GENERALIZED CAYLEY'S Ω-PROCESS

WALTER FERRER SANTOS, ALVARO RITTATORE

Abstract. In this paper we generalize some constructions and results due to Cayley and Hilbert. We define the concept of Ω–process for arbitrary monoids with zero and show how to produce from such a process, operators yielding –in the case of reductive monoids– a quantity of elements of the ring of invariants of the unit group of the monoid, large enough to guarantee its finite generation. Moreover, we give an explicit construction of all Ω–process for general reductive monoids and show, in the case of the monoid of all the n^2 matrices how to obtain the classical Cayley's construction defined in terms of the determinant.

1. Introduction

We assume throughout that the base field $\Bbbk$ is algebraically closed of characteristic zero and that all the geometric and algebraic objetcs are defined over $\Bbbk$. A *linear algebraic monoid* is an affine normal algebraic variety M with an associative product $M \times M \to M$ which is a morphism of algebraic $\Bbbk$–varieties, and equipped with a neutral element $1 \in M$ for this product. In this case, it can be proved that the group of invertible elements of M, $G(M) = \{g \in M \mid \ \exists\, g^{-1}\}$ is an affine algebraic group, that is open in M and usually called the unit group of M. We will concentrate our attention in reductive monoids whose groups of invertibles is given by the set of non-zeroes of a particular character of M. A *reductive monoid* is an irreducible normal linear algebraic monoid whose unit group is reductive.

In [1], S. Doty developed the theory of representations for reductive monoids in the particular case where they have one-dimensional center. He exhibited the relationship existing between the representations of the monoid and those of its unit group. The main technical tool used was the generalization of the concept of induction of representations to this setting, and the extension to the whole monoid of regular functions defined in a convenient subset of the monoid containing the unit group. This is attained by means of an "extension principle", proved by Renner in [8] under the asumption that the center of the monoid is one-dimensional. Later in [9], Renner extended Doty's results for arbitrary reductive monoids.

In this work we generalize the classical definition of Cayley's Ω–process defining it for a given affine monoid with 0 called M, as a semi–equivariant map $\Omega : \Bbbk[M] \to \Bbbk[M]$ –see Section 4–. Then we show how to construct from Ω a family of

Date: July 27, 2005.

The first author would like to thank, Csic-UDELAR and Conicyt-MEC, the second author would like to thank FCE-MEC.

Reynolds operators on the polynomial representations of M and more importantly –following Hilbert's methods in [4]–, we use Ω to prove the finite generation of the invariants of the representations of $G(M)$. Finally, we show how to construct an Ω–process for reductive monoids.

For the basic notations of the theory of affine algebraic groups and in particular for the notion of Reynolds operators, integrals and invariants see [2].

2. Preliminaries

In a similar way than for the case of affine algebraic groups, if M is an affine algebraic monoid, the multiplication of M induces a morphism of algebras $\Delta : \Bbbk[M] \to \Bbbk[M] \otimes \Bbbk[M]$. This map, together with the evaluation at the neutral element $\varepsilon : \Bbbk[M] \to \Bbbk$ induces a bialgebra structure on $\Bbbk[M]$. Moreover, if the monoid M is endowed with a zero element, i.e. an element $0 \in M$ such that $0m = m0 = 0$ for all $m \in M$, then the evaluation at 0 induces an algebra morphism $\nu : \Bbbk[M] \to \Bbbk$ such that $u\nu \star \mathrm{Id} = u\nu = \mathrm{Id} \star u\nu$, where $u : \Bbbk \to \Bbbk[M]$ is the unit of the algebra $\Bbbk[M]$ and $\star$ denotes the convolution product on the endomorphisms of $\Bbbk[M]$: if $T, S \in \mathrm{End}(\Bbbk[M])$, then $T \star S \in \mathrm{End}(\Bbbk[M])$ is defined as $(T \star S)(f) = \sum T(f_1)S(f_2)$ if $\Delta(f) = \sum f_1 \otimes f_2$.

In particular if M and N are affine algebraic monoids, a homomorphism of algebraic monoids $\Phi : M \to N$, is a morphism of algebraic varieties that preserves de product and the unit element of the monoids. Moreover, if the monoids have zero , we assume that $\Phi(0) = 0$. One can easily show in this situation that the associated map $\Phi^* : \Bbbk[N] \to \Bbbk[M]$ is a morphism of bialgebras.

Next, we recall some basic facts about reductive monoids that will be used later.

Let M a *reductive* monoid with unit group G. Recall that in this situation M is an affine algebraic variety and that the action of $G \times G$ on M given by $(a, b) \cdot m = amb^{-1}$ has G as an open orbit, see [10]. Thus, reductive monoids are affine spherical varieties and can be classified in combinatorial terms (see for example [5],[10]).

The following notations will be in force in this article. If G is a reductive group, we call T a maximal torus of G, B a Borel subgroup containing T and B^- its opposite Borel subgroup. We denote as $\mathcal{X}(T)$ the set of weights and $\mathcal{X}_+(T)$ the semigroup of dominant weights with respect to B. We call W the Weyl group associated to T, $C = C(G)$ the Weyl chamber associated to (B, T), $\alpha_1, \dots, \alpha_l$ and $\omega_1, \dots, \omega_l$ the simple roots and fundamental weights associated to (B, T) respectively. Recall that $\mathcal{X}(T)$ is partially ordered by the relation $\lambda \leq \mu$ if and only if $\mu - \lambda \in \mathbb{Z}^+(\alpha_1, \dots, \alpha_l)$.

Finally, we call $\mathcal{X}_*(T)$ the set of one parameter subgroups (1-PS) of T and identify $\mathcal{X}_*(T)$ with $\mathcal{X}(T)$ by means of a W-invariant scalar product $\langle \cdot, \cdot \rangle$, in such a way that $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$, were $\alpha_i^\vee$ is the coroot associated to α_i.

If $\mathcal{Q}(G) = \mathcal{X}_*(T) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathcal{X}(T) \otimes_{\mathbb{Z}} \mathbb{Q}$, then the reductive monoids with unit group G are in one to one correspondence with with the pairs $(\mathcal{C}, \mathcal{F})$ where $\mathcal{F}$ is a subset of the simple roots and $\mathcal{C} \subset \mathcal{Q}(G)$ is a rational poliedral cone generated by $\mathcal{F}$ and a finite number of elements of $-C(G) \subset \mathcal{X}(T)$, such that the cone generated by $\mathcal{C}$ and all the simple roots is strictly convex ([10, §4.2]). This result is a particular case of the classification theory of spherical varieties in terms of colored fans.

3. Comodule structures for polynomial modules

The correct category of representations of an affine algebraic monoid M, is the so called category of polynomial M-modules.

If V is a finite dimensional $\Bbbk$–space, we say that an action $M \times V \to V$ is polynomial if the associated map $M \to \operatorname{End}_{\Bbbk}(V)$ is a morphism of affine algebraic monoids, in particular in the case that M is endowed with $0 \in M$, in accordance with our definitions $0 \cdot v = v$ for all $v \in V$.

The action map $M \times V \to V$ induces a morphism of $\Bbbk$–spaces $\chi_V : V \to V \otimes \Bbbk[M]$, that is a counital comodule structure on V.

The relationship between the structure χ_V and the action of M is given as follows: $\chi_V(v) = \sum v_0 \otimes v_1$ if and only if for all $m \in M$, $m \cdot v = \sum v_1(m) v_0$. Along the paper we consider in general the actions to be on the left side, but it is clear that all the concepts have also a formulation for the action on the right side.

It is easy to prove, in the same way than for affine algebraic groups and their representations, that in this manner we establish an isomorphism of categories, between the category of finite dimensional polynomial M–modules and the finite dimensional counital comodules for the bialgebra $\Bbbk[M]$.

If we drop the hypothesis on the dimension, we obtain the concept of general polynomial M–module that is simply an arbitrary counital comodule V for the bialgebra $\Bbbk[M]$ –the comodule structure in V will be denoted as $\chi_V : V \to V \otimes \Bbbk[M]$.

Similarly than above, this corresponds to a vector space V endowed with an action of M (as an abstract monoid) in such a way that an arbitrary element $v \in V$ is contained in a finite dimensional M–stable submodule of $V_v \subset V$ and with the additional property that the restriction of this action to V_v produces a polynomial finite dimensional M–module.

Definition 1. Let M be a linear algebraic monoid. A (*polynomial*) character of M is a multiplicative morphism $\rho : M \to \Bbbk$, such that $\rho(1) = 1$.

Example 1. 1) In the case that M is the monoid of n^2 matrices with coefficients in $\Bbbk$, $\det : M \to \Bbbk$ is a character.

2) In the case of affine algebraic $\Bbbk$-groups, the above definition of character coincides with the usual one. Moreover, if ρ is a character of M and $a \in G = G(M)$,

then $\rho(a) \neq 0$ –otherwise $1 = \rho(1) = \rho(aa^{-1}) = \rho(a)\rho(a^{-1}) = 0$. Then, the restriction of a character of M yields a character of G. 3) We say that the character ρ is trivial if it only takes the value $1 \in \Bbbk$.

Observation 1. 1) Observe that if $0 \in M$ and ρ is a non–trival polynomial character, then $\rho(0) = \rho(0m) = \rho(0)\rho(m)$ for all $m \in M$, and thus $\rho(0) = 0$.

2) It is clear that the polynomial characters of M correspond bijectively with the polynomial module structures on $\Bbbk$. If λ is a polynomial character of M, it is easy to show that for a polynomial representation V of M an element $v \in V$ is a λ semi–invariant if and only if $\chi_V(v) = v \otimes \lambda$.

In the same manner than for algebraic groups, it is easy to prove that the category of polynomial M–modules is an abelian tensor category, with $\Bbbk$ as unit for the tensor product. Moreover, the finite dimensional polynomial M–modules, admit duals in the usal manner. Observe that if V is a left polynomial M–module, then V^* is a right polynomial M–module.

If V is a polynomial M–module, then the symmetric algebra built on V –that we denote as $\mathrm{S}(V) = \bigoplus_n \mathrm{S}^n(V)$– is also a polynomial module.

It is important to notice that if M is an affine algebraic monoid, $\Bbbk[M]$ can be naturally endowed with two structures of a polynomial M–module, one from the left and the other from the right. In explicit terms if $f \in \Bbbk[M]$ and we denote as usual $\Delta(f) = \sum f_1 \otimes f_2$, then if $m \in M$ we have that $m \cdot f = \sum f_1 f_2(m)$ and $f \cdot m = \sum f_1(m) f_2$.

In particular, in a similar manner than for the case of a group and a subgroup, one can define the induction functor from the representations of a submonoid to the representations of the whole monoid, see [1] and [9] for the definitions in the case of monoids and [2] for the case of groups.

Definition 2. If $N \subset M$ is a submonoid, then the restriction functor from the polynomial M-modules to the polynomial N-modules admits a right adjoint, called the induction functor from N to M , and that is denoted as Ind_N^M .

Given a finite dimensional polynomial N-module V, $\operatorname{Ind}_N^M(V)$ is obtained in the following way –see ([1])–. Call $\operatorname{Mor}(M,V)$ the vector space of all the morphisms of $\Bbbk$-varieties and take the subspace $\operatorname{Ind}_N^M(V) \subset \operatorname{Mor}(M,V)$ given by all $f \in \operatorname{Mor}(M,V)$ such that $f(nm) = n \cdot f(m)$ for all $n \in N$, $m \in M$. We endow this subspace with the polynomial M-module structure defined as follows: $(m' \cdot f)(m) = f(mm')$, $m, m' \in M$, $f \in \operatorname{Ind}_N^M(V)$.

The map $\epsilon_V : \operatorname{Mor}(M,V) \to V$ $\epsilon(f) = f(1)$ restricts to Ind_N^M , and it is a morphism of N-modules, that is the counit of the adjunction.

In the case that V is infinite dimensional, we proceed in the same manner than for affine algebraic groups, first we have to take inside of $\operatorname{Mor}(M,V)$ the subspace $\operatorname{Mor}_{\mathrm{fin}}(M,V) = \big\{f : M \to V : \dim\langle f(M)\rangle_{\Bbbk} < \infty\big\}$ and then take in $\operatorname{Mor}_{\mathrm{fin}}(M,V)$ the N–equivariant maps as before.

Similarly than for algebraic groups, one can also define $\mathrm{Ind}_N^M(V) = \{\sum f_i \otimes v_i : f_i \in \Bbbk[M], v_i \in V$, and $\forall\, n \in N, \sum f_i \cdot n \otimes v_i = \sum f_i \otimes n \cdot v_i\}$ and then the structure of polynomial M–module on $\mathrm{Ind}_N^M(V)$ is given by the action of M on $\Bbbk[M]$ *on the left*.

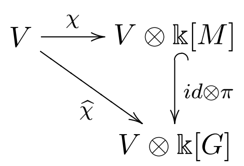
For future reference we write down the universal property of the induction functor. Given a polynomial N–module V, a polynomial M-module U and a homorphism of polynomial N-modules $\varphi : U \to V$, then there exists an unique homomorphism $\widetilde{\varphi} : U \to \mathrm{Ind}_N^M(V)$ such that $\varphi = \epsilon_V \circ \widetilde{\varphi}$.

It is easy to prove the usual properties of the induction functor, for example the transitivity. Given a chain of closed submonoids $S \subset N \subset M$ and a polynomial S-module V , then $\mathrm{Ind}_S^M(V) \cong \mathrm{Ind}_N^M(\mathrm{Ind}_S^N(V))$.

Moreover, the induction functor is left exact –it is a right adjoint of the restriction functor–.

The following considerations will be used later.

Assume that V is a polynomial M–module and call $G = G(M)$ the group of invertible elements. It is clear that by restricting the action from M to G we can view V as a rational G–module. If χ is the associated $\Bbbk[M]$– comodule structure on V, then the associated $\Bbbk[G]$–comodule structure –that we denote as $\widehat{\chi}$ is defined by the commutativity of the diagram below:



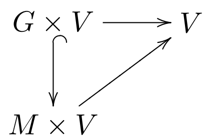
where $\pi : \Bbbk[M] \to \Bbbk[G]$ is the restriction map that can be viewed as an inclusion.

The following definition is relevant to pin down the difference between representations of G and of M.

Definition 3. In the situation above a rational representation θ of G is said to be *polynomial (with respect to M)*, if there exists a polynomial representation χ of M with the property that $\widehat{\chi} = \theta$.

Observation 2. 1) If $\theta : V \to V \otimes \Bbbk[G]$ is a polynomial representation, the polyonomial representation χ of M that yields θ is unique $-(\mathrm{id} \otimes \pi)\chi = \theta$ and π is injective–.

2) An action $G \times V \to V$ is a polynomial representation of G if and only if it can be extended to a polynomial action of M, in other words, if and only if one can find a map $M \times V \to V$ such that the diagram below commutes.



Indeed, in the notations above, if $\theta(v) = \sum v_0 \otimes v_1|_G$ with $v_1 \in \Bbbk[M]$, then the action $g \cdot v = \sum v_0 v_1(g)$ can be extended to the polynomial action $m \cdot v = \sum v_0 v_1(m)$.

3) Equivalently, if V is finite dimensional the morphism $G \to \mathrm{GL}(V)$ produces a polynomial representation of G if and only if there is a multiplicative morphism $M \to \mathrm{End}(V)$ that restricts to the original action of G.

In other words if the diagram below is commutative:

$$\begin{array}{ccc} G & \longrightarrow & \mathrm{GL}(V) \\ \big\downarrow & & \big\downarrow \\ M & \longrightarrow & \mathrm{End}(V) \end{array}$$

4) If V is a polynomial G–module, then it is clear that $S(V)$ is also a polynomial G–module.

In the case that G is an affine algebraic group and M is a monoid equipped with a polynomial character $\rho : M \to \Bbbk$ with the property that $M_\rho = G = G(M)$ –see Lemma ???–, more can be said concerning the relationship between the rational and polynomial representations of G.

Lemma 1. *In the situation above, if $\chi : V \to V \otimes \Bbbk[G]$ is a rational finite dimensional representation of G, there is an exponent $n \geq 0$ with the property that the rational representation of G, $\rho^n|_G\, \chi : V \to V \otimes \Bbbk[G]$ is a polynomial representation of G.*

Proof. The result follows immediately from the fact that $\Bbbk[M]_\rho = \Bbbk[G]$. $\square$

Observation 3. 1) Notice that above we have used the $\Bbbk[G]$–module structure of $V \otimes \Bbbk[G]$ given by multiplication on the second tensorand.

2) Recall that if $- \cdot - : G \times V \to V$ is a finite dimensional rational representation of G and λ is a character of G, we can define a new rational finite dimensional representation of G that we denote as $- \cdot_\lambda - : G \times V \to V$ by the formula $a \cdot_\lambda v = \lambda(a) a \cdot v$ for all $a \in G$ and $v \in V$. With this notation another formulation of the above lemma is the follwing: if $- \cdot - : G \times V \to V$ is a finite dimensional rational representation of G, then for some positive integer n the representation $- \cdot_{\rho^n|_G} - : G \times V \to V$ is polynomial.

3) In the situation above, assume that n_V the minimal exponent with the property that after twisting the orginal action of G on V by ρ^{n_V} we obtain a polynomial action –notice that the exponent n_V may be negative. Then, $n_{V \otimes W} = n_V + n_W$ and $n_{V \oplus W} = \max\{n_V, n_W\}$

Assume that we have an element $\lambda \in \mathcal{X}(M)$ and $v \in V$ such that $\lambda \chi_M(v) = v \otimes 1$. Then v is a $\lambda|_G^{-1}$–semi–invariant for the action of G on V. Indeed, if

$\chi_M(v) = \sum v_0 \otimes v_1$, then $\sum v_0 \otimes \lambda v_1 = v \otimes 1$. Evaluating at $g \in G$ we have that $\lambda(g) \sum v_1(g) v_0 = v$ or equivalently that $\lambda(g)(g \cdot v) = v$.

4. Generalized Cayley's Ω-process

Let M be an affine algebraic monoid and assume that λ is a non–zero character of M.

Definition 4. An Ω*-process (associated to* λ*)* is a non zero linear operator $\Omega : \Bbbk[M] \to \Bbbk[M]$ such that:
$\Omega(f \cdot m) = \lambda(m)\Omega(f) \cdot m$; $\Omega(m \cdot f) = \lambda(m) m \cdot \Omega(f)$ for all $f \in \Bbbk[M]$ and $m \in M$.

In classical nomenclature the above definition is called the "first rule of an Ω–process".

Observation 4. From the above definition one easily concludes that if r is a positive integer, then $\Omega^r(f \cdot m) = \lambda^r(m)\Omega(f) \cdot m$; $\Omega^r(m \cdot f) = \lambda^r(m) m \cdot \Omega(f)$ for all $f \in \Bbbk[M]$ and $m \in M$, in other words if Ω is an Ω–process associated to λ, then Ω^r is an Ω–process associated to λ^r.

Lemma 2. *With the notations above, let* $\mu \in \mathcal{X}(M)$ *be another character of the monoid, and* $g \in \Bbbk[M]$ *a right semi-invariant of weight* μ*. Then for all* $m \in M$

$$\Omega\big((f \cdot m) g\big)\mu(m) = \Omega\big((fg) \cdot m\big) = \lambda(m)\Omega(fg) \cdot m \,, \tag{1}$$

Similarly, if h *is a left semi-invaraint of weight* μ*, then for all* $m \in M$

$$\Omega\big((m \cdot f) h\big)\mu(m) = \Omega\big(m \cdot (fh)\big) = \lambda(m) m \cdot \Omega(fh) \,; \tag{2}$$

In particular,

$$\Omega(f\lambda^s) \cdot m = \Omega\big((f \cdot m)\lambda^s\big)\lambda^{s-1}(m) \qquad \forall s \geq 1 \,, \tag{3}$$

and

$$\Omega m \cdot (f\lambda^s) = \Omega\big((m \cdot f)\lambda^s\big)\lambda^{s-1}(m) \qquad \forall s \geq 1 \,, \tag{4}$$

Proof. It follows easily from the definition of an Ω–process. □

Theorem 1. *In the notations above,*

$$\Omega(\lambda^s) \cdot m = \Omega(\lambda^s)\lambda^{s-1}(m)$$
$$m \cdot \Omega(\lambda^s) = \Omega(\lambda^s)\lambda^{s-1}(m)$$

Moreover, $\Omega(\lambda^s) = \Omega(\lambda^s)(1)\lambda^{s-1} = \alpha_s \lambda^{s-1}$ *with* $\alpha_s = \Omega(\lambda^s)(1) \in \Bbbk$.

Proof. Just put $f = 1$, the constant polynomial in equations (1) and (2)
As for the second assertion, evaluate the above equality at $1 \in M$. □

Corollary 1. *Let $\mu \in \mathcal{X}(M)$ be a polynomial character, and $g \in \Bbbk[M]$ ($h \in \Bbbk[M]$) a right (left) semi-invariant of weight a character μ, then*

$$\Omega^r\big((f \cdot m)g\big)\mu^s(m) = \Omega^r\big((fg) \cdot m\big) = \lambda^r(m)\Omega^r(fg) \cdot m$$

and

$$\Omega^r\big((m \cdot f)h\big)\mu^s(m) = \Omega^r\big(m \cdot (f\mu^s)\big) = \lambda^r(m)m \cdot \Omega^r(fg)\,.$$

In particular,

$$\lambda^r(m)m \cdot \Omega^r(\mu^s) = \Omega^r(\mu^s)\mu^s(m) = \lambda^r(m)\Omega^r(\mu^s) \cdot m\,;$$

$$\Omega^r(\mu^s)(1)\mu^s(m) = \lambda^r(m)\Omega^r(\mu^s)(m)\,;$$

in other words: $\Omega^r(\mu^s)(1)\mu^s = \lambda^r\Omega^r(\mu^s)$.

Proof. Recall that Ω^r is a process associated to λ^r. □

Observation 5. In particular, if $\lambda = \mu$, we obtain that $\Omega^r(\lambda^s)(1)\lambda^s = \Omega^r(\lambda^s)\lambda^r$ and then if we call $\Omega^r(\lambda^s)(1) = \alpha_{r,s}$ –notice that in the notations used before $\alpha_s = \alpha_{1,s}$–, we deduce that $\Omega^r(\lambda^s)\lambda^r = \alpha_{r,s}\lambda^s$.

From $\Omega(\lambda^s) = \alpha_s\lambda^{s-1}$, we deduce that $\Omega^2(\lambda^s) = \alpha_s\alpha_{s-1}\lambda^{s-2}$, if $s \geq 2$. Then $\Omega^r(\lambda^s) = \alpha_s\alpha_{s-1}\dots\alpha_{r-s}\lambda^{r-s}$ if $r \geq s$ and if we call $c_s = \alpha_s\alpha_{s-1}\dots\alpha_1$, then $\Omega^s(\lambda^s) = c_s \in \Bbbk$.

(3) Now, if we write $f \cdot m = \sum f_1(m)f_2$, we have

$$\Omega^r((f \cdot m)\mu^s)\mu^s(m) = \big(\sum f_1(m)\Omega^r(f_2\mu^s)\big)\mu^s(m) = \lambda^r(m)\Omega^r(f\mu^s) \cdot m\,.$$

Evaluating at 0 we have that

$$\lambda^r(m)\Omega^r(f\mu^s)(0) = \big(\sum f_1(m)\Omega^r(f_2\mu^s)(0)\big)\mu^s(m)$$

i.e.,

$$\Omega^r(f\mu^s)(0)\lambda^r = (\sum f_1\Omega^r(f_2\mu^s)(0))\mu^s\,.$$

Similarly

$$\Omega^r((m \cdot f)\mu^s)\mu^s(m) = \big(\sum \Omega^r(f_1\mu^s)f_2(m)\big)\mu^s(m) = \lambda^r(m)m \cdot \Omega^r(f\mu^s)\,.$$

and

$$\Omega^r(f\mu^s)(0)\lambda^r = (\sum \Omega^r(f_1\mu^s)(0)f_2)\mu^s\,.$$

In particular if $\lambda = \mu$ we have that

$$\Omega^r(f\lambda^s)(0)\lambda^r = \lambda^s\sum \Omega^r(f_1\lambda^s)(0)f_2 = \lambda^s\sum f_1\Omega^r(f_2\lambda^s)(0)\,.$$

Moreover, if $r = s = 1$, $\Omega(f\lambda)(0) = \sum f_1\Omega(f_2\lambda)(0) = \sum \Omega(f_1\lambda)(0)f_2$.

Theorem 2. *Let $\mu \in \mathcal{X}(\overline{T})$ and $f \in \Bbbk[M]$ be a $B \times B^-$ semi-invariant of weight $(\mu, -\mu)$. Then $\Omega(f)$ is a semi-invariant of weight $(mu - \lambda, -mu_\lambda)$.*

Proof. In the same manner that before, from the definition of an Ω–process we deduce that for all $b_1 \in B$ and $b_2 \in B^-$,
(5)
$$(b_1, b_2)\cdot\lambda(b_1)\lambda(b_2^{-1}) = \Omega(f)b_1\cdot\Omega(f)\cdot b_2^{-1}\lambda(b_1)\lambda(b_2^{-1}) = \Omega(b_1\cdot f\cdot b_2^{-1}) = \Omega\big((b_1, b_2)f\big) = \mu(b_1)\mu(b_2^{-1})\Omega(f)$$
□

The results of the theorem that follows were known in the classical literature as the "second rule of an Ω–process".

Theorem 3. *If V is a polynomial M-module, consider the map $I_{r,s} : V \to V$, $I_{r,s}(v) = \sum v_0\Omega^r(\lambda^s v_1)(0)$. Then:*

1) The map $I_{r,s}$ is a morphism of $\Bbbk[M]$–modules, with respect to the original action on V twisted by λ^s.

2) For all $v \in V$, $\lambda^s\chi(I_{r,s}(v)) = \lambda^r(I_{r,s}(v) \otimes 1)$.

3) Assume that V is a polynomial G–module with structure θ and consider it as a polynomial M–module with structure $\chi : V \to V \otimes \Bbbk[M]$ with $\theta = (\mathrm{id}\otimes\pi)\chi$. If $I_{r,s}$ is as above, then for all $v \in V$, $I_{r,s}(v)$ is a λ^{r-s} semi–invariant for the action of G on V.

Proof.

1) The assertion concerning the fact that $I_{r,s}$ is a morphism of comodules is equivalent to the commutativity of the diagram below.

$$\begin{array}{ccc} V & \xrightarrow{\ I_{r,s}\ } & V \\ {\scriptstyle\lambda^s\chi}\downarrow & & \downarrow{\scriptstyle\lambda^s\chi} \\ V \otimes \Bbbk[M] & \xrightarrow{\ I_{r,s}\otimes\mathrm{id}\ } & V \otimes \Bbbk[M] \end{array}$$

Then, $\lambda^s\chi\big(I_{r,s}(v)\big) = \lambda^s \sum v_0 \otimes v_1\Omega^r(\lambda^s v_2)(0) = \sum v_0\Omega(\lambda^s v_1)(0) \otimes \lambda^r$.

Moreover, $(I_{r,s} \otimes \mathrm{id})(\lambda^s\chi(v))) = \sum I_{r,s}(v_0) \otimes \lambda^s v_1 = \sum v_0 \otimes \lambda^s\Omega^r(\lambda^s v_1)(0)v_2 = \sum v_0 \otimes \Omega^r(\lambda^s v_1)(0)\lambda^r = \sum v_0\Omega^r(\lambda^s v_1)(0) \otimes \lambda^r$.

Then, the commutativity of the above diagram follows.

2) The assertion of this part is the first of the formulæ proved above.

3) Applying $\mathrm{id}\otimes\pi$ to the equality just proved we obtain that $\lambda^s|_G\,\theta(I_{r,s}(v)) = I_{r,s}(v) \otimes \lambda^r|_G$, and hence we conclude that $I_{r,s}$ is a G–semi–invariant of weight λ^{r-s}.

□

Definition 5. In the situation above we say that the Ω–process is proper if the elements $\alpha_s \in \Bbbk$ are not zero for all $s = 1, 2, \cdots$. Recall that $\Omega(\lambda) = \Omega(\lambda)(1) = \alpha_1, \Omega(\lambda^2)(1) = \alpha_2, \cdots, \Omega(\lambda^s)(1) = \alpha_s$ for all $s \geq 1$.

The definition of integrals in the case of monoids is the same than for groups.

Definition 6. Assume that M is an affine algebraic monoid, a map $J : \Bbbk[M] \to \Bbbk$ is a left normalized integral if it is morphism of M–modules with $J(1) = 1$ and satisfies: $J(f)1 = \sum f_1 J(f_2)$. The equality $J(f)1 = \sum J(f_1)f_2$ characterizes left integrals.

See [2] for the definition and properties of integrals and Reynolds operators for algebraic groups.

In particular, in [2] it is shown how to construct Reynolds operators from integrals. We are considering the obvious adaptation of these concepts to the context of affine algebraic monoids.

Notice also that if we evaluate at x the equality $J(f)1 = \sum f_1 J(f_2)$, we obtain: $J(f) = \sum f_1(x)J(f_2) = \sum J(f_2 f_1(x)) = J(f \cdot x)$.

Theorem 4. *Let M be an affine algebraic monoid with 0 and let λ be a polynomial character of M and assume that Ω is a proper Ω–process with respect to λ. Then, the map $J : \Bbbk[M] \to \Bbbk$, $J(f) = \frac{1}{\Omega(\lambda)}\Omega(f\lambda)(0)$ is a two sided normalized integral.*

Proof. The lemma just proved guarantees that J satisfies the above equality, and clearly $J(1) = 1$. The M–equivariance of J also follows from the above. □

Theorem 5. *With the notations above and assuming that Ω is a proper Ω–process, $\mathcal{R}_V = \frac{1}{\Omega(\lambda)}I_{1,1} : V \to V$, is a Reynolds operator for the category of polynomial M–modules.*

Proof. Indeed, $\sum v_0\Omega(\lambda v_1)(0) = I(v)$ is an M-invariant. Now, if $v \in V^M$, then $\chi(v) = v \otimes 1$ and $I_V(v) = v\Omega(\lambda)(0)/\Omega(\lambda)(0) = v$. One can easily show that if $f : V \to W$ is a morphism of M–modules then the corresponding morphism $\widehat{f} = f|_{V^M} : V^M \to W^M$ is compatible with the operators I_V and I_W in the sense that the following diagram is commutative.

$$\begin{array}{ccc} V & \xrightarrow{I_V} & V^M \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \widehat{f}} \\ W & \xrightarrow{I_W} & W^M \end{array}$$

□

5. Finite generation of invariants

In this section we show that the existence of an Ω–process guarantees the finite generation of the rings of invariants corresponding to linear actions. Our proof is basically the same than the original one due to Hilbert –that appeared in [4]. The required finite generation is guaranteed by the fact that the maps $I_{r,s}$ allows us to produce quite a large number of semi–invariants.

The proof of the theorem that follows is only sketched as it is closely related to the standard methods introduced in [4] –see also [12] for a modern presentation in the case of GL_n.

Theorem 6. *Let M be a linear algebraic monoid with 0 and assume that for some polynomial character $\lambda : M \to \Bbbk$, $G = G(M) = M_\lambda = \{m \in M : \lambda(m) \neq 0\}$. Assume moreover that M admits a proper Ω–process with respect to λ. If V is a finite dimensional rational G–module, then the ring of invariants $\mathrm{S}(V)^G$ is finitely generated.*

Proof. Consider $\mathrm{S}(V)^G \subset \mathrm{S}(V)$ and call $I = \langle \mathrm{S}_+(V)^G \rangle \subset \mathrm{S}(V)$ the ideal of $\mathrm{S}(V)$ generated by the homogeneous invariants of positive degree. Call $\{\xi_1, \cdots, \xi_t\} \subset \mathrm{S}_+(V)^G$ a finite number of homogeneous ideal generators of I, and call d_i the degrees of the ξ_i for $i = 1, \cdots, t$.

Evidently $\Bbbk[\xi_1, \cdots, \xi_t] \subset \mathrm{S}(V)^G$, and we will prove by induction on $d > 0$ that $\mathrm{S}_d(V)^G \subset \Bbbk[\xi_1, \cdots, \xi_t]$.

Assume that for all $e < d$, $\mathrm{S}_e(V)^G \subset \Bbbk[\xi_1, \cdots, \xi_t]$ and take $\xi \in \mathrm{S}_d(V)^G \subset I$. Then, we can find homogeneous elements $f_i \in \mathrm{S}_{e_i}(V)$, $\xi = \sum f_i \xi_i$, where $d = e_i + d_i$, and $e_i < d$ for $i = 1, \cdots, t$.

Call $\theta : V \to V \otimes \Bbbk[G]$ the comodule structure on V. As we already observed for some $n \geq 0$, $\lambda^n\theta : V \to V \otimes \Bbbk[G]$ is polynomial, i.e. it is of the form $\lambda^n\theta = (\mathrm{id} \otimes \pi)\chi$ for some $\chi : V \to V \otimes \Bbbk[M]$, polynomial $\Bbbk[M]$–comodule structure on V.

If we call $\theta_r : \mathrm{S}_r(V) \to \mathrm{S}_r(V) \otimes \Bbbk[M]$, the comodule structure induced by θ on $\mathrm{S}_r(V)$, then $\lambda^{nr}\theta_r$ is polynomial.

Applying θ_d to the above equality we obtain:

$$\xi \otimes 1 = \sum_{i=1}^{t} f_{i0}\xi_i \otimes f_{i1} \,.$$

Multiplying by λ^{nd} we have:

$$\xi \otimes \lambda^{nd} = \sum_{i=1}^{t} f_{i0}\xi_i \otimes \lambda^{nd_i}\lambda^{ne_i} f_{i1} \,,$$

In other words for the coaction $\lambda^{ne_i}\theta_{e_i} : \mathrm{S}_{e_i}(V) \to \mathrm{S}_{e_i}(V) \otimes \Bbbk[M]$ –that sends $\lambda^{ne_i}\theta_{e_i}(f_i) = \sum f_{i0} \otimes g_{i1}$ with $f_{i0} \in \mathrm{S}_{e_i}(V)$ and $g_{i1} \in \Bbbk[M]$– we have that

$$\xi \otimes \lambda^{nd} = \sum_{i=1}^{t} f_{i0}\xi_i \otimes \lambda^{nd_i} g_{i1} \,.$$

Applying Ω^{nd}, then evaluating at zero and recalling that $\Omega^{nd}(\lambda^{nd}) = c_{nd} \in \Bbbk^*$, we obtain

$$c_{nd}\xi = \sum_{i=1}^{t} f_{i0}\Omega^{nd}(\lambda^{nd_i} g_{i1})(0)\xi_i = \sum_{i=1}^{t} I_{nd,nd_i}(f_i)\xi_i \,,$$

where I_{nd,nd_i} is as in Theorem 3 where it is proved that it is a λ^{ne_i} semi–invariant.

As both $\xi, \xi_i, i = 1, \cdots, t$ are in fact invariants, we conclude that $I_{nd,nd_i}(f_i) \in \mathrm{S}^G_{e_i}(V) \subset \Bbbk[\xi_1, \cdots, \xi_t]$. Then as $c_{nd} \neq 0$, $\xi \in \Bbbk[\xi_1, \cdots, \xi_t]$ and the proof is finished. □

6. Representations of reductive monoids

Reductive groups have a well behaved representation theory, which can be extended to reductive monoids. This extension was first performed by S. Doty ([1]) for a special class of reductive monoids –namely the ones with unidimensional center–, and then by Renner in full generality (see [9]).

In this section we present the basic results about the representation theory of reductive monoids, that we will need for the rest of the work, in the laguage of reductive monoids classification as in [10]. In order to make the presentation more clear, some proofs will be sketched.

Observation 6. Since the unique algebraic monoid with unit group a semisimple group G is G itself (see [10]), in what follows we suppose that G is a reductive group with connected center of dimension greater than 0. We denote $G^0 = [G, G]$.

Definition 7. Let M be a reductive monoid of unit group G and denote as $\overline{T}$ the closure of the maximal torus. We define the *polynomial characters* of M as $\mathcal{X}(\overline{T})$ the set of multiplicative morphisms of algebraic monoids from $\overline{T}$ to $(\Bbbk, *)$. It is the submonoid of $\mathcal{X}(T)$ consisting of the characters of T that extend to $\overline{T}$. We define the *dominant polynomial characters* as the intersection $\mathcal{X}_+(\overline{T}) = \mathcal{X}_+(T) \cap \mathcal{X}(\overline{T})$.

Theorem 7. *Let M be a reductive monoid, and let V be a rational G-module all whose T-weights are polynomial. Then the G-module structure of V is polynomial.*

Proof. Just take a basis $\{v_i\}_{i\in I}$ of T-weight vectors of V. Then if $g \cdot v_i = \sum c_{ij}(g) v_j$, $c_{ij} \in \Bbbk[G]$. As $c_{ij}|_T$ extend to $\overline{T}$ for all $i, j \in I$, it follows that c_{ij} extend to M for all $i, j \in I$. □

Consider $\lambda \in \mathcal{X}(\overline{T})$, and consider the one dimensional T-mod $\Bbbk_\lambda$, where $t \cdot \alpha = \lambda(t)\alpha$, $t \in T$, $\alpha \in \Bbbk$. Then $\Bbbk_\lambda$ is a $\overline{T}$-mod, and this structure extends to a structure of $\overline{B}$-mod ($\overline{B}$ is the closure of B in M).

Theorem 8. *Let M be a reductive monoid and consider $\lambda \in \mathcal{X}(\overline{T})$. Then $\operatorname{Ind}_{\overline{B}}^{M} \Bbbk_\lambda$ is non-zero if and only if $\lambda \in \mathcal{X}_+(\overline{T})$.*

In particular, the set of isomorphism classes of simple polynomial M-modules is $L(\lambda) = \operatorname{soc}_G \operatorname{Ind}_B^G \Bbbk_\lambda$, $\lambda \in \mathcal{X}_+(\overline{T})$. □

Next we study the properties of the abstract monoid $\mathcal{X}_+(\overline{T})$.

Theorem 9. *Let M, G, B, T be as above. Then*

$$\mathcal{X}_+(\overline{T}) = \left\{ \lambda \in \mathcal{X}_+(T) \mid \exists f \in \Bbbk[M]^{(B\times B^-)} \ , \ (a,b)\cdot f = \lambda(a)\lambda^{-1}(b) f \ \forall a, b \in T \right\} .$$

PROOF. If $\lambda \in \mathcal{X}_+(T)$, then there exists a nonzero $f \in \Bbbk[G]$ such that $(a,b)\cdot f = \lambda(a)\lambda^{-1}(b)f$, $(a,b) \in B \times B^-$. Hence, $f(x) = ((x,1)\cdot f)(1) = \lambda(x)f(1)$ for all $x \in T$. Suppose that $f \in \Bbbk[M]$, if $f(1) = 0$, then for $x = ab \in BB^-$ we have $f(x) = ((a,b^{-1})\cdot f)(1) = \lambda(a)\lambda^{-1}(b^{-1})f(1) = 0$, and it follows that $f = 0$. Then $f(1) \neq 0$, and we can extend λ to all $\overline{T}$ by the formula $\lambda(x) = \frac{f(x)}{f(1)}$, and thus $\lambda \in \mathcal{X}(\overline{T})$, so $\lambda \in \mathcal{X}_+(\overline{T})$.

On the other hand, let $\lambda \in \mathcal{X}_+(\overline{T})$, and consider $f \in \Bbbk[G]$ such that $(a,b)\cdot f = \lambda(a)\lambda^{-1}(b)f$ for all $(a,b) \in B \times B^-$; we have $f(z) = f(1z) = \lambda(z)f(1)$ for $z \in T$. Let $g : \overline{T} \to \Bbbk$ be the morphism given by $g(z) = \lambda(z)f(1)$. As $g(t) = f(t)$ for all $t \in T$, by the extension principle ([9]) there exists $h \in \Bbbk[M]$ such that $h|_{\overline{T}} = g$ and $h|_G = f$. As $((a,b)\cdot h)(x) = \lambda(a)\lambda^{-1}(b)h(x)$ for all $x \in G \subset M$ and $(a,b) \in B \times B^-$, it follows by continuity that h is a weight vector of weight $(\lambda, -\lambda)$ for $B \times B^-$. □

Theorem 10. *Let M, G, B, T be as above. Then the monoid $\mathcal{X}_+(\overline{T})$ is saturated, that is $n\lambda \in \mathcal{X}_+(\overline{T})$, if and only if $\lambda \in \mathcal{X}_+(\overline{T})$. Moreover, if M has a zero, then $\mathcal{X}_+(\overline{T})$ is an ideal of $\mathcal{X}_+(T)$, i. e. if $\mu \in \mathcal{X}_+(\overline{T})$ and $\lambda \in \mathcal{X}_+(T)$ is such that $\lambda \leq \mu$, then $\lambda \in \mathcal{X}_+(\overline{T})$.* □

Corollary 2. *If Ω is a process for M associated to a polynomial character λ, and $f \in \Bbbk[M]_(\mu, -\mu)$, with $\mu - \lambda \notin \mathcal{C}^\vee$., then $\Omega(f) = 0$.*

7. Characters for linear algebraic monoids

Next we establish a few basic facts about extendible characters for linear algebraic monoids.

Definition 8. Let M be a linear algebraic monoid. A *(polynomial) character* for M is a multiplicative morphism $\rho : M \to \Bbbk$, such that $\rho(1) = 1$.

Example 2. 1) If M the monoid of n^2 matrices with coefficients in $\Bbbk$. Then $\det : M \to \Bbbk$ is a character.

2) In the case of affine algebraic $\Bbbk$-groups, the above definition of character coincides with the usual one.

3) We say that the character ρ is trivial if it only takes the value $1 \in \Bbbk$.

Observation 7. Observe in general, that if $0 \in M$ and ρ is a non–trival character, then $\rho(0) = \rho(0m) = \rho(0)\rho(m)$, then $\rho(0) = 0$.

Moreover, if ρ is a character of M and $a \in G = G(M)$, then $\rho(a) \neq 0$ –otherwise $1 = \rho(1) = \rho(aa^{-1}) = \rho(a)\rho(a^{-1}) = 0$. Then, the restriction of a character of M yields a character of G.

Lemma 3. *Let M be a reductive monoid. Then there exists a non-trivial polynomial character $\lambda \in \mathcal{X}(M)$.*

Proof. Let $(\mathcal{C},\mathcal{F})$ be the colored cone associated to M.

By definition, a rational character $\lambda\in\mathcal{X}(G)$ is polynomial for M if and only if $\lambda\in\Bbbk[M]$. Hence, $\lambda\in\Bbbk[M]$ is a polynomial character if and only $\lambda|_G\in\mathcal{X}(G)$. In this case, λ is a (G,G)-semi-invariant of weight $(\lambda,-\lambda)$. Hence, $\lambda\in\mathcal{X}(G)\subset\mathcal{X}(T)$ is a polynomial character if and only if $\lambda\in\mathcal{C}^\vee$.

We have then to prove that $\mathcal{C}^\vee\cap\mathcal{X}(G)\neq\{0\}$.

Recall from [10] that if $\mathcal{D}$ is the cone geenrated by $\mathcal{C}$ and all the coroots –the colors–, then $\mathcal{D}$ is a strictly convex cone. If we prove that $\mathcal{D}^\vee\cap\mathcal{X}(G)\neq\{0\}$, the result will then follow, since $\mathcal{C}\subset\mathcal{D}$. Let $\mathcal{D}$ be generated by all the coroots and $(w_1,z_1)\dots,(w_s,z_s)\in -C(G)=-C(G_0)\times\mathcal{X}(G)$.

Assume that the cone $\mathcal{E}$ generated by $z_1,\dots,z_s0$ is not strictly convex, and let $0=\sum a_iz_i$, with $a_i\geq 0$ for all $i=1,\dots,s$ and some $a_j>0$. Then $(0,0)\neq(w,0)=\sum_{i=1}^s a_i(w_i,z_i)\in -C(G_0)\times\{0\}$, and it follows that $w=-\sum_{i=1}^l b_i\alpha_i^\vee$, $b_i\geq 0$, with some $b_j>0$, which contradicts the fact that $\mathcal{D}$ is stricty convex. Hence, $\mathcal{E}$ is strcitly convex, and there exists $0\neq z\in\mathcal{X}(G)$ such that $z\in\mathcal{E}^\vee$. It is clear that then $(0,z)\in\mathcal{D}^\vee\cap\mathcal{X}(G)$. □

Lemma 4. *Let G be a reductive group and $\lambda\in\mathcal{X}(G)$ a character. Then there exists an algebraic monoid M with unit group G such that M has a zero, and $G=M_\lambda$.*

Proof. Let $w=\sum_{i=1}^l\omega_i$ and $\mathcal{C}$ the cone generated by (w,λ) and all the coroots. It is clear that $\big(\mathcal{C},\{\alpha_1^\vee,\dots,\alpha_l^\vee\}\big)$ is a colored cone associated to an affine variety M –observe that $(0,\lambda)=(w,\lambda)-(0,\frac{1}{2}\alpha_i)$, and it follows from the classification theory of reductive monoids that M is a reductive monoid with zero.

It follows from the construction that λ is a polynomial character for M. Hence M_λ is a $(G\times G)$-stable divisor, and thus it is the unique $(G\times G)$-stable divisor of M – corresponding to the edge of $\mathcal{C}$ generated by (w,λ). □

8. Existence of Cayley's Ω-process

In this section we describe all the Ω-process associated to a polynomial character $\lambda\in\mathcal{X}(M)$.

Recall that since $\operatorname{char}\Bbbk=0$, then $\Bbbk[M]=\bigoplus V_\mu\otimes V_\mu^*$.

Theorem 11. *Let M be a reductive monoid, $\lambda\in\mathcal{X}(M)$ and Ω a process associated to λ. Then $\Omega=\bigoplus_{\mu\in\mathcal{X}_+(\overline{T})}\frac{a_\mu}{\lambda}\operatorname{Id}_\mu$, where Id_μ is the identity map of $V_\mu\otimes V_\mu^*$, and $a_\mu\in\Bbbk$.*

`en lo que sigue se puede tomar g en vez de s`

If $f_{\mu,-\mu}\in V_\mu\otimes V_\mu^*$ is a semi-invariant, $f_{\mu,-\mu}(1)=1$, define $\Omega(f)=f_{\mu-\lambda,-\mu+\lambda}=f_{\mu,-\mu}f_{\lambda,-\lambda}$, $f_{\mu-\lambda,-\mu+\lambda}(1)=f_{\lambda,-\lambda}(1)=1$ or $f_{\mu-\lambda,-\mu+\lambda}=0$, depending if the multiplicity of $V_{\mu-\lambda}\otimes V_{\mu-\lambda}^*$ euqlas 1 ror 0 respectively. We define $\Omega(s\cdot f\cdot s')=\lambda(s)\lambda(s')s\cdot\Omega(f)\cdot s'$ and extend by linearity.

multiplicty 0 nothing to prove.
multiplicity 1:
If $\sum s_i \cdot f_{\mu,-\mu} \cdot s'_i = \sum h_i \cdot f_{\mu,-\mu} \cdot h'_i$, then $\sum s_i \cdot f_{\mu,-\mu} \cdot s'_i = \sum h_i \cdot f_{\mu,-\mu} \cdot h'_i$ ... □

References

[1] S. Doty, *Representation theory of reductive normal algebraic monoids.* Trans. Amer. Math.Soc. 356 N. 6 (1999) pages 2539–2551.
[2] W. Ferrer Santos and A. Rittatore, *Actions and Invariants of Algebraic Groups.* Series: Pure and Applied Mathematics, 268, Dekker-CRC Press, Florida, (2005).
[3] R. Hartshorne, *Algebraic Geometry.* GTM 52, Springer-Verlag, 1993.
[4] D. Hilbert, *Über die Theorie der algebraischen Formen.* Matematische Annalen, 36 (1890) pages 473–534.
[5] F. Knop, *The Luna-Vust Theory of Spherical Embeddings.* In S. Ramanan *et al*, editors, Procedings of the Hydebarad Conference on Algebraic Groups, pages 225–249. National Board for Higher Mathematics, Manoj, 1991.
[6] O. Mathieu, *Filtrations of G-modules.* Ann. Sci. Éc. Norm. Sup. 23 (1990), pages 625–644.
[7] O. Mathieu, *Tilting modules and their applications.* Adv. Studies in Pure Math. 26 (1998), pages 1–68.
[8] L. Renner, *Classification of Semisimple Algebraic Monoids.* Trans. Amer. Math. Soc. 292 (1985) pages 193–223.
[9] L. Renner, *Linear Algebraic Monoids.* Enc. of Math. Sciences 134, Invariant Theory V., Springer Verlag, 2005.
[10] A. Rittatore, *Algebraic Monoids and group embeddings.* Trans. Groups Vol 3, No 4 (1998), pages 375–396.
[11] A. Rittatore, *Reductive embeddings are Cohen-Macaulay.* Proc. Amer. Math. Soc. 131 (2003), pages 675-684.
[12] B. Sturmfels, *Algorithms in invariant theory.* Texts and Monographs in Symbolic Computation. Springer-Verlag, 1993.

Alvaro Rittatore
Facultad de Ciencias
Universidad de la República
Iguá 4225
11400 Montevideo
Uruguay
e-mail: alvaro@cmat.edu.uy

Walter Ferrer Santos
Facultad de Ciencias
Universidad de la República
Iguá 4225
11400 Montevideo

Uruguay
e-mail: wrferrer@cmat.edu.uy