Rice formulas and Gaussian waves II.

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Abstract

We prove a certain number of results on specular points and dislocations of random waves which we have announced without proof in [1] or for which only an outline of proof has been given in this reference. Along the paper, waves are Gaussian and the basic tools are Rice formulas. The main results are on the first two moments and in some special case, also weak convergence is obtained.

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1 Introduction

This paper is a continuation of [1] in which we study the zeroes of certain random waves that appear in oceanography and optics. Our aim here is to give full proofs of certain results that were stated without proof in that paper, or for which proofs have been only sketched.

Our interest lies in the geometry of the set of zeros of random fields with low-dimensional parameter set (smaller or equal than 3). In general, only a restricted number of geometrical characteristics of these sets can be described with the methods we use, namely the so-called Rice formulas.

When the set of zeros is 0-dimensional, Rice formulas permit to express the moments of the number of zeros by means of certain integrals depending upon a description of the probability law of the random fields. If it is 1-dimensional one can do something similar with length instead of number of zeros, if it is

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2-dimensional with area-measure, and so on. One can also extend the same methods to weighted zeros, that is, compute the moments of total weight in the 0-dimensional case and the integral of a weight function on the 0-level set of the random field in the other cases.

We compute moments that are useful to make statistics on certain parameters appearing in the law of the random field. In some situations we can go further and obtain weak limit theorems for certain re-normalizations of natural functionals of the paths which are of interest.

These are special cases of the general problem of computing moments of the geometric measure of the level sets of random fields. For this purpose, Rice formulas have been developed since the pioneering work of Rice [10]. We refer to the book by Azaïs and Wschebor [2] for an extended presentation of the subject and for proofs of the general formulas we use.

In this paper we will consider two classes of 0-sets of random fields: specular points (Section 2) and dislocations of wave fronts (Section 3). For details not mentioned here and other geometrical properties of waves which can be studied with analogous methods, we refer to [1], [4], [9].

All random fields are assumed to have continuously differentiable paths and to be Gaussian, a hypothesis that is useful to be able to perform the computations associated with Rice formulas, but can fail to approximate physical reality in certain cases.

We use the following notations: $\sigma_d(B)$ the d-dimensional Hausdorff measure of a Borel set B. If f is a function of d variables we denote f_i the partial derivative with respect to the i-th variable. M^T denotes the transpose of a matrix M. ($const$) is a positive constant whose value may change from one occurrence to another. $p_\xi(x)$ is the density of the random variable or vector ξ, whenever it exists. λ_k ($k = 0, 1, 2, ...$) denotes the k-th spectral moment of a stationary random process defined on the real line.

2 Specular points

2.1 Specular points for one-parameter processes

Specular points of a curve are defined as follows: We take cartesian coordinates Oxz in the plane and assume the curve is the graph of a C^1-function $z = W(x)$. A light source placed at $(0, h_1)$ emits a ray that is reflected at the point $(x, W(x))$ of the curve and the reflected ray is registered by an observer placed at $(0, h_2)$.

Using the equality between the angles of incidence and reflexion with respect to the normal vector to the curve - i.e. $N(x) = (-W'(x), 1)$ - an elementary computation gives:

$$W'(x) = \frac{\alpha_2 r_1 - \alpha_1 r_2}{x(r_2 - r_1)} \tag{1}$$

where $\alpha_i := h_i - W(x)$ and $r_i := \sqrt{x^2 + \alpha_i^2}$, i=1,2.

The points $(x, W(x))$ of the curve such that x is a solution of (1) are called "specular points". When the curve is random, one of our aims is to study the probability distribution of the number of specular points such the abcise $x \in A$, where A is a Borel subset of the line.

The following approximation is due to M.S. Longuet-Higgins (see [7], [8]):

Suppose that h_1 and h_2 are big with respect to $W(x)$ and x, then $r_i = \alpha_i + x^2/(2\alpha_i) + O(h_i^{-3})$. Then, (1) can be approximated by

$$W'(x) \simeq \frac{x}{2}\frac{\alpha_1 + \alpha_2}{\alpha_1\alpha_2} \simeq \frac{x}{2}\frac{h_1 + h_2}{h_1 h_2} = kx, \tag{2}$$

where

$$k := \frac{1}{2}\Big(\frac{1}{h_1} + \frac{1}{h_2}\Big).$$

Set $Y(x) := W'(x) - kx$ and $SP(A)$ the number of roots of $Y(x)$ belonging to the set A. We will call $SP(A)$ the "Longuet-Higgins approximation" of the number of specular points, when the parameter k tends to 0.

Assume now that $\{W(x) : x \in \mathbb{R}\}$ is a centered Gaussian stationary process with C^2-paths. In [1] an exact formula has been obtained for the expectation of the number of specular points belonging to the interval $[a, b]$. This is an integral formula, well-adapted to numerical computation and it turns out that $\mathbb{E}(SP([a, b]))$ is a very accurate approximation, for example, for ocean waves.

Also, the Longuet-Higgins approximation is tractable from a mathematical point of view, and one can go much farther than expectation in the description of the law of the number of specular points. More precisely in [1] it is proved that:

1. Adding some hypotheses on the law of the process $\{W(x) : x \in \mathbb{R}\}$ (paths of class C^4 and some mixing condition, such as δ-dependence or a controlled decay of correlation), it follows that

   $$\mathrm{Var}\big[SP(\mathbb{R})\big] = \theta\frac{1}{k} + O(1) \quad \text{as} \quad k \to 0,$$

   where θ is a constant that can be computed by means of an explicit formula from the covariance of the given Gaussian process, which is well-adapted to numerical computation.

This implies that the coefficient of variation of the random variable $SP(\mathbb{R})$ tends to zero in a controlled manner, namely:

$$\frac{\sqrt{\mathrm{Var}(SP(\mathbb{R}))}}{\mathbb{E}(SP(\mathbb{R}))} \sim \sqrt{\frac{\theta \pi k}{2\lambda_4}} \quad \text{as} \quad k \to 0, \tag{3}$$

since

$$\mathbb{E}(SP(\mathbb{R})) \sim \sqrt{\frac{2\lambda_4}{\pi}} \frac{1}{k}$$

((3) corrects a small error in [1]).

2. With some additional requirement on the smoothness of the paths of the process, under the same asymptotic, the natural renormalization of $SP(\mathbb{R})$ tends to the standard normal distribution $\Phi(x)$, that is, for every $x \in \mathbb{R}$:

$$\mathbb{P}\Big(\frac{SP(\mathbb{R}) - \frac{(2\lambda_4/\pi)^{1/2}}{k}}{(\theta/k)^{1/2}} \leq x \Big) \to \Phi(x) \quad \text{as} \quad k \to 0.$$

2.2 Specular points for two-parameter processes

Let us consider in $\mathbb{R}^3$ a coordinate system Oxyz, and a C^1-function $z = W(x, y)$. The following definition of specular points of the graph extends naturally the one we gave above for functions of one real variable.

The source of light is placed at the point $(0, 0, h_1)$ and the observer at $(0, 0, h_2)$. The point (x, y) is said to be a specular point if the normal vector $n(x, y) = (-W_x, -W_y, 1)$ to the graph at $(x, y, W(x, y))$ satisfies the following two conditions:

- the angles with the incident ray $I = (-x, -y, h_1 - W)$ and the reflected ray $R = (-x, -y, h_2 - W)$ are equal (for short the argument (x, y) has been removed),
- it belongs to the plane generated by I and R.

Setting $\alpha_i = h_i - W$ and $r_i = \sqrt{x^2 + y^2 + \alpha_i}$, $i = 1, 2$, as in the one-parameter case we have:

$$\begin{aligned} W_x &= \frac{x}{x^2 + y^2} \frac{\alpha_2 r_1 - \alpha_1 r_2}{r_2 - r_1}, \\ W_y &= \frac{y}{x^2 + y^2} \frac{\alpha_2 r_1 - \alpha_1 r_2}{r_2 - r_1}. \end{aligned} \tag{4}$$

When h_1 and h_2 are large, the system above can be approximated by

$$\begin{aligned} W_x &= kx \\ W_y &= ky, \end{aligned} \tag{5}$$

under the same conditions as in dimension 1. This is the Longuet-Higgins approximation for two-parameter functions.

For each subset Q of $\mathbb{R}^2$, we denote by $SP(Q)$, the number of approximate specular points in the sense of (5) such that $(x, y) \in Q$. In the remaining of this paragraph we limit our attention to this approximation and to the case in which $\{W(x, y) : (x, y) \in \mathbb{R}^2\}$ is a centered Gaussian stationary random field with C^3-paths.

We need some additional notation: μ denotes the spectral measure of the random field, which is a Borel measure on $\mathbb{R}^2$ and $\lambda_{ij}, i, j = 0, 1, 2, ...$ the spectral moments

$$\lambda_{ij} = \int_{\mathbb{R}^2} u^i v^j \mu(du, dv)$$

whenever they are well-defined.

In [1] one can find the statement of certain results on the behavior of expectation and variance of $SP(Q)$ under the asymptotic $k \to 0$. We give full proofs of these results below. For the time being, what is known for variance and coefficient of variation is weaker than in the one-dimensional parameter case.

Let us define:

$$\mathbf{Y}(x, y) := \begin{pmatrix} W_x(x, y) - kx \\ W_y(x, y) - ky \end{pmatrix}. \tag{6}$$

Under the non-degeneracy condition $\lambda_{20}\lambda_{02} - \lambda_{11}^2 \neq 0$, the random field $\{Y(x, y) : x, y \in \mathbb{R}\}$ satisfies the hypotheses of Theorem 6.2. in [2], and we can write the Rice formula:

$$\begin{aligned} \mathbb{E}\big(SP(Q)\big) &= \int_Q \mathbb{E}\big(|\det \mathbf{Y}'(x, y)|\big|\mathbf{Y}(x, y) = 0\big) p_{\mathbf{Y}(x,y)}(\mathbf{0})\; dxdy \\ &= \int_Q \mathbb{E}\big(|\det \mathbf{Y}'(x, y)|\big) p_{\mathbf{Y}(x,y)}(\mathbf{0})\; dxdy, \end{aligned} \tag{7}$$

since for fixed (x, y) the random matrix $\mathbf{Y}'(x, y)$ and the random vector $\mathbf{Y}(x, y)$ are independent, so that the condition in the conditional expectation can be removed.

The density in the right hand side of (7) has the expression

$$\begin{aligned} p_{\mathbf{Y}(x,y)}(\mathbf{0}) &= p_{(W_x, W_y)}(kx, ky) \\ &= \frac{1}{2\pi} \frac{1}{\sqrt{\lambda_{20}\lambda_{02} - \lambda_{11}^2}} \exp\Big[-\frac{k^2}{2(\lambda_{20}\lambda_{02} - \lambda_{11}^2)}\big(\lambda_{02}x^2 - 2\lambda_{11}xy + \lambda_{20}y^2\big)\Big]. \end{aligned} \tag{8}$$

To compute the expectation of the absolute value of the determinant in the right hand side of (7), which does not depend on x, y, we use the method of [3] (see also [6]). Set $\Delta := \det \mathbf{Y}'(x,y) = (W_{xx}-k)(W_{yy}-k) - W_{xy}^2$.

We have

$$\mathbb{E}(|\Delta|) = \mathbb{E}\left[\frac{2}{\pi}\int_0^{+\infty} \frac{1-\cos(\Delta t)}{t^2} dt\right]. \tag{9}$$

Define

$$h(t) := \mathbb{E}\left[\exp\left(it[(W_{xx}-k)(W_{yy}-k) - W_{xy}^2]\right)\right].$$

Then

$$\mathbb{E}(|\Delta|) = \frac{2}{\pi}\Big(\int_0^{+\infty} \frac{1-\mathfrak{Re}[h(t)]}{t^2} dt\Big). \tag{10}$$

To compute $h(t)$ we define

$$A = \begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and Σ the variance matrix of $W_{xx}, W_{yy}, W_{x,y}$

$$\Sigma := \begin{pmatrix} \lambda_{40} & \lambda_{22} & \lambda_{31} \\ \lambda_{22} & \lambda_{04} & \lambda_{13} \\ \lambda_{31} & \lambda_{13} & \lambda_{22} \end{pmatrix}.$$

Let$\Sigma^{1/2} A \Sigma^{1/2} = P \; diag(\Delta_1, \Delta_2, \Delta_3) P^T$ where P is orthogonal. Then by a diagonalization argument

$$h(t) = e^{itk^2}$$
$$\mathbb{E}\Big(\exp\big[it\big((\Delta_1 Z_1^2 - k(s_{11}+s_{21})Z_1) + (\Delta_2 Z_2^2 - k(s_{12}+s_{22})Z_2) + (\Delta_3 Z_3^2 - k(s_{13}+s_{23})Z_3)\big)\big]\Big) \tag{11}$$

where (Z_1, Z_2, Z_3) is standard normal and s_{ij} are the entries of $\Sigma^{1/2} P^T$.

One can check that if ξ is a standard normal variable and τ, μ are real constants, $\tau > 0$:

$$\mathbb{E}\big(e^{i\tau(\xi+\mu)^2}\big) = (1-2i\tau)^{-1/2} e^{\frac{i\tau\mu^2}{(1-2i\tau)}} = \frac{1}{(1+4\tau^2)^{1/4}} \exp\Big[\frac{-2\tau}{1+4\tau^2} + i\big(\varphi + \frac{\tau\mu^2}{1+4\tau^2}\big)\Big],$$

where

$$\varphi = \frac{1}{2}\arctan(2\tau), \quad 0 < \varphi < \pi/4.$$

Replacing in (11), we obtain for $\mathfrak{Re}[h(t)]$ the formula:

$$\mathfrak{Re}[h(t)] = \Big[\prod_{j=1}^{3} \frac{d_j(t,k)}{\sqrt{1+4\Delta_j^2 t^2}}\Big] \cos\Big(\sum_{j=1}^{3}\big(\varphi_j(t) + k^2 t \psi_j(t)\big)\Big) \tag{12}$$

where, for $j = 1, 2, 3$:

- $d_j(t,k) = \exp\Big[-\dfrac{k^2t^2}{2}\;\dfrac{(s_{1j}+s_{2j})^2}{1+4\Delta_j^2t^2}\Big]$,

- $\varphi_j(t) = \dfrac{1}{2}\arctan(2\Delta_j t), \quad 0 < \varphi_j < \pi/4$,

- $\psi_j(t) = \dfrac{1}{3} - t^2\,\dfrac{(s_{1j}+s_{2j})^2\Delta_j}{1+4\Delta_j^2t^2}$.

Introducing these expressions in (10) and using (8) we obtain a new formula which has the form of a rather complicated integral. However, it is well adapted to numerical evaluation.

On the other hand, this formula allows us to compute the equivalent as $k \to 0$ of the expectation of the total number of specular points under the Longuet-Higgins approximation. In fact, a first order expansion of the terms in the integrand gives a somewhat more accurate result, that we state as a theorem:

Theorem 1

$$\mathbb{E}\big(SP(\mathbb{R}^2)\big) = \frac{m_2}{k^2} + O(1), \tag{13}$$

where

$$\begin{aligned} m_2 &= \int_0^{+\infty} \frac{1 - \big[\prod_{j=1}^3 (1+4\Delta_j^2t^2)\big]^{-1/2}\cos\big(\sum_{j=1}^3 \varphi_j(t)\big)}{t^2}\,dt \\ &= \int_0^{+\infty} \frac{1 - 2^{-3/2}\big[\prod_{j=1}^3 \big(A_j\sqrt{1+A_j}\big)\big]\big(1 - B_1B_2 - B_2B_3 - B_3B_1\big)}{t^2}\,dt, \end{aligned} \tag{14}$$

where

$$A_j = A_j(t) = \big(1+4\Delta_j^2t^2\big)^{-1/2}, \quad B_j = B_j(t) = \sqrt{(1-A_j)/(1+A_j)}.$$

Notice that m_2 only depends on the eigenvalues $\Delta_1, \Delta_2, \Delta_3$ and is easily computed numerically.

We now consider the variance of the total number of specular points in two dimensions, looking for analogous results to the one-dimensional case, in view of their interest for statistical applications. It turns out that the computations become much more involved. The statements on variance and speed of convergence to zero of the coefficient of variation that we give below include only the order of the asymptotic behavior in the Longuet-Higgins approximation, but not the constant. However, we still consider them to be useful. If one refines the computations one can give rough bounds on the generic constants in Theorem 2 and Corollary 1 on the basis of additional hypotheses on the random field.

Now we add the following hypothesis to the set already required to study the expectation of the specular points under the Longuet-Higgins asymptotic. We express $W''(\mathbf{0})$ in the reference system xOy of $\mathbb{R}^2$ as the 2×2 symmetric centered Gaussian random matrix:

$$W''(\mathbf{0}) = \begin{pmatrix} W_{xx}(\mathbf{0}) & W_{xy}(\mathbf{0}) \\ W_{xy}(\mathbf{0}) & W_{yy}(\mathbf{0}) \end{pmatrix}$$

The function

$$\mathbf{z} \rightsquigarrow \Delta(\mathbf{z}) = \det\left[\operatorname{Var}\left(W''(0)\mathbf{z}\right)\right],$$

defined on $\mathbf{z} = (z_1, z_2)^T \in \mathbb{R}^2$, is a non-negative homogeneous polynomial of degree 4 in the pair z_1, z_2. We will assume the non-degeneracy condition:

$$\min\{\Delta(\mathbf{z}) : \|\mathbf{z}\| = 1\} = \underline{\Delta} > 0. \tag{15}$$

Theorem 2 *Let us assume that $\{W(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^2\}$ satisfies the above conditions and that it is also δ-dependent, $\delta > 0$, that is, $\mathbb{E}\big(W(\mathbf{x})W(\mathbf{y})\big) = 0$ whenever $\|\mathbf{x} - \mathbf{y}\| > \delta$.*

Then, for k small enough:

$$\operatorname{Var}\left(SP(\mathbb{R}^2)\right) \leq \frac{L}{k^2},$$

where L is a positive constant depending upon the law of the random field.

A direct consequence of Theorems 1 and 2 is the following:

Corollary 1 *Under the same hypotheses of Theorem 2, for k small enough, one has:*

$$\frac{\sqrt{\operatorname{Var}\left(SP(\mathbb{R}^2)\right)}}{\mathbb{E}\big(SP(\mathbb{R}^2)\big)} \leq L_1 k$$

where L_1 is a new positive constant.

Proof of Theorem 2. Let us denote $T = SP(\mathbb{R}^2)$. We have:

$$\operatorname{Var}(T) = \mathbb{E}(T(T-1)) + \mathbb{E}(T) - [\mathbb{E}(T)]^2 \tag{16}$$

We have already computed the equivalents as $k \to 0$ of the second and third term in the right-hand side of (16). Our task in what follows is to consider the first term.

The proof is performed using Rice formula for the second factorial moment of the number of roots of the random field Y. We apply Theorem 6.3. of [2] for dimension $d = 2$ and $k = 2$. Then,

$$\begin{aligned}
&\mathbb{E}(T(T-1)) \\
&= \int\int_{\mathbb{R}^2 \times \mathbb{R}^2} \mathbb{E}\Big(|\det \mathbf{Y}'(\mathbf{x})||\det \mathbf{Y}'(\mathbf{y})| \,\Big|\, \mathbf{Y}(\mathbf{x}) = \mathbf{0}, \mathbf{Y}(\mathbf{y}) = \mathbf{0}\Big) p_{\mathbf{Y}(\mathbf{x}),\mathbf{Y}(\mathbf{y})}(\mathbf{0},\mathbf{0})\, d\mathbf{x}d\mathbf{y} \\
&= \int\int_{\|\mathbf{x}-\mathbf{y}\|>\delta} \ldots\, d\mathbf{x}d\mathbf{y} + \int\int_{\|\mathbf{x}-\mathbf{y}\|\le\delta} \ldots\, d\mathbf{x}d\mathbf{y} = J_1 + J_2.
\end{aligned}$$

For J_1 we proceed as in the proof of Theorem 1 of [1], using the δ-dependence and the evaluations therein. We obtain:

$$J_1 = \frac{m_2^2}{k^4} + \frac{O(1)}{k^2}. \tag{17}$$

Let us show that for small k,

$$J_2 = \frac{O(1)}{k^2}. \tag{18}$$

In view of (16), (13) and (17) this suffices to prove the theorem.

We do not perform all detailed computations. The key point consists in evaluating the behavior of the integrand that appears in J_2 near the diagonal $\mathbf{x} = \mathbf{y}$, where the density $p_{\mathbf{Y}(\mathbf{x}),\mathbf{Y}(\mathbf{y})}(\mathbf{0},\mathbf{0})$ degenerates and the conditional expectation tends to zero.

For the density, using the invariance under translations of the law of $W'(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^2$, we have:

$$\begin{aligned} p_{\mathbf{Y}(\mathbf{x}),\mathbf{Y}(\mathbf{y})}(\mathbf{0},\mathbf{0}) &= p_{W'(\mathbf{x}),W'(\mathbf{y})}(k\mathbf{x}, k\mathbf{y}) \\ &= p_{W'(\mathbf{0}),W'(\mathbf{y}-\mathbf{x})}(k\mathbf{x}, k\mathbf{y}) \\ &= p_{W'(\mathbf{0}),[W'(\mathbf{y}-\mathbf{x})-W'(\mathbf{0})]}(k\mathbf{x}, k(\mathbf{y}-\mathbf{x})). \end{aligned}$$

Perform the Taylor expansion, for small $\mathbf{z} = \mathbf{y} - \mathbf{x} \in \mathbb{R}^2$:

$$W'(\mathbf{z}) = W'(\mathbf{0}) + W''(\mathbf{0})\mathbf{z} + O(\|\mathbf{z}\|^2).$$

Using the non-degeneracy assumption (15) and the fact that $W'(\mathbf{0})$ and $W''(\mathbf{0})$ are independent, we can show that for $\mathbf{x}, \mathbf{z} \in \mathbb{R}^2$, $\|\mathbf{z}\| \le \delta$:

$$p_{\mathbf{Y}(\mathbf{x}),\mathbf{Y}(\mathbf{y})}(\mathbf{0},\mathbf{0}) \le \frac{C_1}{\|\mathbf{z}\|^2} \exp\big[-C_2 k^2 (\|\mathbf{x}\| - C_3)^2\big]$$

where C_1, C_2, C_3 are positive constants.

Let us consider the conditional expectation. For each pair $\mathbf{x}, \mathbf{y}$ of different points in $\mathbb{R}^2$, denote by τ the unit vector $(\mathbf{y}-\mathbf{x})/\|\mathbf{y}-\mathbf{x}\|$ and $\mathbf{n}$ a unit vector orthogonal to τ. We denote respectively by $\partial_\tau \mathbf{Y}, \partial_{\tau\tau}\mathbf{Y}, \partial_{\mathbf{n}}\mathbf{Y}$ the first and second partial derivatives of the random field in the directions given by τ and $\mathbf{n}$.

Under the condition

$$\mathbf{Y}(\mathbf{x}) = \mathbf{0}, \mathbf{Y}(\mathbf{y}) = \mathbf{0}$$

we have the following simple bound on the determinant, based upon its definition and Rolle's Theorem applied to the segment $[\mathbf{x}, \mathbf{y}] = \{\lambda\mathbf{x} + (1-\lambda)\mathbf{y}\}$:

$$\big|\det \mathbf{Y}'(\mathbf{x})\big| \le \|\partial_\tau \mathbf{Y}(\mathbf{x})\|\|\partial_{\mathbf{n}}\mathbf{Y}(\mathbf{x})\| \le \|\mathbf{y}-\mathbf{x}\| \sup_{\mathbf{s}\in[\mathbf{x},\mathbf{y}]} \|\partial_{\tau\tau}\mathbf{Y}(\mathbf{s})\|\|\partial_{\mathbf{n}}\mathbf{Y}(\mathbf{x})\| \tag{19}$$

So,

$$\begin{aligned}
&\mathbb{E}\Big(|\det \mathbf{Y}'(\mathbf{x})||\det \mathbf{Y}'(\mathbf{y})|\Big|\ \mathbf{Y}(\mathbf{x})=\mathbf{0},\mathbf{Y}(\mathbf{y})=\mathbf{0}\Big)\\
&\quad\le \|\mathbf{y}-\mathbf{x}\|^2\mathbb{E}\Big[\sup_{\mathbf{s}\in[\mathbf{x},\mathbf{y}]}\|\partial_{\tau\tau}\mathbf{Y}(\mathbf{s})\|^2\|\partial_{\mathbf{n}}\mathbf{Y}(\mathbf{x})\|.\|\partial_{\mathbf{n}}\mathbf{Y}(\mathbf{y})\|\Big|W'(\mathbf{x})=k\mathbf{x},W'(\mathbf{y})=k\mathbf{y}\Big]\\
&\quad= \|\mathbf{z}\|^2\mathbb{E}\Big[\sup_{\mathbf{s}\in[\mathbf{0},\mathbf{z}]}\|\partial_{\tau\tau}\mathbf{Y}(\mathbf{s})\|^2\|\partial_{\mathbf{n}}\mathbf{Y}(\mathbf{0})\|.\|\partial_{\mathbf{n}}\mathbf{Y}(\mathbf{z})\|\Big|W'(\mathbf{0})=k\mathbf{x},\frac{W'(\mathbf{z})-W'(\mathbf{0})}{\|\mathbf{z}\|}=k\tau\Big],
\end{aligned}$$

where the last equality is again a consequence of the stationarity of the random field $\{W(\mathbf{x}):\mathbf{x}\in\mathbb{R}^2\}$.

At this point, we perform a Gaussian regression on the condition. For the condition, use again Taylor expansion, the non-degeneracy hypothesis and the independence of $W'(\mathbf{0})$ and $W''(\mathbf{0})$. Then, use the finiteness of the moments of the supremum of bounded Gaussian processes (see for example [2], Ch. 2), take into account that $\|z\|\le\delta$ to get the inequality:

$$\mathbb{E}\Big(|\det \mathbf{Y}'(\mathbf{x})||\det \mathbf{Y}'(\mathbf{y})|\Big|\ \mathbf{Y}(\mathbf{x})=\mathbf{0},\mathbf{Y}(\mathbf{y})=\mathbf{0}\Big)\le C_4\ \|\mathbf{z}\|^2\big(1+k\|\mathbf{x}\|\big)^4 \tag{20}$$

where C_4 is a positive constant. Summing up, we have the following bound for J_2:

$$\begin{aligned}
J_2 &\le C_1C_4\ \pi\delta^2\int_{\mathbb{R}^2}\big(1+k\|\mathbf{x}\|\big)^4\exp\big[-C_2k^2(\|\mathbf{x}\|-C_3)^2\big]d\mathbf{x}\\
&= C_1C_4\ 2\pi^2\delta^2\int_0^{+\infty}\big(1+k\rho\big)^4\exp\big[-C_2k^2(\rho-C_3)^2\big]\rho d\rho
\end{aligned} \tag{21}$$

Performing the change of variables $w=k\rho$, (18) follows. ∎

3 Dislocation of wave fronts

Dislocations are phase singularities of wavefronts. They correspond to lines of darkness in light propagation, or threads of silence in sound (see Berry and Dennis [3]). In a mathematical framework they can be defined as the loci of points where the amplitude of waves vanishes.

We represent the wave as

$$W(\mathbf{x},t)=\xi(\mathbf{x},t)+i\eta(\mathbf{x},t),\ \text{ where } \mathbf{x}\in\mathbb{R}^d.$$

The dislocations are the intersection of the two random surfaces $\xi(\mathbf{x},t)=0$, $\eta(\mathbf{x},t)=0$. We consider a fixed time, for instance $t=0$.

For random waves, when $d = \mathbf{2}$ we will study the expectation of the random variable

$$\#\{\mathbf{x} \in S : \xi(\mathbf{x}, 0) = \eta(\mathbf{x}, 0) = 0\}.$$

When $d = \mathbf{3}$ one important quantity is the length of the level curve

$$\mathcal{L}\{\mathbf{x} \in S : \xi(\mathbf{x}, 0) = \eta(\mathbf{x}, 0) = 0\}.$$

In what follows, we will re-formulate some results in optics in the standard form of probability theory and give complete proofs for them. A short presentation can be found in [1].

3.1 Mean number of dislocation points

Let us consider a space variable $\mathbf{x}$ in $\mathbb{R}^2$ and a random wave with real part $\xi(\mathbf{x})$ and imaginary part $\eta(\mathbf{x})$. We define $\{\mathbf{Z}(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^2\}$ as the random field taking values in $\mathbb{R}^2$, with coordinates $\xi(\mathbf{x}), \eta(\mathbf{x})$. We assume that these two coordinates are independent centered Gaussian stationary isotropic random fields with C^2-paths and the same distribution. With no loss of generality, we also assume that $\mathrm{Var}(\xi(\mathbf{x})) = 1$.

First, we are interested in the expectation of the number of dislocation points

$$d_2 := \mathbb{E}[\#\{\mathbf{x} \in S : \xi(\mathbf{x}) = \eta(\mathbf{x}) = 0\}],$$

where S is a subset of the parameter space having area equal to 1, for simplicity.

Then, using the Rice formula for Gaussian fields ([2] Theorem 6.2) we get:

$$d_2 = \int_S \mathbb{E}[|\det(\mathbf{Z}'(\mathbf{x}))| \big| \mathbf{Z}(\mathbf{x}) = 0] p_{\mathbf{Z}(\mathbf{x})}(0) dx, \tag{22}$$

where $p_{\mathbf{Z}(\mathbf{x})}(.)$ is the density of $\mathbf{Z}(\mathbf{x})$. One can easily check that this density is non-degenerate. Moreover, one has (use Proposition 6.5. of [2]) $\mathbb{P}(\exists \mathbf{x}, \mathbf{Z}(\mathbf{x}) = 0, \det[\mathbf{Z}'(\mathbf{x}) = 0]) = 0$. These two conditions imply the validity of (22).

Set $\lambda_2 = \mathrm{Var}(\xi_i(\mathbf{x})) = \mathrm{Var}(\eta_i(\mathbf{x}))$, $i = 1, 2$. The stationarity implies, first, that the integrand in (22) is constant and, second, that $\mathbf{Z}(\mathbf{x})$ and $\mathbf{Z}'(\mathbf{x})$ are independent, so that the conditional expectation is in fact an ordinary expectation.

The entries of $\mathbf{Z}'(\mathbf{x})$ are four independent centered Gaussian variables with variance λ_2, so that, up to the factor λ_2, $|\det(\mathbf{Z}'(\mathbf{x}))|$ is the area of the parallelogram generated by two independent standard Gaussian variables in $\mathbb{R}^2$. Using invariance of the distribution, the distribution of this volume is the product of independent square roots of a $\chi^2(2)$ and a $\chi^2(1)$ distributed random variables. An elementary calculation gives then: $\mathbb{E}[|\det(\mathbf{Z}'(\mathbf{x}))|] = \lambda_2$. Finally, we get

$$d_2 = \frac{1}{2\pi} \lambda_2.$$

This quantity is equal to $\frac{K_2}{4\pi}$ in Berry and Dennis [3] notation, giving their formula (4.6).

3.2 Mean length of dislocation curve

Now suppose that the space variable is of dimension **3** and the random field $\{\mathbf{Z}(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^3\}$ satisfies the same hypotheses as in the 2-dimensional parameter case. Generically the dislocation points form a curve $\mathcal{C}$:

$$\mathcal{C} = \{\mathbf{x} : \mathbf{Z}(\mathbf{x}) = 0\}$$

Our aim is to compute for each measurable subset S of $\mathbb{R}^3$:

$$d_3 = \mathbb{E}[\mathcal{L}(\mathcal{C} \cap S)]$$

where $\mathcal{L}$ is the length of the curve which is always defined, at least, as the Hausdorff measure of dimension 1. The Rice formula to be applied is now [2] Th 6.8 that reads

$$d_3 = \int_S \mathbb{E}[(\det \mathbf{Z}'(\mathbf{x})\ \mathbf{Z}'(\mathbf{x})^T)^{1/2} \big| \mathbf{Z}(\mathbf{x}) = 0] p_{\mathbf{Z}(\mathbf{x})}(0) dx].$$

and the verification of the validity is performed in a similar way to the 2-dimensional case above. For simplicity, we may assume again that S has Lebesgue measure equal to 1. The expression can be simplified using the stationarity and the normalization of the variance, to get

$$d_3 = \frac{1}{2\pi} \mathbb{E}[(\det \mathbf{Z}'(\mathbf{x})\mathbf{Z}'(\mathbf{x})^T)^{1/2}],$$

with

$$\mathbb{E}[(\det(\mathbf{Z}'(\mathbf{x})\mathbf{Z}'(\mathbf{x})^T)^{1/2}] = \lambda_2 \mathbb{E}(V),$$

where V is the surface area of the parallelogram generated by two standard Gaussian variables in $\mathbb{R}^3$. The projection method gives

$$\mathbb{E}(V) = \mathbb{E}(XY) = \frac{4}{\sqrt{2\pi}} \sqrt{\frac{\pi}{2}} = 2,$$

Here X and Y are independent and X (resp. Y) is the square root of a $\chi^2(3)$-distributed (resp. $\chi^2(2)$-distributed) random variable.

So,

$$d_3 = \frac{\lambda_2}{\pi}.$$

In Berry and Dennis' notations [3] the last quantity is denoted by $\frac{k_2}{3\pi}$ giving their formula (4.5).

3.3 Variance

In this section, we limit ourselves to dimension **2** and the random field satisfies the hypotheses we introduced to compute the expectation of the number of dislocation points. We further assume that for $\mathbf{s}_1, \mathbf{s}_2 \in \mathbb{R}^2, \mathbf{s}_1 \neq \mathbf{s}_2$ the joint distribution of $\xi(\mathbf{s}_1), \xi(\mathbf{s}_2)$ does not degenerate. Let S be again a measurable subset of $\mathbb{R}^2$ having Lebesgue measure equal to 1.

The variance of the number of dislocations points is an important issue that can be obtained via the second factorial moment of the number of zeroes. More precisely:

$$\operatorname{Var}\left(N_S^{\mathbf{Z}}(\mathbf{0})\right) = \mathbb{E}\big(N_S^{\mathbf{Z}}(\mathbf{0})\big(N_S^{\mathbf{Z}}(\mathbf{0}) - 1\big)\big) + d_2 - d_2^2.$$

and using Theorem 6.3 of [2], we can write the formula:

$$\mathbb{E}\big(N_S^{\mathbf{Z}}(\mathbf{0})\big(N_S^{\mathbf{Z}}(\mathbf{0}) - 1\big)\big) = \int_{S\times S} A(\mathbf{s}_1, \mathbf{s}_2) d\mathbf{s}_1 d\mathbf{s}_2.$$

where

$$A(\mathbf{s}_1, \mathbf{s}_2) = \mathbb{E}\Big(|\det \mathbf{Z}'(\mathbf{s}_1) \det \mathbf{Z}'(\mathbf{s}_2)| \big| \mathbf{Z}(\mathbf{s}_1) = \mathbf{Z}(\mathbf{s}_2) = 0\Big) p_{\mathbf{Z}(\mathbf{s}_1, \mathbf{s}_2)}(0,0)$$

Taking into account that the law of the random field is invariant under translations and orthogonal transformations of $\mathbb{R}^2$, we have

$$A(\mathbf{s}_1, \mathbf{s}_2) = A\big((0,0),(r,0)\big) = A(r) \quad \text{whith } r = \|\mathbf{s}_1 - \mathbf{s}_2\|.$$

The Rice's function $A(r))$ has two intuitive interpretations. First it can be viewed as

$$A(r) = \lim_{\epsilon \to 0} \frac{1}{\pi^2 \epsilon^4} \mathbb{E}\big[N\big(B((0,0),\epsilon)\big) \times N\big(B((r,0),\epsilon)\big)\big].$$

Second it is the density of the Palm distribution (a generalization of horizontal window conditioning of [5]) of the number of zeroes of $\mathbf{Z}$ per unit of surface, locally around the point $(r,0)$ given that there is a zero at $(0,0)$. $A(r)/d_2^2$ is called the "correlation function" in [3].

To compute $A(r)$, we recall that $\xi_1, \xi_2, \eta_1, \eta_2$ denote the partial derivatives of ξ, η with respect to the first and second coordinate. So,

$$\begin{aligned} A(r) &= \mathbb{E}\big[|\det \mathbf{Z}'(0,0) \det \mathbf{Z}'(r,0)| \big| \mathbf{Z}(0,0) = \mathbf{Z}(r,0) = \mathbf{0}_2\big] p_{\mathbf{Z}(0,0),\mathbf{Z}(r,0)}(\mathbf{0}_4) \\ &= \mathbb{E}\Big[\big|\big(\xi_1\eta_2 - \xi_2\eta_1\big)(0,0)\big(\xi_1\eta_2 - \xi_2\eta_1\big)(r,0)\big| \Big| \mathbf{Z}(0,0) = \mathbf{Z}(r,0) = \mathbf{0}_2\Big] \\ &\qquad p_{\mathbf{Z}(0,0),\mathbf{Z}(r,0)}(\mathbf{0}_4) \end{aligned} \tag{23}$$

where $\mathbf{0}_p$ denotes the null vector in dimension p.

The density is easy to compute

$$p_{\mathbf{Z}(0,0),\mathbf{Z}(r,0)}(\mathbf{0}_4) = \frac{1}{(2\pi)^2(1-\rho^2(r))}, \text{ where } \rho(r) = \int_0^\infty J_0(kr)\Pi(dk).$$

Here, J_0 is the Bessel function of the first kind of order 0. The spectral measure μ is invariant under the isometries of $\mathbb{R}^2$, so that the measure Π on $\mathbb{R}^+$ is defined to be such that for every $w \geq 0$, $\mu(\tau : \tau \in \mathbb{R}^2, \|\tau\| \leq w) = 2\pi\Pi([0, w])$.

To compute the conditional expectation of the product of the absolute value of the determinants, we use again the same device as in [3], as well as the same notations. We have:

$$|w| = \frac{1}{\pi}\int_{-\infty}^{+\infty} (1 - \cos(wt)t^{-2}dt. \tag{24}$$

$$\begin{cases} C := \rho(r) \\ E = \rho'(r) \\ H = -E/r \\ F = -\rho\text{''}(r) \\ F_0 = -\rho\text{''}(0) \end{cases}$$

The regression formulas imply that the conditional variance matrix of the vector

$$\mathbf{W} = \Big(\xi_1(\mathbf{0}), \xi_1(r,0), \xi_2(\mathbf{0}), \xi_2(r,0), \eta_1(\mathbf{0}), \eta_1(r,0), \eta_2(\mathbf{0}), \eta_2(r,0)\Big),$$

is given by

$$\Sigma = Diag\Big[\mathcal{A}, \mathcal{B}, \mathcal{A}, \mathcal{B}\Big]$$

with

$$\mathcal{A} = \begin{pmatrix} F_0 - \frac{E^2}{1-C^2} & F - \frac{E^2C}{1-C^2} \\ F - \frac{E^2C}{1-C^2} & F_0 - \frac{E^2}{1-C^2} \end{pmatrix}$$

$$\mathcal{B} = \begin{pmatrix} F_0 & H \\ H & F_0 \end{pmatrix}$$

Using formula (24) the expectation we have to compute is equal to

$$\frac{1}{\pi^2}\int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} dt_2 t_1^{-2} t_2^{-2}\Big[1 - \frac{1}{2}T(t_1, 0) - \frac{1}{2}T(-t_1, 0) - \frac{1}{2}T(0, t_2) - \frac{1}{2}T(0, -t_2)$$
$$+ \frac{1}{4}T(t_1, t_2) + \frac{1}{4}T(-t_1, t_2) + \frac{1}{4}T(t_1, -t_2) + \frac{1}{4}T(-t_1, -t_2)\Big] \tag{25}$$

where

$$T(t_1, t_2) = \mathbb{E}\big[\exp\big(i(w_1t_1 + w_2t_2)\big)\big]$$

with

$$w_1 = \xi_1(\mathbf{0})\eta_2(\mathbf{0}) - \eta_1(\mathbf{0})\xi_2(\mathbf{0}) = \mathbf{W}_1\mathbf{W}_7 - \mathbf{W}_3\mathbf{W}_5$$
$$w_2 = \xi_1(r,0)\eta_2(r,0) - \eta_1(r,0)\xi_2(r,0) = \mathbf{W}_2\mathbf{W}_8 - \mathbf{W}_4\mathbf{W}_6.$$

$T(t_1, t_2) = \mathbb{E}\big(\exp(i\mathbf{W}^T\mathcal{H}\mathbf{W})\big)$ where $\mathbf{W}$ has the distribution $N(0, \Sigma)$ and

$$\mathcal{H} = \begin{bmatrix} 0 & 0 & 0 & \mathcal{D} \\ 0 & 0 & -\mathcal{D} & 0 \\ 0 & -\mathcal{D} & 0 & 0 \\ \mathcal{D} & 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{D} = \frac{1}{2}\left[\begin{array}{cc} t_1 & 0 \\ 0 & t_2 \end{array}\right].$$

A standard diagonalization argument shows that

$$T(t_1,t_2) = \mathbb{E}\big(\exp(i\mathbf{W}^T\mathcal{H}\mathbf{W})\big) = \mathbb{E}\big(\exp(i\sum_{j=1}^{8}\lambda_j\zeta_j^2)\big),$$

where the ζ_j's are independent with standard normal distribution and the λ_j are the eigenvalues of $\Sigma^{1/2}\mathcal{H}\Sigma^{1/2}$. Using the characteristic function of the $\chi^2(1)$ distribution:

$$\mathbb{E}\big(\exp(i\mathbf{W}^T\mathcal{H}\mathbf{W})\big) = \prod_{j=1}^{8}(1-2i\lambda_j)^{-1/2}. \tag{26}$$

Clearly

$$\Sigma^{1/2} = Diag\Big[\mathcal{A}^{1/2},\mathcal{B}^{1/2},\mathcal{A}^{1/2},\mathcal{B}^{1/2}\Big]$$

and

$$\Sigma^{1/2}\mathcal{H}\Sigma^{1/2} = \left[\begin{array}{cccc} 0 & 0 & 0 & \mathcal{M} \\ 0 & 0 & -\mathcal{M}^{\mathcal{T}} & 0 \\ 0 & -\mathcal{M} & 0 & 0 \\ \mathcal{M}^{\mathcal{T}} & 0 & 0 & 0 \end{array}\right]$$

with $\mathcal{M} = \mathcal{A}^{1/2}\mathcal{D}\mathcal{B}^{1/2}$.

Let λ be an eigenvalue of $\Sigma^{1/2}\mathcal{H}\Sigma^{1/2}$. It is easy to check that λ^2 is an eigenvalue of $\mathcal{M}\mathcal{M}^T$. Respectively if λ_1^2 and λ_2^2 are the eigenvalues of $\mathcal{M}\mathcal{M}^T$, those of $\Sigma^{1/2}\mathcal{H}\Sigma^{1/2}$ are $\pm\lambda_1$(twice) and $\pm\lambda_2$ (twice).

Note that λ_1^2 and λ_2^2 are the eigenvalues of $\mathcal{M}\mathcal{M}^T = \mathcal{A}^{1/2}\mathcal{D}\mathcal{B}\mathcal{D}\mathcal{A}^{1/2}$ or equivalently, of $\mathcal{D}\mathcal{B}\mathcal{D}\mathcal{A}$. Using (26)

$$\mathbb{E}\big(\exp(i\mathbf{W}^T\mathcal{H}\mathbf{W})\big) = \big(1{+}4(\lambda_1^2{+}\lambda_2^2){+}16\lambda_1^2\lambda_2^2\big)^{-1} = \big(1{+}4\,tr(\mathcal{D}\mathcal{B}\mathcal{D}\mathcal{A}){+}16\det(\mathcal{D}\mathcal{B}\mathcal{D}\mathcal{A})\big)^{-1}$$

where

$$\mathcal{D}\mathcal{B}\mathcal{D}\mathcal{A} = \frac{1}{4}\left[\begin{array}{cc} t_1^2F_0(F_0-\frac{E^2}{1-C^2}) + t_1t_2H(F-\frac{E^2C}{1-C^2}) & t_1^2F_0(F-\frac{E^2C}{1-C^2}) + t_1t_2H(F_0-\frac{E^2}{1-C^2}) \\ t_1t_2H(F_0-\frac{E^2}{1-C^2}) + t_2^2F_0(F-\frac{E^2C}{1-C^2}) & t_1t_2H(F-\frac{E^2C}{1-C^2}) + t_2^2F_0(F_0-\frac{E^2}{1-C^2}) \end{array}\right]$$

So,

$$4\;tr(\mathcal{D}\mathcal{B}\mathcal{D}\mathcal{A}) = (t_1^2+t_2^2)F_0(F_0-\frac{E^2}{1-C^2}) + 2t_1t_2H(F-\frac{E^2C}{1-C^2}) \tag{27}$$

$$16\det(\mathcal{D}\mathcal{B}\mathcal{D}\mathcal{A}) = t_1^2t_2^2\big[F_0^2-H^2\big]\big[(F_0-\frac{E^2}{1-C^2})^2-(F-\frac{E^2C}{1-C^2})^2\big] \tag{28}$$

giving

$$\begin{aligned} T(t_1,t_2) &= \mathbb{E}\big(\exp(i\mathbf{W}^T\mathcal{H}\mathbf{W})\big) \\ &= \Big(1+(t_1^2+t_2^2)F_0(F_0-\frac{E^2}{1-C^2})+2t_1t_2H(F-\frac{E^2C}{1-C^2}) \\ &\quad + t_1^2t_2^2\big[F_0^2-H^2\big]\big[(F_0-\frac{E^2}{1-C^2})^2-(F-\frac{E^2C}{1-C^2})^2\big]\Big)^{-1} \end{aligned} \tag{29}$$

Performing the change of variable $t' = \sqrt{A_1}t$ with $A_1 = F_0(F_0 - \frac{E^2}{1-C^2})$ the integral (25) becomes

$$\frac{A_1}{\pi^2}\int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} dt_2 t_1^{-2} t_2^{-2}$$

$$\Big[1 - \frac{1}{1+t_1^2}\frac{1}{1+t_2^2} + -\frac{1}{2}\Big\{\frac{1}{1+(t_1^2+t_2^2)-2A_2t_1t_2+t_1^2t_2^2Z} + \frac{1}{1+(t_1^2+t_2^2)+2A_2t_1t_2+t_1^2t_2^2Z}\Big\}\Big]$$

$$\begin{aligned} &= \frac{A_1}{\pi^2}\int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} dt_2 t_1^{-2} t_2^{-2} \\ &\Big[1 - \frac{1}{1+t_1^2} - \frac{1}{1+t_2^2} + \frac{1+(t_1^2+t_2^2)+t_1^2t_2^2Z}{\Big(1+(t_1^2+t_2^2)+t_1^2t_2^2Z\Big)^2 - 4A_2^2t_1^2t_2^2}\Big] \end{aligned} \tag{30}$$

where

$$\begin{cases} A_2 = \frac{H}{F_0}\frac{F(1-C^2)-E^2C}{F_0(1-C^2)-E^2} \\ Z = \frac{F_0^2-H^2}{F_0^2}\Big[1-(F-\frac{E^2C}{1-C^2})^2.(F_0-\frac{E^2}{1-C^2})^{-2}\Big]. \end{cases}$$

In this form, and up to a sign change, this result is equivalent to Formula (4.43) of [3] (note that $A_2^2 = Y$ in [3]).

In order to compute the integral (30), first we obtain

$$\int_{-\infty}^{\infty}\frac{1}{t_2^2}\big[1-\frac{1}{1+t_2^2}\big]dt_2 = \pi.$$

We split the other term into two integrals, thus we have for the first one

$$\frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{t_2^2}\big[\frac{1}{1+(t_1^2+t_2^2)-2A_2t_1t_2+t_1^2t_2^2Z} - \frac{1}{1+t_1^2}\big]dt_2$$

$$\begin{aligned} &= -\frac{1}{2(1+t_1^2)}\int_{-\infty}^{\infty}\frac{1}{t_2^2}\frac{(1+t_1^2Z)t_2^2-2A_2t_1t_2}{1+t_1^2-2A_2t_1t_2+(1+t_1^2Z)t_2^2}dt_2 \\ &= -\frac{1}{2(1+t_1^2)}\int_{-\infty}^{\infty}\frac{1}{t_2^2}\frac{t_2^2-2Z_1t_1t_2}{t_2^2-2Z_1t_1t_2+Z_2}dt_2 = I_1, \end{aligned}$$

where $Z_2 = \frac{1+t_1^2}{1+Zt_1^2}$ and $Z_1 = \frac{A_2}{1+Zt_1^2}$.

Similarly, for the second integral we get

$$\frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{t_2^2}\big[\frac{1}{1+(t_1^2+t_2^2)+2A_2t_1t_2+t_1^2t_2^2Z}-\frac{1}{1+t_1^2}\big]dt_2$$

$$=-\frac{1}{2(1+t_1^2)}\int_{-\infty}^{\infty}\frac{1}{t_2^2}\frac{t_2^2+2Z_1t_1t_2}{t_2^2+2Z_1t_1t_2+Z_2}\,dt_2=I_2$$

$$\begin{aligned}I_1+I_2&=-\frac{1}{2(1+t_1^2)}\int_{-\infty}^{\infty}\frac{1}{t_2^2}\big[\frac{t_2^2-2Z_1t_1t_2}{t_2^2-2Z_1t_1t_2+Z_2}+\frac{t_2^2+2Z_1t_1t_2}{t_2^2+2Z_1t_1t_2+Z_2}\big]dt_2\\&=-\frac{1}{(1+t_1^2)}\int_{-\infty}^{\infty}\frac{t_2^2+(Z_2-4Z_1^2t_1^2)}{t_2^4+2(Z_2-2Z_1^2t_1^2)t_2^2+Z_2^2}\,dt_2\\&=-\frac{1}{(1+t_1^2)}\frac{\pi(Z_2-2Z_1^2t_1^2)}{Z_2\sqrt{(Z_2-Z_1^2t_1^2)}}.\end{aligned}$$

In the third line we have used the formula provided by the method of residues. In fact, if the polynomial X^2-SX+P with $P>0$ has not root in $[0,\infty)$, then

$$\int_{-\infty}^{\infty}\frac{t^2-\gamma}{t^4-St^2+P}\,dt=\frac{\pi}{\sqrt{P(-S+2\sqrt{P})}}(\sqrt{P}-\gamma).$$

In our case $\gamma=-(Z_2-4Z_1^2t_1^2)$, $S=-2(Z_2-2Z_1^2t_1^2)$ and $P=Z_2^2$.

Therefore we get

$$A(r)=\frac{A_1}{4\pi^3(1-C^2)}\int_{-\infty}^{\infty}\frac{1}{t_1^2}\big[1-\frac{1}{(1+t_1^2)}\frac{(Z_2-2Z_1^2t_1^2)}{Z_2\sqrt{(Z_2-Z_1^2t_1^2)}}\big]dt_1.$$

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