ALGEBRAIC MONOIDS AND GROUP EMBEDDINGS

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Abstract. We study the geometry of algebraic monoids. We prove that the group of invertible elements of an irreducible algebraic monoid is an algebraic group, open in the monoid. Moreover, if this group is reductive, then the monoid is affine. We then give a combinatorial classification of reductive monoids by means of the theory of spherical varieties.

1. Introduction

Let k be an algebraically closed field of arbitrary characteristic. An *algebraic monoid* is an algebraic variety S over k with an associative product $m : S \times S \to S$ which is a morphism of varieties, and such that there exists an element $1 \in S$ with $m(1,s) = m(s,1) = s$ for all $s \in S$. We write $m(s,s') = ss'$ for all $s, s' \in S$. Let $G(S)$ be the *unit group* of S :

$$G(S) = \{g \in S \ : \ \exists g' \in S \ , \ gg' = g'g = 1\} \ .$$

We have the following picture:

$$\begin{array}{ccc} \{ \text{ algebraic monoids } \} & \supset & \{ \text{ algebraic groups } \} \\ \cup & & \cup \\ \{ \text{ affine alg. monoids } \} & \supset & \{ \text{ affine alg. groups } \} \end{array}$$

It is known that if S is an affine algebraic monoid, then $G(S)$ is an algebraic group, open in S ([10]). We extend this result to irreducible algebraic monoids. Moreover, the natural action of $G(S) \times G(S)$ on S by left and right multiplication has only one closed orbit : S is a *simple embedding* of $G(S)$. Furthermore, if we require $G(S)$ to be a reductive group, then the monoid S is affine. The proof of this result is based on the following remark:

If S is a reductive monoid, then by the Bruhat decomposition there exist opposite Borel subgroups B, B^- of $G(S)$ such that BB^- is open in S. So S is a spherical $(G(S) \times G(S))$-variety. It follows that if S has a zero, then S is affine. We reduce the general case to that one by using techniques of the theory of spherical varieties.

Moreover, using the dictionary between simple spherical varieties and a family of combinatorial objects, the *colored cones* ([6]), we classify reductive monoids in arbitrary characteristic, in a way dual to that one of [14] (in characteristic zero).

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2. Algebraic monoids and group embeddings

We say that an algebraic monoid S is *irreducible* (resp. *normal*) if its underlying variety is irreducible (resp. normal).

An element $e \in S$ is an *idempotent (element)* of S if $e^2 = e$. We denote $E(S)$ the set of idempotents of S.

We say that a monoid S has a *zero* if there exists an element $0 \in S$ such that $0s = s0 = 0$ for all $s \in S$.

An *ideal* of S is a non-empty subset $I \subset S$ such that $SIS \subset I$.

If there exists an ideal $Y \subset S$ contained in all the ideals of S, we say that S has *kernel* Y.

If the unit group of a monoid is a reductive algebraic group, we say that the monoid is *reductive.*

A morphism $\pi : S \to S'$ between two algebraic monoids is a *morphism of algebraic monoids* if $\pi(ss') = \pi(s)\pi(s')$, and $\pi(1_S) = 1_{S'}$. It is clear that $\pi(G(S)) \subset G(S')$. If S and S' have a zero, we impose that $\pi(0_S) = 0_{S'}$.

These conditions are not automatically satisfied:

Example 1. Let $M(n)$ be the monoid of $n \times n$ matrices with coefficients in k. We can identify $M(n)$ with the monoid of endomorphisms of a k-vector space V of dimension n. Its unit group is $\mathrm{Gl}(n)$, the group of invertible matrices; it is isomorphic to the group of automorphisms of V. The monoid $M(n)$ is an affine, smooth, irreducible, reductive monoid with zero.

The morphism $M(n) \to M(n)$, $A \mapsto 0$, commutes with the multiplication, but it is not a morphism of algebraic monoids. The morphism $M(n) \to M(n) \times M(n)$, $A \mapsto (A, \mathrm{id})$, is not a morphism of algebraic monoids. Indeed, the image of the zero matrix is not the zero of $M(n) \times M(n)$.

Notation. From now on we suppose that all the algebraic monoids are irreducible, unless otherwise mentioned.

Definition 1. Let G be an algebraic group. Consider the action of $G \times G$ on G given by

$$(a, b) \cdot g = agb^{-1} \quad , \quad a, b, g \in G \ .$$

This action is transitive, and the isotropy group of 1 is $\Delta(G)$, the diagonal of G. Moreover, the action satisfies the equation:

$$(a, 1) \cdot b = ab = (1, b^{-1}) \cdot a \ . \tag{1}$$

We say that a $(G \times G)$-variety X is an *embedding of* G if there exists an element $x \in X$ such that the orbit $\mathcal{O}_x$ is open in X and isomorphic to $G \cong (G \times G)/\Delta(G)$. If there exists only one closed orbit, we say that X is a *simple embedding.*

Theorem 1. *Let S be an algebraic monoid. Then $G(S)$ is a connected algebraic group, open in S, and S is a simple embedding of $G(S)$. Moreover, the unique closed $(G(S)\times G(S))$-orbit in S is the kernel of S.*

Proof. Let $m : S\times S \to S$ be the product in S, and $p_1, p_2 : S\times S\to S$ the projections. Then

$$G = p_1(m^{-1}(1)) \cap p_2(m^{-1}(1)) \ , \tag{2}$$

and G is constructible in S. If we prove that G is open and dense in S, it will follow in particular that G is connected, because S is irreducible.

From equation (2), we have

$$G \supset p_1(m^{-1}(1)^\circ) \cap p_2(m^{-1}(1)^\circ) \ ,$$

where $m^{-1}(1)^\circ$ is an irreducible component of $m^{-1}(1)$ that contains $(1,1)$.

The product m is surjective, and S is irreducible, so ([5, prop. 6.4.5])

$$\dim m^{-1}(1)^\circ \geq \dim S \ .$$

Let $\rho_i = p_i|_{m^{-1}(1)^\circ}$, $i=1,2$. In order to show that G is dense in S, it is sufficient to prove that the morphisms ρ_i are dominant: then there exists a non-empty open subset of S contained in $\rho_1(m^{-1}(1)^\circ)\cap\rho_2(m^{-1}(1)^\circ)\subset G$.

The fibre $\rho_1^{-1}(1)$ is composed of the couples $(x,y)\in S\times S$ such that $x=1$ and $xy=1$, so $\rho_1^{-1}(1)=\{(1,1)\}$.

Because of the upper-semicontinuity of the function $x\mapsto \dim\rho^{-1}(\rho(x))$, there exists a non-empty open subset $U\subset m^{-1}(1)^\circ$ such that $\dim\rho^{-1}(\rho(x))=0$ for all $x\in U$. Therefore,

$$\dim S \leq \dim m^{-1}(1)^\circ = \dim \overline{\rho_1(m^{-1}(1)^\circ)} \leq \dim S \ .$$

We have then equality, and $\dim\overline{\rho_1(m^{-1}(1)^\circ)} = \dim S$, and thus ρ_1 is a dominant morphism. By symmetry, ρ_2 is a dominant morphism too.

The left multiplication $\ell_g : G\times S\to S$ by an element $g\in G$ is an automorphism of S, with inverse $\ell_{g^{-1}}$, such that $\ell_g(G)=G$. It follows immediately that G is open in S, and that every point in G is smooth in S.

Next, we show that $g\mapsto g^{-1}$ is a morphism. Before doing that, we prove that if $s\in S$ has a left inverse, then s is a unit.

Let $s'\in S$ be such that $s's=1$. Then ℓ_s has $\ell_{s'}$ as a left inverse, so ℓ_s is injective. The group G being open in S, $\ell_s(G)$ is constructible in S and $\dim\ell_s(G)=\dim S$. Then $\ell_s(G)$ contains a non-empty open subset U of S. But $U\cap G\neq\emptyset$, so $\ell_s(G)\cap G\neq\emptyset$. If $g\in G\cap\ell_s(G)$, there exists $g'\in G$ such that $g=sg'$, and thus $s=gg'^{-1}\in G$. Therefore, we have:

$$m^{-1}(1) = \{(x,y)\in S\times S \ : \ x\in G\,,\ y=x^{-1}\} \ ,$$

and $m^{-1}(1)=m^{-1}(1)^\circ$ is irreducible, so that $p_1|_{m^{-1}(1)}=\rho_1 : m^{-1}(1)\to G$ is bijective. If ρ_1 is separable, then it follows from Zariski's Main Theorem that ρ_1

is an isomorphism. Analogously, ρ_2 is an isomorphism. The inverse map $g \mapsto g^{-1}$ is obtained by the composition $\rho_2 \circ \rho_1^{-1}$, so it is an isomorphism.

So we must prove that ρ_1 is separable. Every point of G is smooth in S, so we can consider the differential of the product m at the point $(1,1) \in G \times G$:

$$T_{(1,1)}m : T_{(1,1)}(S \times S) \cong T_1S \times T_1S \to T_1S \quad , \quad (u,v) \mapsto u+v \; .$$

Indeed, $m|_{\{1\}\times S}(s) = s$ and $m|_{S\times\{1\}}(s) = s$ for all $s \in S$; the restriction of $T_{(1,1)}m$ to the subspaces $T_1S \times \{0\}$ and $\{0\} \times T_1S$ is then the identity.

In particular, $T_{(1,1)}m$ is surjective, and the subvariety $m^{-1}(1) \subset S \times S$ is smooth at $(1,1)$, with tangent space equal to

$$T_{(1,1)}m^{-1}(1) = \ker T_{(1,1)}m = \{(u,v) \in T_1S \times T_1S \ : \ u+v=0\} \; .$$

The differential of ρ_1 at the point $(1,1)$ is the restriction to $T_{(1,1)}m^{-1}(1)$ of the differential of the projection. It follows that $T_{(1,1)}\rho_1$ is surjective, and ρ_1 is separable.

We now show that S is a simple embedding of $G = G(S)$. We consider the action of $G \times G$ over S by left and right multiplication :

$$(g_1, g_2) \cdot s = g_1 s g_2^{-1} \quad , \quad g_1, g_2 \in G \ , \ s \in S \; .$$

The $(G \times G)$-orbit passing by 1 is G, so S is an embedding of G. If Y is a closed $(G \times G)$-orbit, then

$$SYS = \overline{G}Y\overline{G} \subset \overline{GYG} = \overline{Y} = Y \; .$$

It follows that every closed orbit is an ideal. If Y and Y' are closed orbits, then $\emptyset \neq YY' \subset Y \cap Y'$, and thus $Y = Y'$, because Y and Y' are orbits. Let Y be the unique closed orbit. If $I \subset S$ is an ideal, then $\emptyset \neq IY \subset Y \cap I$. It follows that $Y \subset I$, so Y is the kernel of S. □

We remark that in the proof of the preceding theorem we have proved :

Corollary 1. *Let S be an algebraic monoid. If $s \in S$ has a left (or right) inverse, then s is an unit.* □

On the other side, we have a partial converse to theorem 1 (which was proved by Vinberg in characteristic 0 – see [14]):

Proposition 1. *Let G be an algebraic group and S be an affine embedding of G. Then the product on G extends to a product $\widetilde{m} : S \times S \to S$, in such a way that $G(S) = G$.*

Proof. If the product $m : G \times G \to G$ extends to a morphism $\widetilde{m} : S \times S \to S$, then $\widetilde{m}$ is a product, because $\widetilde{m}|_{G\times G} = m$ is associative. For convenience, we denote $\widetilde{m} = m$. It is clear that the identity $1 \in G$ satisfies $1s = s1 = s$ for all

$s \in S$. Indeed, this holds for the elements of the open subset $G \subset S$. Consider the right and left actions of G given by :

$$\begin{array}{ll} G \times S \to S & g \cdot s = (g,1)\cdot s \\ S \times G \to S & s \cdot g = (1, g^{-1}) \cdot s \end{array}$$

By construction and equation (1) these actions coincide on $(G \times G) \subset (S \times S)$ with the product in G . Then there exists a morphism

$$m : (G \times S) \cup (S \times G) = U \to S$$

which extends the product of G . Therefore, it suffices to prove that every regular function defined in U can be extended to $S \times S$.

If V is a vector space and $W \subset V$ a subspace, an easy calculation shows that $W \otimes W = (V \otimes W) \cap (W \otimes V)$. Applying this to $k[S] \subset k[G]$, we get:

$$k[S \times S] \cong k[S] \otimes k[S] = (k[S] \otimes k[G]) \cap (k[G] \otimes k[S]) \ .$$

On the other hand, a function $f \in k[G] \otimes k[G]$ belongs to $k[S] \otimes k[G]$ if and only if it can be extendend to a function defined in $S \times G$. The same happens for $k[G] \otimes k[S]$. It follows that every function defined in U can be extended to a function defined in $S \times S$.

Finally, we prove that $G(S) = G$. The inclusion $G \subset G(S)$ is trivial. Let $x \in S$ be a unit. Then $xG \cap G \neq \emptyset$, and there exists $g \in G$ such that $x \cdot g \in G$. It follows that $x = (x \cdot g) \cdot g^{-1} \in G$. □

In order to study the geometry of monoids it is useful to suppose that they are normal. The following result allows us to doing so without loss of generality:

Lemma 1. ([11], [13]) *Let S be an algebraic monoid, and $\pi : \widetilde{S} \to S$ its normalization. Then there exists an unique associative product $\widetilde{m} : \widetilde{S} \times \widetilde{S} \to \widetilde{S}$ such that $\pi \circ \widetilde{m} = m \circ (\pi, \pi)$. Moreover, the following proprieties are verified:*

i) *$I \subset S$ is an ideal if and only if $\pi^{-1}(I) \subset \widetilde{S}$ is an ideal.*

ii) *S has a zero 0_S if and only if $\pi^{-1}(0_S) = \{0_{\widetilde{S}}\}$, where $0_{\widetilde{S}}$ is a zero of $\widetilde{S}$.*

iii) *$\widetilde{S}$ is an algebraic monoid, with $G(\widetilde{S}) = \pi^{-1}(G(S))$, and*

$$\pi|_{G(\widetilde{S})} : G(\widetilde{S}) \to G(S)$$

is an isomorphism of groups.

iv) *If $e \in E(S)$, then $E(\widetilde{S}) \cap \pi^{-1}(e) \neq \emptyset$.*

It follows that π is a morphism of (algebraic) monoids. In particular, π is equivariant for the actions of $G(\widetilde{S}) \times G(\widetilde{S})$ and $G(S) \times G(S)$.

Proof. We only show how to define the product of the normalization, for the proof of the assertions, we refer the reader to [13] and [10].

We define the multiplication $\widetilde{m} : \widetilde{S} \times \widetilde{S} \to \widetilde{S}$ by means of the universal propriety of normalizations: $\widetilde{m}$ is the unique morphism such that the diagram

$$\begin{array}{ccc} \widetilde{S}\times\widetilde{S} & \xrightarrow{\ \widetilde{m}\ } & \widetilde{S} \\ {\scriptstyle \pi\times\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ S\times S & \xrightarrow[m]{} & S \end{array}$$

is commutative.

By the universal property of normalizations, we can easily prove that $\widetilde{m}$ is associative, and by construction, π is then a morphism of algebraic semigroups.

□

Proposition 2. *Let S be a (not necessarily irreducible) monoid, such that $G = G(S)$ is dense in S. Then any two irreducible components of S are isomorphic, and their intersection is contained in $S \setminus G$.*

PROOF. Let S_1 be an irreducible component of S passing by 1. If m denotes the product in S, then $m(S_1 \times S_1)$ is irreducible and contains S_1, hence $m(S_1 \times S_1) = S_1$. It follows that S_1 is an algebraic monoid with unit group $G_1 = S_1 \cap G$. From Theorem 1 we deduce that G_1 is the connected component of G passing by 1. It follows that $S_1 = \overline{G_1}$ is the unique irreducible component of S passing by 1.

Let S_0 be an irreducible component of S. Then there exists $g \in G \cap S_0$. An easy calculation shows that $\ell_g|_{S_1} : S_1 \to S_0$ is an isomorphism of inverse $\ell_{g^{-1}}|_{S_0}$. In order to prove the last assertion, we observe that $S_0 \cap G = gG_1$, hence S_0 is the unique irreducible component passing by g.

□

3. Reductive monoids

We restrict ourselves to the study of reductive (irreducible) monoids. All the reductive groups are supposed connected.

Definition 2. If G is a reductive group, a homogeneous space G/H is *spherical* if there exists a Borel subgroup B of G such that $BH \subset G$ is open. If X is a G-variety with an open orbit isomorphic to G/H, then X is called an *embedding of G/H.* We say that X is *simple* if the action of G over X has only one closed orbit. If X is normal, we say that X is a *spherical variety.*

If G is a reductive group, it follows from the Bruhat decomposition that the normal G-embeddings are spherical varieties so, by Theorem 1, normal reductive monoids are simple spherical varieties. The aim of this section is to study the geometry of reductive monoids from the point of view of the theory of spherical varieties. In particular, we will show that reductive monoids are affine.

Remark. In the litterature it is imposed to G-embeddings to be normal. We don't do this in order to be able to state results valid for all reductive monoids.

We begin by proving the following

Lemma 2. *Let S be a (not necessarily irreducible) reductive monoid with zero. Then S is affine.*

Proof. More generally, if X is a simple spherical variety with closed orbit a point, then X is affine (see proposition 5 below). By Proposition 2, we can suppose that S is irreducible. If S is not normal it suffices to take the normalization $\pi : \widetilde{S} \to S$. It follows from Lemma 1 that $\widetilde{S}$ is a normal monoid with zero. So $\widetilde{S}$ is affine and it follows that S is. Indeed, the morphism of the normalization π is a finite surjective morphism, so $\widetilde{S}$ is affine if and only if S is ([3, p. 63]). □

In order to continue our study we must give *a priori* proofs of results concerning idempotents in reductive monoids, which are known in the affine case. We recall that an *one parameter subgroup* (1-PS) of a group G is a multiplicative morphism $\lambda : k^* \to G$. We denote $\Xi_*(G)$ the group of 1-PS of G.

Lemma 3. *Let S be a reductive monoid with unit group G, and let $T \subset G$ be a maximal torus. Then every $(G \times G)$-orbit contains an idempotent element belonging to $\overline{T}$.*

Proof. First, we show that we can approach every $(G \times G)$-orbit of S by an 1-PS of G.

If $x \in S$ there exists a curve germ $\xi \in G_{((t))} = \mathrm{Hom}(\mathrm{Spec}\, k((t)), G)$, where $k((t))$ denotes the ring of Laurent series in one variable with coefficients in k, such that $\lim_{t \to 0} \xi(t) = x$. By Iwahori's Theorem ([8]), there exists an 1-PS λ of G such that $\lambda = f\xi f'$, where f and f' belong to $G_{k[[t]]} = \mathrm{Hom}(\mathrm{Spec}\, k[[t]], G)$. Let $g = f(0)$ and $g' = f'(0)$. Then,

$$\lim_{t \to 0} \lambda(t) = \lim_{t \to 0} f\xi(t)f' = gxg' \in \mathcal{O}_x \ .$$

The 1-PS λ has its values in a maximal torus T'. As all maximal tori of G are conjugated, there exists $g_0 \in G$ such that $g_0\lambda(t)g_0^{-1} \in T$ for all $t \in k^*$. Then $\lambda_0(t) = g_0\lambda(t)g_0^{-1} \in \Xi_*(T)$ has a limit in $\mathcal{O}_x$.

Next, we observe that if $\lim_{t \to 0} \lambda_0(t) = e$, then

$$e^2 = \lim_{t \to 0} \lambda(t) \lim_{t \to 0} \lambda(t) = \lim_{t \to 0} \lambda(t)^2 = \lim_{t \to 0} \lambda(t^2) = e \ ,$$

so e is the idempotent we are looking for. □

Lemma 4. *Let S be a reductive monoid with unit group G. Let $e \in E(S)$, and $Y = \mathcal{O}_e = GeG$. If Y is a monoid with identity e, then e is a central element of S, and Y is an algebraic group. The isotropy group of e satisfies:*

$$(G \times G)_e = (G_e \times G_e)\Delta(G) \ ,$$

where G_e is the isotropy group of e for the action of G by left multiplication. Moreover, G_e is a normal subgroup of G. In particular, $(G \times G)_e$ is a reductive group.

Proof. As e is the identity of Y it follows that $ge = e(ge) = (eg)e = eg$ for all $g \in G$. Then $(g,g)\cdot e = geg^{-1} = gg^{-1}e = e$, and $\Delta(G) \subset (G\times G)_e$. Moreover, $G \subset S$ is dense in S, so $se = es$ for all $s \in S$. Thus, e is a central element of S.

In particular $Y = GeG = Ge$, and $\varphi_0 : G/G_e \to Y$ is a bijective morphism. Observe that the multiplication in Y has an inverse, namely $(ge)^{-1} = g^{-1}e$.

If $g_1 \in G_e$ and $g \in G$, then

$$(gg_1g^{-1})\cdot e = gg_1g^{-1}e = gg_1eg^{-1} = geg^{-1} = gg^{-1}e = e\ .$$

It follows that G_e is normal in G. Thus, φ_0 is compatible with products.

On the other hand, it is clear that $(G_e \times G_e)\Delta(G) \subset (G\times G)_e$. If $(a,b) \in (G\times G)_e$, then

$$e = (a,b)\cdot e = aeb^{-1} = ab^{-1}e = (ab^{-1},1)\cdot e\ .$$

It follows that $ab^{-1} \in G_e$, and $(a,b) = (ab^{-1},1)(b,b) \in (G_e\times G_e)\Delta(G)$. □

Remark. The multiplication in Y may not be induced by that one of $G\times G$ through $G\times G \to (G\times G)/(G\times G)_e \to Y$. Indeed, $(G\times G)_e$ is not always normal in $G\times G$.

The following result of Waterhouse shows some properties of unit groups of affine algebraic monoids.

Proposition 3. ([16],[11, thm. 3.3.6]) *Let G be a connected algebraic group (not necessarily reductive). Then the following properties are equivalent:*

i) The character group of G is non trivial.

ii) The rank of the radical of G is not equal to zero.

iii) There exists an affine algebraic monoid which unit group is G and such that $G \neq S$. □

If moreover G is reductive, we show that the *connected center* of S (i.e. the closure in S of Z, the connected center of G) meets the kernel of S :

Proposition 4. *Let S be a reductive monoid with unit group $G \neq S$; let Z be the connected center of G, and Y be the kernel of S. Then $\overline{Z}\cap Y \neq \emptyset$. In particular, the connected center of G is non trivial.*

Proof. Let T be a maximal torus of G. By Lemma 3 there exists an 1-PS $\lambda \in \Xi_*(T)$ whose limit exists and is an idempotent $e \in Y \cap \overline{T}$.

If $W = N_G(T)/C_G(T)$ is the Weyl group of G, then W acts on T by conjugation. We obtain then an action of W over $\Xi_*(T)$ as follows :

$$w\cdot(\mu(t)) = w\mu(t)w^{-1} \in \Xi_*(T) \quad \forall \mu \in \Xi_*(T)\,,\ w \in W\ .$$

From $W\cdot T \subset T$ we obtain $W\cdot\overline{T} \subset \overline{T}$. Let $\mu = \sum_{w\in W} w\cdot\lambda \in \Xi_*(T)$; it is a W-invariant 1-PS, with limit $e = \prod_{w\in W} wew^{-1} \in E(Y)$. In this way, we have constructed a 1-PS μ of the center of G (because it is W-invariant) whose limit e belongs to Y. In particular, μ is non trivial, and $e \in Y \cap \overline{Z}$: the proposition is proved. □

It is known that every affine embedding of a semi-simple group is trivial ([10]). This result follows immediately from the proposition above:

Corollary 2. *If the unit group of a monoid S is a semi-simple group G_0, then $G_0 = S$.* □

Theorem 2. *Let S be a reductive monoid. Then S is affine.*

Proof. First, we consider the case where S is normal. Consider the kernel Y of S, $G = G(S)$, $T \subset G$ a maximal torus and $Z \subset G$ the connected center. Let $e \in \overline{Z} \cap Y$ as in Proposition 4. Then e is a central element of S, so $es = se$ for all $s \in S$. An element $y \in Y$ can be written as a product $y = geg' = gg'e = egg'$. In particular, Y is an algebraic monoid with identity $e \in Y$.

Consider the morphism $\psi : S \to Y$, $s \mapsto se$. We have:

$$\begin{cases} \psi((g,g')\cdot s) = \psi(gs{g'}^{-1}) = gs{g'}^{-1}e = gse{g'}^{-1} = (g,g')\cdot(se) = (g,g')\cdot\psi(s) \\ \\ \psi(ss') = ss'e = ss'ee = ses'e = \psi(s)\psi(s') \end{cases}$$

It follows that ψ is an $(G \times G)$-equivariant morphism of monoids. The fibre $F_e = \{s \in S \mid se = e\}$ is then stable under multiplication and under the action of $(G \times G)_e$.

The identity 1 belongs to F_e; hence the fibre F_e is a monoid with zero (equal to e). If $G_e = \{g \in G \ : \ ge = e\}$, from Lemma 4 we deduce that $(G \times G)_e = (G_e \times G_e)\Delta(G)$. Moreover, $F_e \cap G = G_e$, and it follows that G_e is an open subset of F_e. Let F_0 be an irreducible component of F_e. As $\dim GF_0 = \dim S$, we have $F_0 \cap G \neq 0$ and G_e is dense in F_e. As G_e is a reductive group, it follows form lemma 2 that F_e is affine.

We consider the variety $F = (G \times G) \times F_e$. The reductive group $(G \times G)_e$ acts in F:

$$(a,b)\cdot((g_1,g_2),s) = ((g_1a^{-1}, g_2b^{-1}), asb^{-1}) \ ,$$

where $(a,b) \in (G \times G)_e$, $g_1, g_2 \in G$, $s \in F_e$. The quotient $(G \times G) *_{(G\times G)_e} F_e = F /\!/ (G \times G)_e$ is then a affine variety. We consider the $(G \times G)$-morphism

$$\gamma : F \to S \quad , \quad \gamma((a,b),s) = (a,b)\cdot s = asb^{-1} \ .$$

As γ is constant over the $(G \times G)_e$-orbits of F, then γ induces a morphism $\widetilde{\gamma} : (G \times G) *_{(G\times G)_e^0} F_e \to S$. The group $G \times G$ acts on $(G \times G) *_{(G\times G)_e} F_e$ by left multiplication in the left factor, in such a way that $\widetilde{\gamma}$ is $(G \times G)$-equivariant.

An easy calculation shows that $\widetilde{\gamma}$ is bijective. We affirm that it is birational, then it follows from Zariski's Main Theorem that $\widetilde{\gamma}$ is isomorphism. Finally, S is isomorphic to an affine variety, so S is affine *a fortiori*.

$\widetilde{\gamma}$ *is birational*

As $\dim F = \dim G$ and $\widetilde{\gamma}$ is injective, we have $G \cap \widetilde{\gamma}(F) \neq \emptyset$. But $\widetilde{\gamma}$ is $(G\times G)$-equivariant, so $G \subset \widetilde{\gamma}(F)$. It follows that there exists $x \in F$ such that $\widetilde{\gamma}(x) = 1$, and that $\widetilde{\gamma}|_{(G\times G)\cdot x} : \mathcal{O}_x \to G$ is an isomorphism with inverse $g \mapsto g \cdot x$; that is, $\widetilde{\gamma}$ is birational.

For the general case, observe that if S reductive, non-normal monoid, its normalization $\widetilde{S}$ is a reductive normal monoid by Lemma 1. It follows that $\widetilde{S}$, and therefore S, is affine. Indeed, the normalization $\pi : \widetilde{S} \to S$ is an finite surjective morphism.

□

Corollary 3. *The kernel of a normal reductive monoid is a reductive group.*

Proof. We have seen that if S is a normal reductive monoid and Y its kernel, then Y is a group, with identity $e \in \overline{Z} \cap Y$. Moreover, if we keep the notation of the proof of theorem 2, then $S \cong (G\times G) *_{(G\times G)_e} F_e$. It follows that

$$Y \cong (G\times G)\cdot \overline{((1,1),e)} \cong (G\times G)/(G\times G)_e\ .$$

But $(G\times G)/(G\times G)_e \cong G/G_e$, and Y is a reductive group. □

Remarks. i) It follows from Porposition 2 that if S is a (reducible) reductive normal monoid, then $\overline{G(S)}$ is affine.

ii) It is known that any irreducible affine monoid can be embedded as a closed submonoid of the monoid of endomorphisms of some vector space V of finite dimension ([10]). It follows from Theorem 2 that any reductive monoid S can be seen as the closure in $\mathrm{End}(V)$ of a closed reductive subgroup of $\mathrm{Gl}(V)$, which is then isomorphic to $G(S)$.

We can sumarize our results in the following

Theorem 3. *The reductive monoids are exactly the affine embeddings of reductive groups. The commutative reductive monoids are exactly the affine embeddings of tori.*

Proof. We have only to prove the last statement. It is clear that if S is a affine embedding of a torus then S is commutative. Conversely, if S is a reductive commutative monoid, then its unit group is a reductive commutative connected group, i.e. a torus. □

4. Classification of reductive monoids

In [12], Renner classified all (affine) reductive normal monoids under some restrictions on the unit group. This classification was extended by Vinberg in [14], where he classified all reductive normal monoids in characteristic zero in terms of the decomposition of their algebra of regular functions for the action of the unit group. This classification is dual of that of affine embeddings of a reductive group (see [15]). From the work of Knop ([6]), we know how to classify affine embeddings of reductive groups in arbitrary characteristic. In order to do that, we begin by recalling some results from the theory of spherical embeddings and in particular *reductive embeddings* (i.e. embeddings of reductive groups), which proofs can be found in [6]. Next, we classify (normal) reductive monoids. In view of Lemma 1, we suppose that all varieties considered are normal.

4.1. **Reductive embeddings.**

Notation. Let X be an algebraic variety. We denote by $k(X)$ the field of rational functions over X .

Definition 3. Let G be a reductive group, G/H a spherical homogeneous space, B a Borel subgroup such that BH is open in G , and X a simple spherical embedding of G/H , with unique closed orbit Y . We denote $\mathcal{D}=\mathcal{D}(G/H)$ the *set of colors,* i.e. the set of B-stable irreducible divisors of G/H . We call the *(set of) colors* of X , and we denote by $\mathcal{F}(X)$, the subset of $\mathcal{D}$ consisting of the irreducible divisors such that their closure contains Y .

We denote by $\mathcal{B}(X)$ the set of irreducible G-stable divisors of X .

Let $k(G/H)^{(B)}$ be the set of eigenvectors for the action of B , and $\Lambda_{G/H}$ be the set of weights of $k(G/H)^{(B)}$, we define

$$\mathcal{Q}(G/H)=\operatorname{Hom}_{\mathbb{Z}}(\Lambda_{G/H},\mathbb{Q})\cong \operatorname{Hom}_{\mathbb{Z}}(\Lambda_{G/H},\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Q} .$$

We call the *rank* of G/H the dimension of the $\mathbb{Q}$-vector space $\mathcal{Q}(G/H)$.

Let X be an algebraic variety. A *valuation* of the field $k(X)$ is a function $\nu : k(X)\to \mathbb{Q}\cup\{-\infty\}$ with the following properties:

i) $\nu(f_1+f_2)\geq \min\{\nu(f_1),\nu(f_2)\}$ for all $f_1,f_2\in k(X)$.

ii) $\nu(f_1f_2)=\nu(f_1)+\nu(f_2)$ for all $f_1,f_2\in k(X)$

iii) $\nu(k^*)=0$, $\nu(0)=+\infty$.

It is easy to prove that $\nu(k(X)\setminus\{0\})\cong\mathbb{Z}$. We say that the valuation ν is *normalized* if $\nu(k(X)\setminus\{0\})=\mathbb{Z}$. Two valuations ν and μ are *equivalent* if $\nu^{-1}(\mathbb{Q}^+\setminus\{0\})=\mu^{-1}(\mathbb{Q}^+\setminus\{0\})$.

If X is a G-variety, a valuation ν of $k(X)$ is called G-invariant if $\nu(g\cdot f)=\nu(f)$ for all $g\in G$, $f\in k(X)$, where the G-action over $k(X)$ is given by $(g\cdot f)(x)=f(g^{-1}x)$ for all $x\in X$.

Let G/H be an spherical homogeneous space, ν a valuation of $k(G/H)$ and $\chi\in\Lambda_{G/H}$ a weight. If $f_\chi, f'_\chi\in k(G/H)^{(B)}$ are two rational functions of weight χ , then the function f_χ/f'_χ is constant over the open B-orbit , and thus it is constant over G/H. It follows that $\nu(f_\chi) = \nu(f'_\chi)$. We set $\rho_\nu(\chi) = \nu(f_\chi)$. If f_χ and $f_{\chi'}$ are two rational functions of weight χ and χ' respectively, then $f_\chi f_{\chi'}$ is a rational function of weight $\chi+\chi'$, and $\rho_\nu(\chi+\chi') = \rho_\nu(\chi)+\rho_\nu(\chi')$. Thus, every valuation ν of $k(G/H)$ defines an element $\rho_v \in \mathcal{Q}(G/H)$.

If $\mathcal{V} = \mathcal{V}(G/H)$ is the set of G-invariant valuations of $k(G/H)$, the restriction of ρ to $\mathcal{V}$ gives an injection $\rho : \mathcal{V} \hookrightarrow \mathcal{Q}(G/H)$ ([6]). We will identify $\mathcal{V}$ with its image in $\mathcal{Q}(G/H)$.

The set $\mathcal{V}$ is a polyhedral cone (i.e. a cone generated by a finite number of elements of the lattice $\mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$), we call it the *valuation cone.*

Let $D \in \mathcal{B}(X)$ be a G-stable irreducible divisor of X, and ν_D the (normalized) G-invariant valuation associated to D. As D is determined by ν_D, the map

$$\rho : \mathcal{B}(X) \to \mathcal{V}(G/H) \quad , \qquad D \mapsto \rho_{\nu_D}$$

is injective. We will identify $\mathcal{B}(X)$ with its image by ρ.

On the other hand, there is a natural mapping $\mathcal{D}(G/H) \to \mathcal{Q}(G/H)$: to each color D we associate ρ_{ν_D}, where ν_D is the valuation of $k(G/H)$ associated to D. This mapping, also denoted by ρ, is not necessarily injective ([6]).

Definition 4. Let G/H be a spherical homogeneous space. A *colored cone* of $\mathcal{Q}(G/H)$ is a pair $(\mathcal{C}, \mathcal{F})$, where $\mathcal{F} \subset \mathcal{D}(G/H)$ is a subset such that:

i) $\rho(\mathcal{F})$ does not contain 0,

ii) $\mathcal{C} \subset \mathcal{Q}(G/H)$ is a polyhedral cone generated by $\rho(\mathcal{F})$, and a finite number of elements of $\mathcal{V}$,

iii) $\mathrm{int}(\mathcal{C}) \cap \mathcal{V} \neq \emptyset$, where $\mathrm{int}(\mathcal{C})$ denotes the relative interior of $\mathcal{C}$.

A colored cone is *strictly convex* if $\mathcal{C}$ is.

A *colored face* of the colored cone $(\mathcal{C}, \mathcal{F})$ is a colored cone $(\tau, \mathcal{F}')$, with τ a face of $\mathcal{C}$, and $\mathcal{F}' = \mathcal{F} \cap \rho^{-1}(\tau)$.

In the following Proposition we resume some well known properties of the classification of spherical varieties in terms of colored cones.

Proposition 5. (see [6]) i) *There exists a bijection between the simple embeddings of G/H and the strictly convex colored cones of $\mathcal{Q}(G/H)$. This bijection associates to each variety X the colored cone $(\mathcal{C}(X), \mathcal{F}(X))$, where $\mathcal{C}(X)$ is the cone generated by $\mathcal{B}(X) \cup \rho(\mathcal{F}(X))$. We call $(\mathcal{C}(X), \mathcal{F}(X))$ the* colored cone *associated to X. We denote $\mathcal{V}(X) = \mathcal{C}(X) \cap \mathcal{V}(G/H)$.*

ii) *The G-orbits of a simple spherical variety X are in bijection with the* colored faces *of its associated colored cone. This bijection is constructed as follows:*

Let $Y \subset X$ be a G-orbit. We denote by $\mathcal{C}_Y(X)$ the cone generated by the image by ρ of the B-stable irreducible divisors which contain Y. If $\mathcal{F}_Y(X)$ is the set of colors such that their closure contains Y, then $(\mathcal{C}_Y(X), \mathcal{F}_Y(X))$ is the colored face we are looking for.

iii) *If* $P\subset G$ *is the parabolic subgroup given by the intersection of the stabilizers of the colors in* $\mathcal{D}\setminus\mathcal{F}(Y)$ *, then*

$$\dim Y=\operatorname{rk}(G/H)-\dim\mathcal{C}_Y(X)+\dim G/P\ .$$

In particular, the unique closed orbit is a point if and only if the dimension of $\mathcal{C}(X)$ *is equal to the rank of* G/H *, and the set of colors of the embedding is all* $\mathcal{D}$ *.*

iv) *A spherical variety* X *is affine if and only if* X *is simple and there exists* $\chi\in\Lambda_{G/H}$ *such that*

(a) $\chi|_{\mathcal{V}(G/H)}\leq 0$
(b) $\chi|_{\mathcal{C}(X)}=0$
(c) $\chi|_{\rho(\mathcal{D}(G/H)\setminus\mathcal{F}(X))}>0$.

v) Morphisms. *Let* G/H *and* G/H' *be two spherical homogeneous spaces and* $\varphi: G/H\to G/H'$ *a dominant* G*-equivariant morphism. Then* φ *induces an injection* $\varphi^*:\Lambda_{G/H'}\to\Lambda_{G/H}$ *; and thus a morphism* $\varphi_*:\mathcal{Q}(G/H)\to\mathcal{Q}(G/H')$ *. The valuation cone* $\mathcal{V}(G/H)$ *goes under this morphism* φ_* *over* $\mathcal{V}(G/H')$ *. Moreover, if* $\mathcal{F}_\varphi$ *denotes the set of colors* $D\in\mathcal{D}(G/H)$ *such that* $\overline{\varphi(D)}=G/H$ *, then* $\varphi_*(\mathcal{D}(D/H)\setminus\mathcal{F}_\varphi)\subset\mathcal{D}(G/H')$ *.*

Furthermore, let X *and* X' *be spherical varieties of open orbit* G/H *and* G/H' *respectively. Then* φ *extends to a morphism* $\varphi:X\to X'$ *, if and only if* $\varphi_*(\mathcal{C}(X))\subset\mathcal{C}(X')$ *, and* $\varphi_*(\rho_{G/H}(\mathcal{F}(X))\subset\rho_{G/H'}(\mathcal{F}(X'))$ *. Remark that such a morphism is necessarily* G*-equivariant.* □

In [15], Vust established the combinatorial data associated to symmetric spaces, in particular to the homogeneous space $G\cong(G\times G)/\Delta(G)$, where G is a reductive connected group, and $\operatorname{char}k=0$. In order to carry out the classification of reductive monoids we must generalize his results to arbitrary characteristic.

Let us fix the notation:

We denote G_0 the commutator of G . Thus, $G=ZG_0$, where $Z=Z(G)$ is the connected center of G . Moreover, $Z_0=G_0\cap Z$ is finite; G is then isomorphic to $(G_0\times Z)/Z_0$:

$$k[G]\cong k[(G_0\times Z)/Z_0]\cong(k[G_0]\otimes_k k[Z])^{Z_0}\ .$$

We fix T_0 , a maximal torus of G_0 , and $B_0\subset G_0$, a Borel subgroup that contains T_0 . Let $\alpha_1,\dots,\alpha_l$ and $\omega_1,\dots,\omega_l$ be the simple roots and fundamental weights associated to (B_0,T_0) respectively. We set $T=T_0Z$, a maximal torus of G . We denote W the Weyl group associated to (G,T) , and $C=C(G)$ the Weyl chamber of G associated to B .

If $B=B_0Z$, the weight lattice of G associated to (B,T) , denoted by $\Xi(T)$, satisfies:

$$\Xi(T)\cong(\Xi(T_0)\times\Xi(Z))^{Z_0}=\{(\lambda,\mu)\in\Xi(T_0)\times\Xi(Z)\ :\ \mu|_{Z_0}=\lambda|_{Z_0}\}\ .$$

It is a sub-lattice of finite index of $\Xi(T_0)\times\Xi(Z)$.

The reflection associated to a simple root α is denoted as usual s_α . We denote $\Xi_+(T)$ the set of dominants weights of G for W .

Remark. We identify $\Xi_*(T)$ with $\Xi(T)$ by means of a W-invariant form $\langle\cdot,\cdot\rangle$, in such a way that $\langle\omega_i,\alpha_j^\vee\rangle=\delta_{ij}$.

From the Bruhat decomposition, it is easy to prove the following

Lemma 5. *Let G be a reductive group. If we keep the preceding notations and identify* $\mathrm{Hom}_{\mathbb{Z}}(\Xi(T),\mathbb{Z})$ *with* $\Xi_*(T)\cong\Xi(T)$ *, then*

$$\begin{aligned}\mathcal{Q}((G\times G)/\Delta(G))= \quad & \mathcal{Q}(G)\cong\Xi_*(T)\otimes\mathbb{Q}\cong\Xi(T)\otimes\mathbb{Q}\cong\\ & (\Xi(T_0)\otimes\mathbb{Q})\oplus(\Xi(Z)\otimes\mathbb{Q})=\mathcal{Q}(G_0)\oplus\mathcal{Q}(Z)\ ,\end{aligned}$$

where the lattice considered is $(\Xi(T_0)\times\Xi(Z))^{Z_0}\subset\Xi(T_0)\times\Xi(Z)$.

Moreover, the colors of G are the divisors $D_i=\overline{Bs_{\alpha_i}B^-}$, $i=1,\ldots,l$.

Proof. The set Λ_G of weights of $k(G)^{(B\times B^-)}$ is isomorphic to $\Xi(T)$. It follows $\mathrm{Hom}_{\mathbb{Z}}(\Lambda_G,\mathbb{Z})\cong\Xi_*(T)$, and

$$\mathcal{Q}(G)=\mathrm{Hom}_{\mathbb{Z}}(\Xi(T),\mathbb{Q})\cong\Xi(T)\otimes\mathbb{Q}=(\Xi(T_0)\times\Xi(Z))^{Z_0}\otimes\mathbb{Q}\,.$$

In order to calculate $\mathcal{D}(G)$, we deduce form the Bruhat decomposition that the $(B\times B^-)$-orbits of codimension one in G are $Bs_{\alpha_i}B^-$, $i=1,\ldots,l$. □

Suppose that G is a reductive group such that the algebra $k[G]$ is factorial. Then if $f\in k[G]\setminus\{0\}$ is irreducible, we can consider the valuation of $k(G)$ that associates to one rational function $\varphi/\psi\in k(G)$, $\varphi,\psi\in k[G]$, the number of times that f appears as a factor in the decomposition of φ/ψ . the center of this valuation is the set of zeros of f . This fact allows us to simplify the calculation of the colored data associated to an arbitrary reductive group, by means of the following

Proposition 6. ([13],[14]) *Let G be a reductive group and $\Gamma\subset G$ a finite central subgroup. Then*

i) *$\mathcal{Q}(G/\Gamma)=\mathcal{Q}(G)$, where we consider the lattice $\Xi_*(T/\Gamma)=\Xi_*(T)^\Gamma\subset\Xi_*(T)$.*
ii) *If $\pi:G\to G/\Gamma$ is the canonical projection, then π induces a bijection between $\mathcal{D}(G)$ and $\mathcal{D}(G/\Gamma)$.*
iii) *Under the identification i), $\rho_G(\mathcal{D}(G))=\rho_{G/\Gamma}(\mathcal{D}(G/\Gamma))$, and the valuation cones $\mathcal{V}(G)$ and $\mathcal{V}(G/\Gamma)$ coincide. In particular, π_* is the identity.*
iv) *If S is a monoid with unit group G , then S/Γ is a reductive monoid, with unit group G/Γ . Moreover, if we denote $\pi:S\to S/\Gamma$ the quotient, then $\pi_*:\mathcal{Q}(G)\to\mathcal{Q}(G')$ is the identity, and under this identification we have $\mathcal{C}(S/\Gamma)=\mathcal{C}(S)$.* □

We study now the relationship between the one parameter subgroups of G and the $(G\times G)$-invariant valuations of $k(G)$. Next we use this information in order to find the cone of valuations associated to G.

The following result of Knop allows us to work in arbitrary characteristic:

Lemma 6. [6, Cor. 2.7] *Let G be a reductive group, and X a spherical G-variety, of open orbit G/H. Let $x_0 \in X$ be such that $B\cdot x_0$ is open in X. Denote by p the characteristic exponent of k: $p = \operatorname{char}(k)$ if $\operatorname{char}(k) > 0$, $p = 1$ if $\operatorname{char}(k) = 0$. Then for all $f \in k[B\cdot x_0]$ and $\nu_0 \in \mathcal{V}(G/H)$ there exists $n = p^N$ and $f' \in k(G/H)^{(B)}$ such that:*

$$\begin{array}{rcll} \nu_0(f') & = & \nu_0(f^n) & \\ \nu(f') & \geq & \nu(f^n) & \forall \nu \in \mathcal{V}(G/H) \\ \nu_D(f') & \geq & \nu_D(f^n) & \forall D \in \mathcal{D}(G/H)\ . \end{array}$$

□

Definition 5. Let G/H be a spherical homogeneous space. A G/H-embedding is called *elementary* if it contains two G-orbits, one of them of codimension one.

Let X be an elementary embedding. Then it follows from the definition that its associated colored cone is of the form $(\mathbb{Q}^+\nu,\emptyset)$, where $\nu \in \mathcal{V}(G/H)$.

Proposition 7 ([7])**.** *Let G/H be a spherical homogeneous space. Then the (normalised) G-invariant valuations are in bijection with the elementary embeddings of G/H.* □

Let G be a reductive group and $\lambda \in \Xi_*(T)$ an 1-PS. Then λ induces a valuation of $k(G)$ as follows:

We consider the k^* action over $k[G]$ induced by that one of $G\times G$: $t\cdot f = \lambda(t)\cdot f$. We have then a decomposition:

$$k[G] = \bigoplus_{n\in\mathbb{Z}} k[G]_n \quad , \quad k[G]_n = \{f \in k[G] \ :\ t\cdot f = t^n f \quad \forall t \in k^*\}\ .$$

If $f = \sum_{n\in\mathbb{Z}} f_n$, $f_n \in k[G]_n$, then $t\cdot f = \sum_{n\in\mathbb{Z}} t^n f_n$. We define

$$\nu_\lambda(f) = \inf\{n \ :\ f_n \neq 0\}\ .$$

An easy calculation shows that ν_λ induces a valuation of $k(G)$.

Remark that if $f \in k[G]^{(B\times B^-)}$, then $\nu_\lambda(f) = \langle \lambda, \chi_f\rangle$, because $\lambda(t)\cdot f = \chi_f(\lambda(t))f$, and $\chi_f(\lambda(t)) = t^{\langle\lambda,\chi_f\rangle}$.

Proposition 8. *Let G be a reductive group, and ν a $(G\times G)$-invariant valuation of $k(G)$. Let X be the elementary embedding associated to ν, and let $Y \subset X$ be the unique closed orbit of X. Consider $\lambda \in \Xi_*(T)$ such that $\lim_{t\to 0}\lambda(t)$ exists in X and belongs to the open $(B\times B^-)$-orbit of Y. Then ν_λ is equivalent to ν. Recall that B^- is the opposite Borel subgroup of B.*

Proof. Consider $x = \lim_{t\to 0}\lambda(t) \in Y$. It is known ([6]) that

$$((B \times B^-)\cdot 1) \cup ((B \times B^-)\cdot x) = BB^- \cup ((B \times B^-)\cdot x) = X_0$$

is an affine variety. As BB^- is open in X , $k(X_0) = k(BB^-) = k(G) = k(X)$. It follows that ν is a valuation of $k(X_0)$, the field of fractions of $k[X_0]$.

In order to prove that the valuation ν_λ assocated to λ is equivalent to ν , we must verify the next two conditions:

i) $k[X_0] \subset \mathcal{O}_{\nu_\lambda} = \{f \in k(X_0) \mid \nu_\lambda(f) \geq 0\}$.
ii) $k[X_0] \cap \mathcal{M}_{\nu_\lambda} = \{f \in k(X_0) \mid \nu_\lambda(f) > 0\}$ is the ideal in $k[X_0]$ associated to the divisor $\overline{(B \times B^-)\cdot x}$.

By Lemma 6 it suffices to show that for all $f \in k[X_0]^{(B\times B^-)} \setminus \{0\}$, $\nu_\lambda(f) \geq 0$, and that $\nu(f) > 0$ if and only if $\nu_\lambda(f) > 0$.

If a function $f \in k[X_0]^{(B\times B^-)}$ is such that $f(1) = 0$, we deduce from equivariance that it is constant and equal to 0 over all X_0 . It follows that a function $f \in k[X_0]^{(B\times B^-)} \setminus \{0\}$ verifies $f(1) \neq 0$. As $(\lambda(t)\cdot f) = \chi_f(\lambda(t))f$, we have $\chi_f(\lambda(t)) = t^{\nu_\lambda(f)}$. Then

$$\lim_{t\to 0} t^{\nu_\lambda(f)} f(1) = \lim_{t\to 0}(\lambda(t)\cdot f)(1) = \lim_{t\to 0} f(\lambda(t)) = f(x) \ ,$$

From this we deduce that $\nu_\lambda(f) \geq 0$, and $\nu_\lambda(f) > 0$ only if $f(x) = 0$, that is $\nu(f) > 0$. □

Proposition 9. *Let G be a reductive group. Under the preceding identification, $\mathcal{Q}(G) = \Xi(T)\otimes\mathbb{Q}$, $\mathcal{Q}(G)^* = \Xi(T)\otimes\mathbb{Q}$, and the valuation cone $\mathcal{V}(G) \subset \mathcal{Q}(G)$ is opposite to $C(G)$. Moreover, the valuation associated to the color $\overline{Bs_\alpha B^-}$, α a simple root, identifies to $(\alpha^\vee, 0) \in (\Xi(T_0)\times\Xi(Z))^{Z_0}$.*

Proof. From Proposition 6, we can suppose that G is the direct product of a semi-simple group by a torus: $G = G_0 \times Z$. Moreover, we can suppose that G_0 is simply connected.

Let $\alpha_1, \dots, \alpha_l$ be the simple roots of G . Then the colors are $\overline{Bs_{\alpha_i}B^-}$, $i = 1, \dots, l$.

We remark that the algebra of regular functions $k[G]$ is factorial. Consider the fundamental weight ω_i and let V_{ω_i} be a simple G-module of highest weight ω_i . Let $v_i \in V_{\omega_i}$ be an eigenvector of B , of weight ω_i , and $v_i^* \in V_{\omega_i}^*$ its dual. Then $g \mapsto \langle g\cdot v_i^*, v_i\rangle$ is a regular function of G , wich will be denoted f_i . This function f_i is an eigenvector of $B \times B^-$, of weight $(\omega_i, -\omega_i)$. We affirm that f_i is an equation for the color $D_i = \overline{Bs_{\alpha_i}B^-}$.

Indeed, if $bs_{\alpha_j}b' \in Bs_{\alpha_j}B^-$, $b = tu$, $b' = t'u'$, $t, t' \in T$, $u \in U$, $u' \in U^-$, then

$$f_i(bs_{\alpha_j}b') = ((tu, t'u')\cdot f_i)(s_{\alpha_j}) = \omega_i(tt'^{-1})f_i(s_{\alpha_j}) \ .$$

The vector $s_{\alpha_j}\cdot v_i$ is an eigenvector of T , of weight $s_{\alpha_j}\omega_i$. As $s_{\alpha_j}\omega_i = \omega_i$ if $i \neq j$, and that $s_{\alpha_i}\omega_i \neq \omega_i$, we deduce that $f_i(s_{\alpha_j}) = \langle s_{\alpha_j}\cdot v_i^*, v_i\rangle = 0$ if and only if $i = j$. From the Bruhat decomposition we get that f_i is an equation for D_i . The algebra $k[G]$ is factorial, so if $\chi = \sum n_i(\omega_i, -\omega_i)$, then $f_\chi = \prod f_i^{n_i}$, and it follows that the valuation associated to the color $D_i = \overline{Bs_{\alpha_i}B^-}$ is $(\alpha_i^\vee, 0)$.

Next, we must calculate the cone of valuations of G . First of all we reduce again the problem, by observing that

$$\begin{aligned} \mathcal{V} = \mathcal{V}(G) &= \mathcal{V}(G_0) \times \mathcal{V}(Z) = \mathcal{V}(G_0) \times \mathcal{Q}(Z) \ , \\ C = C(G) &= C(G_0) \times \mathcal{Q}(Z) \ . \end{aligned}$$

We can then suppose that G is a semi-simple simply connected group.

Let $\nu \in \mathcal{V}$ be a normalised valuation and X be the elementary embedding of G associated to ν . Consider $\lambda \in \Xi_*(T)$, a 1-PS such that the limit $\lim_{t\to 0} \lambda(t)$ exists and belongs to $X \setminus G$. With a convenient choice of $w \in W$, we have $w\lambda(t)w^{-1} \in -C$, so we can suppose $\lambda \in -C$. We affirm that the orbit $(B\times B^-)\cdot x$, $x = \lim_{t\to 0}\lambda(t)$, is open in $X \setminus G$. From Proposition 8, this implies that ν_λ is equivalent to ν .

We follow [2, Appendix] :

Let us find the isotropy group $(G\times G)_x$. Consider the subset $G(\lambda) \subset G$ of the $g \in G$ such that the limit $\lim_{t\to 0}\lambda(t)g\lambda(t)^{-1}$ exists in G . Then $G(\lambda)$ is a parabolic subgroup of G , whose unipotent radical is:

$$R_uG(\lambda) = \{g \in G \ : \ \lim_{t\to 0} \lambda(t)g\lambda(t)^{-1} = 1\} \ .$$

Moreover, the Levi subgroup of $G(\lambda)$ is the centralizer $L(\lambda)$ of the image of λ . It is easy to see that $G(\lambda) \supset B^-$.

If $g \in R_uG(\lambda)$, then:

$$\lambda(t) = (\lambda(t), 1)\cdot((g, g)\cdot 1) = (\lambda(t)g\lambda(t)^{-1}, g)(\lambda(t), 1)\cdot 1 \ .$$

Taking limits with t going to 0 , we have $x = (1, g)\cdot x$. In an analogous way, we have $(g, 1)\cdot x = x$. It follows that $R_uG(\lambda)\times R_uG(\lambda)^- \subset (G\times G)_x$.

On the other hand, if $g \in L(\lambda)$ then $(g, g)\cdot\lambda(t) = \lambda(t)$, so $(g, g)\cdot x = x$. It follows that

$$(G\times G)_x \supset G'_\lambda \overset{def}{=} (R_uG(\lambda)\times R_uG(\lambda)^-)\Delta(L(\lambda))(\lambda(k^*)\times\{1\}) \ .$$

As $(B\times B^-)G'_\lambda$ is open in $G\times G$, $(B\times B^-)\cdot x$ is open in $\mathcal{O}_x$. Moreover, $(G\times G)_x^0$, the irreducible component of $(G\times G)_x$ passing by the identity, is equal to G'_λ . In an analogous way, we get $(B\times B^-)_x^0 = \Delta(T)(\lambda(k^*)\times\{1\})$.

It remains to prove that if $\lambda \in -C$, then ν_λ is a $(G\times G)$-invariant valuation. Consider a faithfull representation $G \hookrightarrow \mathrm{Gl}(V) \subset \mathrm{End}(V)$. Then $G \to G/Z_0 = G\cdot[\mathrm{Id}] \subset \mathbb{P}(\mathrm{End}(V))$, where $[\mathrm{Id}]$ denotes the equivalence class in $\mathbb{P}(\mathrm{End}(V))$ of $\mathrm{Id} \in \mathrm{End}(V)$. It follows that the limit $\lim_{t\to 0}\lambda(t)$ exists in $X = \overline{G/Z_0}$. Let x be that limit. Consider $\mu \in \mathcal{V}(G)$ belonging to the relative interior of $\mathcal{C}_{\mathcal{O}_x}(X)$, the colored cone associated to the simple embedding $\{y \in X \ : \ \mathcal{O}_x \subset \overline{\mathcal{O}_y}\}$. It follows from Proposition 5 that there exists a morphism $\varphi : X_\mu \to X$ such that

$\varphi(X_\mu) = G/Z_0 \cup \mathcal{O}_x$. As φ is a $(G \times G)$-morphism birational over its image, to calculate the $(G\times G)$-invariant valuations of $k(X_\mu)$ is equivalent to calculate those ones of $k(G \cup \mathcal{O}_x)$. The preceeding argument shows that the limit $\lim_{t\to 0} \lambda(t) = x$ belongs to the open $(B \times B^-)$-orbit of $\mathcal{O}_x$. It follows that ν_λ is the valuation associated to X . As in Proposition 8, this shows that ν_λ is a $(G \times G)$-invariant valuation of $k(G)$. □

Corollary 4. *Let S be a monoid with unit group G . If $\lambda \in \Xi_*(T)$ is a 1-PS of G , then the limit $\lim_{t\to 0} \lambda(t)$ exists in S if and only if $\lambda \in \cup_{w\in W} w \cdot \mathcal{V}(S)$.*

Proof. The cone $-C(G)$ is a fundamental domain for the action of W . On the other hand, the limit $\lim_{t\to 0} \lambda(t)$ exists in S if an only if the limits $\lim_{t\to 0} w\lambda(t)w^{-1}$, $w \in W$, exist in S . We can then suppose that $\lambda \in -C(G) = \mathcal{V}(G)$.

Consider $\lambda \in \mathcal{V}(S) = \mathcal{C}(S) \cap \mathcal{V}(G)$, and consider the elementary embedding X_λ associated to λ . Then there exists a morphism $\varphi : X_\lambda \to S$. We deduce from Proposition 8 that the limit $\lim_{t\to 0} \lambda(t)$ exists in X_λ , so the limit $\lim_{t\to 0} \varphi(\lambda(t)) = \lim_{t\to 0} \lambda(t)$ exists in S .

Suppose that $\lambda \in \mathcal{V}(G)$ is an 1-PS such that the limit $\lim_{t\to 0} \lambda(t) = x$ exists in S . If $\mu \in \mathcal{V}(G)$ belongs to the relative interior of $\mathcal{C}_{\mathcal{O}_x}(S)$, then there exists a morphism $\varphi : X_\mu \to S$, where X_μ is the elementary embedding associated to μ , and $\varphi(X_\mu) = G \cup \mathcal{O}_x$. Then the valuation ν_λ is equivalent to ν_μ , and it follows that λ and μ are proportionals. So $\lambda \in \mathcal{C}_{\mathcal{O}_x}(S) \subset \mathcal{C}(S)$, because $\varphi_*(\mu) = \mu \in \mathcal{C}(S)$. □

Remark. In the case of reductive embeddings, the application $\rho : \mathcal{D} \to \mathcal{Q}(G)$ is injective. We identify then $\mathcal{D}$ with its image by ρ .

From this, we can recover the fact that a reductive monoid is a quasi-direct product of a reductive group and a reductive monoid with zero ([10],[13]):

Proposition 10. *Let S be a reductive monoid with unit group G . Then there exist a reductive group G_1 and a reductive monoid with zero S_0 such that*

$$S \cong (G_1 \times S_0)/\Gamma ,$$

where Γ is a finite central subgroup of G . In particular, if $G_2 = G(S_0)$, then $G \cong (G_1 \times G_2)/\Gamma$. □

An easy caculation shows that, as in the case of toric varieties, we can restrict ourselves to the study of reductive monoids with zero:

Proposition 11. *Let S be a reductive monoid with unit group G . Let G_1, G_2, S_0 and Γ be as above. Then, under the identification $\mathcal{Q}(G) = \mathcal{Q}(G_1) \oplus \mathcal{Q}(G_2)$, we have:*

$$(\mathcal{C}(S), \mathcal{F}(S)) = (\{0\} \times \mathcal{C}(S_0), \{0\} \times \mathcal{D}(G_2)) \ . \tag{3}$$

□

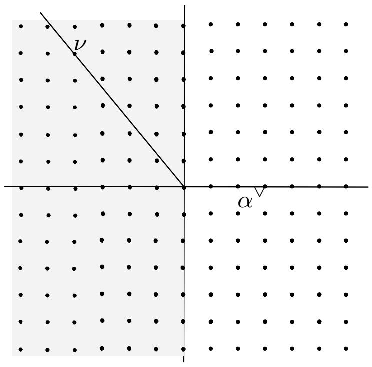
Corollary 5. *Let S be a reductive monoid with unit group G. Let $(\mathcal{C}(S), \mathcal{F}(S))$ be the colored cone associated to S. Then $\mathcal{C}(S) + \mathbb{Q}^+\mathcal{D}$ is a strictly convex cone.*

□

The converse to this corollary is not true, as the following example shows:

Example 2. Consider $G = \mathrm{Sl}(2) \times k^*$. We identify $\mathcal{Q}(G)$ to $\mathbb{Q}^2$, and $\Xi_*(T)$ to the lattice $\mathbb{Z}^2 \subset \mathbb{Q}^2$. The affine embeddings of G are exactly those where the closed orbit is a point (with colored cone of the form $(\mathcal{C}, (\alpha^\vee, 0))$), and the *elementary embeddings* associated to the half lines $(\mathbb{Q}^+(0,1), \emptyset)$ and $(\mathbb{Q}^+(0,-1), \emptyset)$.

Indeed, let $(\mathcal{C}, \emptyset)$ be the colored cone associated to an affine embedding without colors. Then there exists $\chi \in \Lambda$ such that $\chi(\alpha^\vee, 0) > 0$, $\chi(\nu) \leq 0$ for all $\nu \in \mathcal{V}$. It follows that $\chi(\nu) = 0$ if and only if ν belongs to the y-axis. Let $\nu \in \mathcal{V}$ such that the cone $\mathcal{C}'$ generated by ν and $(\alpha^\vee, 0)$ is strictly convex. Then the embedding S associated to $(\mathcal{C}', (\alpha^\vee, 0))$ is affine (just take $\chi \equiv 0$). In particular, S is a reductive monoid (with zero: the set of colors of S is all $\mathcal{D}$).



On the other hand, if we take $\tau = \mathcal{C}' \cap \mathcal{V}$, then $\tau + \mathbb{Q}^+(\alpha^\vee, 0) = \mathcal{C}'$ is strictly convex, but the embedding associated to the colored cone $(\tau, \emptyset)$ is not affine.

An immediate consequence of the preceding proposition is that the colored cone of a reductive monoid S is determined by $\mathcal{V}(S)$, or even $\mathcal{B}(S)$. Moreover, we can find the decomposition of S as a quasi-direct product of a reductive monoid with zero and a reductive group (cf. Proposition 10) as follows:

If we keep the notation of Proposition 10, the space $\mathcal{Q}(G_2)$ is $\langle \mathcal{C}(S) \rangle$, the subspace of $\mathcal{Q}(G)$ generated by $\mathcal{C}(S)$. The lattice considered contains $\Xi(T) \cap \langle \mathcal{C}(S) \rangle$ as a subgroup of finite index.

A simply connected covering of the commutator of the normal subgroup G_1 (resp. G_2) is the semi-simple simply connected group whose root system is given by the Dynkin diagram of G restricted to $\mathcal{F}(S)$ (resp. $\mathcal{D}(G) \setminus \mathcal{F}(S)$).

In terms of the combinatorics of the colored cones of $\mathcal{Q}(G)$, this remark gives the following easy result:

Proposition 12. *Let G be a reductive group. Given a polyhedral cone τ contained in $\mathcal{V}(G) = -C$, such that $\tau + \mathbb{Q}^+\mathcal{D}$ is strictly convex, there exists an unique subset of colors $\mathcal{F} \subset \mathcal{D}$ such that $(\tau + \mathbb{Q}^+\mathcal{F}, \mathcal{F})$ is the colored cone associated to an affine embedding of G.* □

Corollary 6. *Let G be a reductive group. Let S and S' be two monoids with unit group G, such that $\mathbb{Q}^+\mathcal{B}(S) = \mathbb{Q}^+\mathcal{B}(S')$. Then S and S' are isomorphic. In particular, if $\mathcal{V}(S) = \mathcal{V}(S')$, then S and S' are isomorphic.* □

Remark. Let us give a geometric interpretation of the corollary above. If S is a reductive monoid with unit group G and $(\mathcal{C}(F), \mathcal{F}(S))$ its colored cone, there exists a proper G-morphism $\widetilde{S} \to S$, where $\widetilde{S}$ is the embedding associated to $(\mathcal{V}(S), \emptyset)$ ([1]). Moreover, the embedding $\widetilde{S}$ is minimal for this property; it is called the *decoloration* of S. Corollary 6 says then that two monoids are isomorphic if and only if their decolorations are.

In conclusion, we have the following classification:

Theorem 4. *Let G be a reductive group. The isomorphism classes of algebraic monoids with unit group G are in bijection with the strictly convex polyhedral cones of $\mathcal{Q}(G)$ generated by all the colors and a finite set of elements of $\mathcal{V}(G)$.* □

5. Some consequences

We end up with some applications of the classification obtained above.

First, we describe the algebra of regular functions on a reductive monoid, generalising the result obtained by Vinberg in characteristic zero ([14]).

Notation. Let G be a reductive group. We keep the notations of the last section. If M is a G-module, we denote $M^{(B)}$ the set of eigenvectors of M for the action of B.

Proposition 13. *Let S be a monoid with unit group G. Then,*

$$k\left[S \setminus \overline{\cup_{D \notin \mathcal{F}(S)} D}\right]^{(B\times B^-)} = \left\{ f \in k(G)^{(B\times B^-)} \ : \ \chi_f \in \mathcal{C}(S)^\vee \right\} , \quad \textit{and}$$

$$k[S]^{(B\times B^-)} = \left\{ f \in k(G)^{(B\times B^-)} \ : \ \chi_f \in \mathcal{C}(S)^\vee \cap \Xi_+(T) \right\} .$$

Moreover, the set $\mathcal{L}_S = \mathcal{C}(S)^\vee \cap \Xi_+(T)$ of weights of $k[S]^{(B\times B^-)}$ generates $\Xi(T)$, the set of weights of $k(G)^{(B\times B^-)} = k(S)^{(B\times B^-)}$.

In particular, if $\operatorname{char} k = 0$, *the decomposition of $k[S]$ into simple $(G \times G)$-modules is the following:*

$$k[S] = \bigoplus_{\chi \in \mathcal{C}(S)^\vee \cap \Xi_+(G)} V_\chi \otimes V_\chi^* .$$

Proof. The first equality follows from [6, thm. 3.5]. In order to prove the second one, observe that a function $f \in k\left[S \setminus \overline{\cup_{D\in\mathcal{F}(S)} D}\right]^{(B\times B^-)}$ extends to a regular function on S if and only if $\nu_D(f) \geq 0$ for all $D \in \mathcal{F}(S)$. By Lemma 5, this means that $\langle \chi_f, \alpha_i^\vee\rangle = \nu_{D_i}(f) \geq 0$, i.e. χ_f is a dominant weight.

Let $f = \psi/\varphi \in k(S)^{(B\times B^-)}$, where $\varphi, \psi \in k[S]$. The $(G\times G)$-module $V \subset k[S]$ generated by φ is of finite dimension. Let $\{g_1\cdot\varphi, \dots, g_r\cdot\varphi\}$, $g_i \in G\times G$, be a system of generators of V. Then, there exists a $(B\times B^-)$-eigenvector $h = \sum_{i=1}^r t_i g_i \cdot \varphi \in V$, $t_i \in k$. Consider the $(B\times B^-)$-eigenvector $hf = (\sum_{i=0}^r t_i g_i\cdot\varphi)\psi/\varphi \in k[S]$; its weight is $\chi_{hf} = \chi_h + \chi_f$. We have $f = hf/h$, where $hf, h \in k[S]^{(B\times B^-)}$, and $\chi_f = \chi_{fh} - \chi_h$. It follows that $\mathcal{L}_S$ generates $\Xi(T)$.

Finally, the last equality is a direct consequence of the complete reducibility of the rational representations of $G\times G$ in characteristic 0. □

Proposition 14 ([4, §29]). *Let G be a reductive group. If we keep the preceding notations, the stabilizer in $G\times G$ of the color $D_i = \overline{Bs_{\alpha_i}B^-}$, $i = 1, \dots, l$, is generated by the parabolic subgroups $(P_{\alpha_j} \times P^-_{\alpha_j})$, $j \neq i$, where P_α denotes the minimal parabolic associated to the simple root α, and $P^-_{\alpha_j}$ is the oposite parabolic subgroup of P_{α_j}.* □

Corollary 7. *Let S be a reductive monoid with unit group G. If $Y \subset S$ is a $(G\times G)$-orbit, whose combinatorial data is $(\mathcal{C}_Y(S), \mathcal{F}_Y(S))$, its dimension is given by:*

$$\dim Y = l + \dim Z(G) - \dim \mathcal{C}_Y(S) + 2\dim G/P_{\mathcal{D}\setminus\mathcal{F}(Y)}$$

where l is the rank of G_0, and $P_{\mathcal{D}\setminus\mathcal{F}(Y)}$ is the parabolic subgroup generated by the parabolics P_α, $\alpha \in \mathcal{D}\setminus\mathcal{F}(Y)$.

Proof.

Apply Propositions 5 (iii) and 14 in order to calculate the intersection of the parabolic subgroups which stabilize the colors belonging to $\mathcal{D}\setminus\mathcal{F}(Y)$. □

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