claim that the map $\varphi : G_0 \times Z'_I \rightarrow H$, $\varphi(g, z) = \overline{(g, z)}$ is surjective, which implies the assertion. Indeed, let $h \in H$ and take $(g, z) \in G_0 \times \pi^{-1}(Z_I) \subset G_0 \times Z$ a representant of h . Then there exists $z_0 \in Z_0$ such that $zz_0 \in Z'_I$, and $\varphi(gz_0^{-1}, zz_0) = \overline{(gz_0^{-1}, zz_0)} = \overline{(g, z)} = h$. \square

4.2. Asymptotic semigroup associated to a semisimple group.

We keep the notations of the preceding sections. In this paragraph we study the fibre $\pi^{-1}(0)$ of $\pi : \text{Env}(G_0) \rightarrow \mathbb{A}^l$, which will give a generalisation to arbitrary characteristic of the *asymptotic semigroup* defined by Vinberg in [16].

Definition 5. As before, let G_0 be a semisimple simply connected group, X its wonderful embedding, and let Y be the unique closed orbit of X . We define the *asymptotic semigroup associated to G_0* as:

$$\text{As}(G_0) = \text{Spec} \left(\bigoplus_{\lambda \in \Xi_+(T_0)} H^0(Y, L_\lambda) \right).$$

If $\Gamma \subset G_0$ is a finite central subgroup of G_0 , we put:

$$\text{As}(G_0/\Gamma) = \text{As}(G_0)/\Gamma.$$

In order to describe the combinatorial data associated to the asymptotic semigroup of a semisimple group, it suffices to describe the data associated to its simply connected cover ([11],[12]), so from now on, we suppose G_0 simply connected.

Theorem 8. *If $\pi : \text{Env}(G_0) \rightarrow \mathbb{A}^l$ is the abelianisation of the enveloping semigroup of G_0 , and $\pi_0 : S_0 \rightarrow A_0$ the abelianisation of S_0 , then*

$$\text{As}(G_0) \cong \pi^{-1}(0) \cong \pi_0^{-1}(0).$$

In particular, $\text{As}(G_0)$ is an algebraic semigroup with zero.

Let $\varphi : E \rightarrow S_0$ be the decoloration of S_0 , and let $\varrho : E \rightarrow X$ the bundle over the wonderful embedding of G_0 constructed in the preceding paragraph. Consider the closed $(G \times G)$ -stable subset of E :

$$E_Y = \varrho^{-1}(Y) = E_{\{1, \dots, l\}, \{1, \dots, l\}} \cong L_1^\vee|_Y \oplus \dots \oplus L_l^\vee|_Y,$$

where $Y \cong G_0/B_0 \times G_0/B_0^-$ is the unique closed $(G_0 \times G_0)$ -orbit of X . Then E_Y is a simple spherical $(G_0 \times G_0)$ -variety with open orbit isomorphic to $(G_0 \times G_0)/((U^- \times U)\Delta(T_0))$, and the restriction of φ to E_Y is a birational proper $(G_0 \times G_0)$ -morphism from E_Y over $\text{As}(G_0)$:

$$\varphi|_{\varrho^{-1}(Y)} = \varphi_Y : E_Y \rightarrow \pi_0^{-1}(0) = \text{As}(G_0).$$

In particular, $\text{As}(G_0)$ is a simple $(G_0 \times G_0)$ -spherical variety, with open orbit isomorphic to $(G_0 \times G_0)/((U^- \times U)\Delta(T_0))$, and thus of dimension $\dim \text{As}(G_0) = \dim G_0$.

PROOF. Consider the $(G_0 \times G_0)$ -action on $\text{As}(G_0)$ induced by the action over the sections $H^0(Y, L_\lambda|_Y)$, $\lambda \in \Xi_+(T_0)$. By Theorem 7, $\text{As}(G_0)$ is isomorphic to $\pi^{-1}(0)$ as a $(G_0 \times G_0)$ -variety.

On the other hand, the same arguments used in Theorem 7 allow to prove that $\pi_0^{-1}(0) \cong \text{As}(G_0)$.

By construction, $\varphi_Y : E_Y \rightarrow \text{As}(G_0)$ is a birational proper $(G_0 \times G_0)$ -morphism. Let us study the geometry of the $(G \times G)$ -spherical variety E_Y .

Recall that the closed $(G \times G)$ -stable subsets of E_Y are the subsets $E_{I, Y} = E_{I, \{1, \dots, l\}} = \bigoplus_{i \in I} L_i^\vee|_Y$, $I \subset \{1, \dots, l\}$. The line bundles L_i^\vee are $(G_0 \times G_0)$ -linearized, so there exist global sections s_i , $i = 1, \dots, l$, such that they do not

vanish over the affine open set $(B_0 \times B_0^-) \cdot y_0$, where $y_0 \in Y$ is the unique point of Y fixed by $B_0^- \times B_0$. It follows that

$$\varrho^{-1}((B_0 \times B_0^-) \cdot y_0) \cong ((B_0 \times B_0^-) \cdot y_0) \times \mathbb{A}^l.$$

More generally, if $I \subset \{1, \dots, l\}$ then

$$\varrho^{-1}((B_0 \times B_0^-) \cdot y_0) \cap E_{I,Y} = \bigoplus_{i \in I} L_i^\vee|_Y \cong ((B_0 \times B_0^-) \cdot y_0) \times \mathbb{A}^{\#I}.$$

In particular, $((B_0 \times B_0^-) \cdot \sum_{i \in I} s_i)(y_0)$ is a open $(B_0 \times B_0^-)$ -orbit of $E_{I,Y}$.

If $x_I = \sum_{i \in I} s_i(y_0)$ then $(G_0 \times G_0) \cdot x_I$ is an open $(G_0 \times G_0)$ -orbit of $E_{I,Y}$. So the $(G_0 \times G_0)$ -orbits and the $(G \times G)$ -orbits of E_Y coincide; in particular, E_Y is a simple $(G_0 \times G_0)$ -spherical variety, with $\theta(Y)$ as unique closed orbit, where θ is the zero section.

Let us calculate the isotropy group of $x_0 = x_{\{1, \dots, l\}}$.

If $(a, b) \in G_0 \times G_0$, then $(a, b) \cdot x_0 = \sum((a, b) \cdot s_i)((a, b) \cdot y_0)$, so $(a, b) \cdot x_0 = x_0$ if and only if $(a, b) \in (G_0 \times G_0)_{y_0}$, and $(a, b) \cdot s_i(y_0) = s_i(y_0)$ for all $i = 1, \dots, l$. It follows that we have $a \in B_0^-$, $b \in B_0$, and $\omega_i(a^{-1}b) = 1$, so $\omega_i(a) = \omega_i(b)$ for all $i = 1, \dots, l$. Then $(a, b) \in (G_0 \times G_0)_{x_0}$ if and only if $(a, b) \in (U^- \times U)\Delta(T_0)$.

On the other hand, as φ_Y is a birational $(G_0 \times G_0)$ -morphism, $\text{As}(G_0)$ is a $(G_0 \times G_0)$ -spherical variety, with open orbit isomorphic to $(G_0 \times G_0)/((U^- \times U)\Delta(T_0))$. Moreover, the $(G_0 \times G_0)$ -action extends onto a $(G \times G)$ -action in such way that the orbit of both actions coincide: just take the action induced by the $(G \times G)$ -action on E_Y .

Finally, the dimension of E_Y is equal to the dimension of Y plus l , so $\dim E_Y = \dim G_0$ (recall that l is equal to the number of simple roots). \square

Corollary 2. *The fibres of π and π_0 are equidimensional, with dimension equal to $\dim G_0$.* \square

The asymptotic semigroup being a $(G_0 \times G_0)$ -spherical variety, it seems natural to describe the combinatorial data associated to this variety. We begin by the calculation of the combinatorial data associated to the spherical homogeneous space $(G_0 \times G_0)/((U^- \times U)\Delta(T_0))$ (the open orbit of $\text{As}(G_0)$).

Proposition 6. *The combinatorial data associated to the spherical homogeneous space $(G_0 \times G_0)/H$, $H = (U^- \times U)\Delta(T_0)$, is the following:*

- i) *The space $\mathcal{Q}((G_0 \times G_0)/H)$ is isomorphic to $\Xi(T_0) \otimes \mathbb{Q}$.*
- ii) *The valuation cone $\mathcal{V}((G_0 \times G_0)/H)$ is all the space $\mathcal{Q}((G_0 \times G_0)/H)$.*
- iii) *The set of colors \mathcal{D} has $2l$ elements, namely :*

$D_\alpha = ((B_0 s_\alpha B_0^-) \times G_0)H$ and $D'_\alpha = (G_0 \times (B_0^- s_{-\alpha} B_0))H$, where α is a simple root of G_0 associated to (B_0, T_0) .

Moreover,

$$\rho_{(G_0 \times G_0)/H}(\nu_{D_\alpha}) = \rho_{(G_0 \times G_0)/H}(\nu_{D'_\alpha}) = \alpha^\vee.$$

PROOF. A weight $(\omega, \lambda) \in \Xi(G_0 \times G_0)$ is always $(U^- \times U)$ -invariant, and it is $\Delta(T)$ -invariant if and only if $\lambda = -\omega$, so $\mathcal{Q}((G_0 \times G_0)/H) \cong \Xi(T_0) \otimes \mathbb{Q}$.

On the other hand, $U^- \times U$ is a unipotent maximal subgroup of $G_0 \times G_0$, so $\mathcal{V}((G_0 \times G_0)/H) = \mathcal{Q}((G_0 \times G_0)/H)$ ([10], [7, cor. 7.2]).

In order to calculate the colors, observe that if $y_0 = (1, 1) \in (G_0 \times G_0)/H$, then

$$(((G_0 \times G_0)/H) \setminus (B_0 \times B_0^-)) \cdot y_0 = \bigcup_{i=1}^l D_{\alpha_i} \cup D'_{\alpha_i}.$$

It follows that the colors are the divisors D_α and D'_α , α a simple root.

Consider the projection $p : G_0 \times G_0 \rightarrow (G_0 \times G_0)/H$. It has reduced (as schemes) and irreducible fibres, so by [7, §5] in order to calculate the valuations associated to the colors D_α and D'_α it suffices to calculate the valuations of $k(G_0 \times G_0) = k(G_0) \otimes k(G_0)$ associated to the colors of $G_0 \times G_0$. Those colors are the divisors $B_0 s_{\alpha_i} B_0^- \times G_0$ and $G_0 \times B_0^- s_{-\alpha_i} B_0$, $i = 1, \dots, l$. The valuation of $k(G_0)$ associated to $B_0 s_{\alpha_i} B_0^-$ is α_i^\vee , so $\rho_{G_0 \times G_0}(\nu_{D_\alpha}) = (\alpha_i^\vee, 0)$, and $\rho_{G_0 \times G_0}(\nu_{D'_\alpha}) = (0, -\alpha_i^\vee)$. If $p : G_0 \times G_0 \rightarrow (G_0 \times G_0)/H$ is the projection, then $p_*(\rho_{G_0 \times G_0}(\nu_{D_\alpha})) = p_*(\alpha_i^\vee, 0) = \alpha_i^\vee$, so $\rho_{(G_0 \times G_0)/H}(\nu_{D_\alpha}) = \alpha_i^\vee$. Analogously, $\rho_{(G_0 \times G_0)/H}(\nu_{D'_\alpha}) = \alpha_i^\vee$. \square

To finish, we describe the colored cone associated to $\text{As}(G_0)$:

Proposition 7. *Let G_0 be a semisimple simply connected group, and $H = (U^- \times U)\Delta(T_0)$ as before. Then*

$$\varphi_Y = \varphi|_{E_Y} : E_Y = E_{\{1, \dots, l\}, \{1, \dots, l\}} \rightarrow \text{As}(G_0)$$

is the decoloration of $\text{As}(G_0)$.

Moreover, the colored cone of $\mathcal{Q}((G_0 \times G_0)/H)$ associated to $\text{As}(G_0)$ is :

$$(\mathcal{C}(\text{As}(G_0)), \mathcal{F}(\text{As}(G_0))) = (\mathbb{Q}^+ \{\alpha_1^\vee, \dots, \alpha_l^\vee\}, \{D_{\alpha_i}, D'_{\alpha_i} \mid i = 1, \dots, l\}).$$

PROOF. The asymptotic semigroup is an affine spherical variety. In particular, it is simple. The $(G_0 \times G_0)$ -action on $\text{As}(G_0)$ is induced by the left and right multiplication on S_0 , so the unique closed orbit is the zero of S_0 . It follows that the set of colors $\mathcal{D}(\text{As}(G_0))$ is all $\mathcal{D}((G_0 \times G_0)/(U^- \times U)\Delta(T_0))$.

On the other hand, E_Y is without colors. Indeed, the closure in E_Y of the colors are obtained by taking the preimages by ϱ of the $(B_0 \times B_0^-)$ -stable divisors of Y . But E_Y is a simple spherical variety of unique closed orbit $\theta(Y)$, the zero section of the bundle $E_Y \rightarrow Y$, and it is clear that this orbit is not contained in the closure of any color.

As φ_Y is a proper morphism, $\mathcal{C}(E_Y) = \mathcal{V}(E_Y) = \mathcal{V}(\text{As}(G_0))$, so $\varphi_Y : E_Y \rightarrow \text{As}(G_0)$ is the decoloration of the asymptotic semigroup.

Now, if we show that $\text{As}(G_0)$ contains no $(G_0 \times G_0)$ -stable divisors, the proposition is proved. Such a divisor is the image by φ of a $(G_0 \times G_0)$ -stable divisor $D \subset E_Y$.

Recall that the $(G_0 \times G_0)$ -stable divisor $E_{\{i\},Y}$, $i = 1, \dots, l$, of E_Y is obtained by taking the zero section of the bundle L_i^\vee :

$$E_{\{i\},Y} = E_{\{1, \dots, \widehat{i}, \dots, l\}, \{1, \dots, l\}} = L_1^\vee|_Y \oplus \cdots \oplus \widehat{L_i^\vee|_Y} \oplus \cdots \oplus L_l^\vee|_Y,$$

where $\widehat{L_i^\vee|_Y}$ means that L_i^\vee is not taken into account.

By construction,

$$\varphi_Y(E_{\{i\},Y}) \cong \text{Spec} \left(\bigoplus_{n_j \geq 0} H^0(Y, L_{\sum_{j \neq i} n_j \omega_j}) \right)$$

Under this isomorphism, the inclusion $\varphi_Y(E_{\{i\},Y}) \subset \text{As}(G_0)$ is induced by:

$$H^0(Y, \bigotimes_i L_i^{\otimes n_i}) \longrightarrow \begin{cases} H^0(Y, \bigotimes_{j \neq i} L_j^{\otimes n_j}) & \text{if } n_i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

The kernel of this morphism is the ideal associated to $\varphi_Y(E_{\{i\},Y})$, and verifies:

$$\mathcal{P}_{\varphi_Y(E_{\{i\},Y})} = \bigoplus_{\substack{n_j \geq 0, j \neq i \\ n_i > 0}} H^0(Y, L_1^{\otimes n_1} \otimes \cdots \otimes L_l^{\otimes n_l}).$$

In particular, this ideal contains $H^0(Y, L_i)$.

Recall that the bundle $L_i|_Y$ is isomorphic to $\mathcal{L}(\omega_i) = L(\omega_i) \boxtimes L(-\omega_i)$. Consider a non-zero section s of $L(\omega_i) \rightarrow G_0/B_0$, (i.e. s does not vanish over an open subset of G_0/B_0). Let s' be a non-zero section of $L(-\omega_i) \rightarrow G_0/B_0^-$. then $\varphi_Y(D'_i)$ is contained in the intersection of the zeroes of s et s' . By construction, these varieties intersect in a variety of codimension greater or equal to 2, so $\text{codim } \varphi_Y(D_i) \geq 2$, and the proposition is proved. \square

4.3. Minimality properties of the enveloping semigroup.

In this paragraph we prove that the enveloping semigroup of a semisimple group G_0 is a minimal element of the family $\mathcal{FM}(G_0)$, in the sense of [15, Thm. 5]. In particular, $\text{Env}(G_0)$ is a very flat monoid, and thus the abelianisation $\pi : \text{Env}(G_0) \rightarrow \mathbb{A}^l$ gives a flat deformation of $G - 0$ towards $\text{As}(G_0)$. We keep the notations of the preceding paragraphs.

Proposition 8. *Let G_0 be a semisimple group. Then $\text{Env}(G_0)$ is a very flat monoid.*

PROOF. Recall that reductive monoids are Cohen-Macaulay varieties ([13]), and that the abelianisation of $\text{Env}(G_0)$ is $\pi : \text{Env}(G_0) \rightarrow \mathbb{A}^l$, which is a smooth variety. Let \mathcal{P} be a prime ideal of $k[x_1, \dots, x_l] = k[\mathbb{A}^l]$ and $\mathcal{P}k[\mathbb{A}^l] \subset \mathcal{Q} \subset k[\text{Env}(G_0)]$ be a prime ideal in the preimage of \mathcal{P} by $\pi : \text{Env}(G_0) \rightarrow \mathbb{A}^l$; we

consider $A = k[\text{Env}(G_0)]_{\mathcal{Q}}$ and $R = k[\mathbb{A}^l]_{\mathcal{P}}$. From [4, thm. 18.16], it follows that A is a flat R -module if and only if $\dim A = \dim R + \dim A/\mathcal{M}_R A$, where $\mathcal{M}_R = \mathcal{P}R$ is the unique maximal ideal of R .

By Corollary 2 and Theorem 7, π is equidimensional with reducibles (as schemes) and irreducibles fibres, so if V is the subvariety associated to \mathcal{Q} , then $\dim A - \dim R = \dim \pi^{-1}\pi(V) = \dim A/\mathcal{P}A$. \square

Theorem 9. *Let G_0 be a semisimple group. Let S be a reductive monoid such that its unit group G' has commutator G_0 . Let Z' be the connected center of G' , and $Z'_0 = T_0 \cap Z'$ (so $G' \cong (G_0 \times Z')/Z'_0$). Then:*

i) *If S is very flat, then there exists $\theta : Z' \rightarrow T_0$ such that $Z'|_{Z'_0} = \text{Id}_{Z'_0}$ and*

$$\mathcal{C}(S) = \mathbb{Q}^+ \{(-\theta_*(f), f) \in \mathcal{Q}(G_0) \times \mathcal{Q}(Z') : f \in \mathcal{C}(S) \cap \mathcal{Q}(Z')\} + \mathbb{Q}^+ \mathcal{F} ,$$

where $\mathcal{F} \subset \mathcal{D}(G')$ is the set of colors of S . Moreover, $\mathcal{C}(A_S) = \mathcal{C}(S) \cap \mathcal{Q}(Z')$ and $\theta_*(\mathcal{C}(A_S)) \subset C_0$, where C_0 denotes as before the Weyl chamber of G_0 . In particular, $\mathbb{Q}^+ \mathcal{B}(S) = \mathbb{Q}^+ \{(-\theta_*(f), f) : f \in \mathcal{C}(S) \cap \mathcal{Q}(Z')\}$.

ii) *Conversely, suppose that there exists a morphism $\theta : Z' \rightarrow T_0$ such that the restriction to Z'_0 is the identity, and such that*

$$\mathcal{C}(S) = \mathbb{Q}^+ \{(-\theta_*(f), f) : f \in \mathcal{C}(S) \cap \mathcal{Q}(Z')\} + \mathbb{Q}^+ \mathcal{F} ,$$

with $\mathcal{C}(A_S) = \mathcal{C}(S) \cap \mathcal{Q}(Z')$, and $\theta_*(\mathcal{C}(A_S)) \subset C_0$. Then S is a very flat monoid.

In particular, S is very flat if and only if $S \cong A_S \times_{\mathbb{A}^l} \text{Env}(G_0)$.

PROOF. The proof of i) is carried on straightforward from [15, §4].

In order to prove ii), we observe that if $\varphi : A_S \rightarrow \mathbb{A}^l$ is the morphism induced by $\theta_* : \mathcal{C}(A_S) \rightarrow C_0 = \mathcal{C}(\mathbb{A}^l)$, then by Propositions 2 and 1, $\widehat{S} = (A_S \times_{\mathbb{A}^l} \text{Env}(G_0))$ is a very flat monoid, with abelianisation A_S (recall that $\pi : \text{Env}(G_0) \rightarrow \mathbb{A}^l$ has normal irreducible fibres). We claim that $\widehat{S} \cong S$. In particular, the last assertion of the theorem is proved.

In order to prove the assertion, first we show that \widehat{G} , the unit group of \widehat{S} , is isomorphic to G' .

We know that $G = G(\text{Env}(G_0)) \cong (G_0 \times T_0)/Z_0$. In order to avoid confusion, we note Z the connected center of G , and $\kappa : Z \rightarrow T_0$ the induced isomorphism. By Proposition 1, $\widehat{G} = G(A_S \times_{\mathbb{A}^l} \text{Env}(G_0)) = (Z'/Z'_0) \times_{Z/Z_0} G$. Consider

$$\gamma : (G_0 \times Z')/Z'_0 \rightarrow (Z'/Z'_0) \times_{Z/Z_0} G = \widehat{G} \quad , \quad \overline{(g_0, z')} \mapsto \left(\overline{z'}, \overline{(g_0, \kappa^{-1}\theta(z'))} \right) .$$

The morphism γ is well defined. Indeed, for all $z' \in Z'$, $z'_0 \in Z'_0$ and for all $g_0 \in G_0$ (recall that $Z'_0 \subset Z_0$),

$$\pi \left(\overline{(z'_0 g_0, \kappa^{-1}\theta(z' z_0^{-1}))} \right) = \overline{\kappa^{-1}\theta(z' z_0^{-1})} = \overline{\kappa^{-1}\theta(z')} .$$

It follows that

$$\left(\overline{z'}, \overline{(g_0, \kappa^{-1}\theta(z'))}\right) = \left(\overline{z'z_0^{-1}}, \overline{(z_0'g_0, \kappa^{-1}\theta(z'z_0^{-1}))}\right) .$$

We claim that γ is an isomorphism. Indeed,

$$\left(\overline{z'}, \overline{(g_0, \kappa^{-1}\theta(z'))}\right) = \left(\overline{1}, \overline{(1, 1)}\right)$$

if and only if $z' \in Z_0'$ and $\overline{(1, 1)} = \overline{(g_0, \kappa^{-1}\theta(z'))}$. Then, $\kappa^{-1}\theta(z') = z'$, and $\overline{(1, 1)} = \overline{(g_0, z')} = \overline{(g_0z'^{-1}, 1)}$. It follows that $g_0z'^{-1} = 1$, and $\overline{(g_0, z')} = \overline{(1, 1)}$; i.e., γ is injective.

Consider $\left(\overline{z'}, \overline{(g_0, z)}\right) \in (Z'/Z_0') \times_{Z/Z_0} G$; then $\overline{\kappa^{-1}\theta(z')} = \gamma(\overline{z'}) = \pi\left(\overline{(g_0, z)}\right) = \overline{z}$, that is there exists $z_0 \in Z_0$ such that $\kappa^{-1}\theta(z') = zz_0$, so

$$\left(\overline{z'}, \overline{(g_0, z)}\right) = \left(\overline{z'}, \overline{(g_0z_0^{-1}, zz_0)}\right) = \gamma(g_0z_0^{-1}, z') .$$

It follows that γ is surjective, so it is an isomorphism.

By Proposition 1, $\widehat{Z} = (Z'/Z_0') \times_{Z/Z_0} Z = (Z'/Z_0') \times_{Z/Z_0} \kappa^{-1}\theta(Z')$, which is isomorphic to Z' . This isomorphism is given by $Z' \ni z' \mapsto \left(\overline{z'}, \overline{(1, \kappa^{-1}\theta(z'))}\right)$. Moreover,

$$\mathbb{Q}^+\mathcal{B}(\widehat{S}) = \left\{(-\widehat{\kappa}_*(f), f) \in \mathcal{Q}(G_0) \times \mathcal{Q}(\widehat{Z}) : f \in \mathcal{C}(A_S)\right\}$$

where $\widehat{\kappa}(r, z) = \kappa(z)$. Then, $\widehat{\kappa} \circ \gamma(z') = \kappa(\kappa^{-1}\theta(z')) = \theta(z')$ for all $z' \in Z'$, so $\mathbb{Q}^+\mathcal{B}(S) = \mathbb{Q}^+\mathcal{B}(\widehat{S})$. It follows that $S \cong \widehat{S}$. \square

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