On tightness and weak convergence in the approximation of the occupation measure of Fractional Brownian Motion.

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Abstract

In this paper we consider approximations of the occupation measure of the Fractional Brownian motion by means of some functionals defined on regularizations of the paths. In a previous article Berzin and León proved a cylindrical convergence to a Wiener process of conveniently rescaled functionals. Here we show tightness of the approximation in the space of continous functions endowed with the topology of uniform convergence on compact sets. This allows us to simplify the identification of the limit.

Key words: Occupation measure, fractional Brownian motion, limit theorem.

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1 Introduction and notations

Let $\mathcal{X} = \{X(t) : t \in \mathbb{R}\}$ be a standard Fractional Brownian Motion (FBM) with Hurst exponent $H, 0 < H < 1$, that is, a centered Gaussian process with covariance

$$E(X(s)X(t)) = \frac{1}{2}\big[|s|^{2H} + |t|^{2H} - |t-s|^{2H}\big]. \tag{1}$$

In the paper [3] by C. Berzin and J.R. León, the authors study the speed of approximation of the occupation measure of the process FBM by means of certain functionals defined on regularizations of the paths. We start with a description of their results.

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Let $\psi : \mathbb{R} \to \mathbb{R}^+$ be a $\mathcal{C}^1$-kernel with support contained in $[-1, 1]$ and $\int_{-\infty}^{+\infty} \psi(x)dx = 1$. By the "regularized process" we mean the convolution

$$X^\varepsilon(t) = \frac{1}{\varepsilon} \int_{-\infty}^{+\infty} \psi(\frac{t-s}{\varepsilon}) X(s)ds.$$

We set for $t \geq 0, \ \varepsilon > 0$:

$$\zeta_\varepsilon(t) = \frac{\varepsilon^{1-H} \dot{X}^\varepsilon(t)}{\sigma_H}$$

$\dot{X}^\varepsilon$ denotes the derivative of the function X^ε and σ_H is the positive normalizing constant:

$$\sigma_H^2 = -\frac{1}{2} \int_{-1}^{1} \int_{-1}^{1} \dot{\psi}(v)\dot{\psi}(w)|v-w|^{2H} dvdw.$$

Let $g : \mathbb{R} \to \mathbb{R}$ be a function in $L^2(\phi(x)dx)$, where $\phi(x) = (2\pi)^{-1/2} \exp(-x^2/2)$ is the standard normal density. g has an expansion in the Hermite polynomials ($H_n(x) = \exp(x^2/2)(-d/dx)^n \exp(-x^2/2)$, $n = 0, 1, 2, ...$), having the form $g(x) = \sum_{n=N_g}^{+\infty} a_n H_n(x)$ (N_g is called the "Hermite index of g"). With no loss of generality, we assume that $N_g \geq 1$; if this were not the case, we replace the function g by $g - a_0 = g - E[g(\xi)]$ where ξ denotes here and in what follows a standard normal random variable.

We will also assume in what follows that the function g is an even function. This will simplify somewhat our computations. In particular, it implies that $a_1 = 0$, so that $N_g \geq 2$.

Finally, for $t \geq 0$, let us define:

$$S_\varepsilon(t) = \frac{1}{\alpha_{H,N_g}(\varepsilon)} \int_0^t g\big(\zeta_\varepsilon(u)\big)du \tag{2}$$

Define the normalization $\alpha_{H,N_g}(\varepsilon)$ by means of:

1. $\alpha_{H,N_g}(\varepsilon) = \varepsilon^{1/2} \quad if \ \ 0 < H < 1 - \frac{1}{2N_g}$
2. $\alpha_{H,N_g}(\varepsilon) = [\varepsilon \ |\ln(\varepsilon)|]^{1/2} \quad if \ \ H = 1 - \frac{1}{2N_g}$
3. $\alpha_{H,N_g}(\varepsilon) = \varepsilon^{N_g(1-H)} \quad if \ \ 1 - \frac{1}{2N_g} < H < 1$

The main results proved in [3] are contained in the next statement.

Theorem 1. *With the above notations, as* $\varepsilon \to 0$*:*

1. *In cases 1. and 2. the random process* $\{S_\varepsilon(t) : t \geq 0\}$ *converges cylindrically to a known multiple of the Wiener process.*

2. *In case 3, for each t, $S_\varepsilon(t)$ converges in L^2 to a random variable, which can be described by means of a certain Itô-Wiener integral driven by a standard Wiener process which can be defined in terms of the FBM $\mathcal{X}$.*

The following example can be useful to exhibit the interest in this kind of result. Let $f : I \to \mathbb{R}$ be a real-valued function defined on an interval of the line and denote by $N_u(f, I)$ the number of roots lying in I of the equation $f(t) = u$.

Then, for any compact interval I and any continuous function $h : \mathbb{R} \to \mathbb{R}$, almost surely

$$\begin{aligned}\sqrt{\frac{\pi}{2}}\frac{\varepsilon^{1-H}}{\sigma_H}\int_{-\infty}^{+\infty} h(x)N_x\big(X^\varepsilon, I)dx \\ = \sqrt{\frac{\pi}{2}}\int_I h(X^\varepsilon(t))|\zeta_\varepsilon(t)|dt \to \int_I h(X(t))dt = \int_{-\infty}^{+\infty} h(x)\ell_I^X(x)dx.\end{aligned} \tag{3}$$

In the right-hand side of (3), $\ell_I^X(.)$ denotes the local time of the FBM on the interval I, that is, the Radon-Nikodym derivative of the occupation measure $\mu_I(B) = \lambda(\{t \in I : X(t) \in B\})$ with respect to the Lebesgue measure λ. The first equality in (3) holds true for any continuous h and any C^1-function X^ε and its proof is elementary.

In other words, (3) says that, almost surely, as $\varepsilon \to 0$, the normalized number of crossings of the regularized path with the level x, namely:

$$\sqrt{\frac{\pi}{2}}\frac{\varepsilon^{1-H}}{\sigma_H}N_x\big(X^\varepsilon, I)$$

converges (in the above mentioned weak topology) to the local time of the FBM at x. A proof of (3) along with extensions of this kind of results to general classes of random processes, can be found in [2].

Notice that if we put

$$g_0(x) = \sqrt{\frac{\pi}{2}}|x| - 1, \tag{4}$$

(3) can be rewritten as

$$\int_I h(X^\varepsilon(t)g_0(\zeta_\varepsilon(t))dt \to 0 \quad \text{almost surely.} \tag{5}$$

Berzin and León theorem allows to compute the speed of convergence in (5) as well as the limit, in the sense of cylindrical convergence. This is a useful if one is willing to use this kind of result to make statistical inference (for example, on the value of the Hurst exponent H) on the basis of data arising from the observation of the smoothed path. We will keep (4) as a guiding example to understand the conditions that the functions g should verify. Notice that $N_{g_0} = 2$.

There is still a point to be mentioned on this: we have stated these results choosing the function h to be identically equal to 1 (see the definition of $S_\varepsilon(t)$). In fact, this is essentially sufficient for our purpose, since if one knows how to proceed for constant functions, it is not hard to pass to general test functions h having some regularity properties.

The primary aim of this paper is to prove tightness in the space of continuous functions, of the set of processes $\{S_\varepsilon(.) : 0 < \varepsilon < 1\}$.

A proof of tightness has been given in [5] under the additional restriction that the Hermite coefficients of g satisfy:

$$\sum_{n=N_g}^{+\infty} 3^{n/2}\sqrt{n!}|a_n| < \infty.$$

This condition is obviously verified if g is a polynomial, since in this case a_n vanishes when n is larger than the degree. However, it fails to hold in our basic example g_0. In fact, in this case, a_n vanishes for odd n and an elementary computation gives

$$a_{2k} = \frac{(-1)^k}{\sqrt{\pi}2^k k!},$$

which implies that the Chambers and Slud series is divergent. Our main Theorem 2 below states that tightness follows from a general simple condition on the function g which is obviously satisfied by g_0.

A by-product of tightness is that it helps at the same time to simplify substantially the identification of the limit law of the random process $\{S_\varepsilon(t) : t \geq 0\}$ as $\varepsilon \to 0$, when $0 < H \leq 3/4$. In fact, it reduces this problem to a computation of second order moments, instead of the more complicated tools in Berzin-León, based upon Wiener chaos expansions (See Theorem 4).

From a technical point of view, one of the main points in the proof below is that it provides a new method - as far as the authors know - that appears to have an independent interest, when one needs to compute the expectation of crossed moments of functions of Gaussian random variables. The key step in the proof is that the function F defined in (21) below is real-analytic in some neighborhood of the origin as a function of the various covariances, and that one can compute its Taylor expansion.

This paper refers only to Fractional Brownian Motion. In a forthcoming paper, the authors will consider extensions of these results to more general processes, such as multiparameter and multifractal random fields, where the same questions will be adressed, as well as their use in various inference problems.

2 Main theorems.

Theorem 2. *Assume that the function g is even and polynomially bounded (that is,*

$$|g(x)| \le K\ (|x|+1)^M \tag{6}$$

for some positive constants K, M). We assume further that $a_0 = 0$ and $a_2 \neq 0$. Then, for each $T > 0$, the set of random processes $\{S_\varepsilon(t) : 0 \le t \le T\}_{0<\varepsilon<1}$ with the normalization $\alpha_{H,N_g}(\varepsilon)$ given above, is tight in the space $\mathcal{C}\big([0,T],\mathbb{R}\big)$.

Remark Notice that $a_0 = E(g(\xi)), a_2 = E(g(\xi)(\xi^2-1))$. We are assuming that the Hermite index N_g is equal to 2. In fact, a similar proof with minor changes works for $N_g > 2$.

For the proof of Theorem 2 we first prove the following one, which is interesting by itself. In Theorem 3 below we replace the derivatives $\dot{X}^\varepsilon(t)$ of the regularized process, by the quotient of increments of the original process $\frac{X(t+\varepsilon)-X(t)}{\varepsilon}$. This amounts to making the convolution of the path $X(.)$ with the kernel $\psi(x) = \mathbf{1}_{[-1,0]}$ which is not $\mathcal{C}^1$, so that the next statement is not actually included in Theorem 2. However, the proof of Theorem 2 will be an easy adaptation of the one of the next theorem.

Theorem 3. *Assume the same hypotheses of Theorem 2 on the function g. We define:*

$$Z_\varepsilon(t) = \frac{X(t+\varepsilon)-X(t)}{\varepsilon^H}$$

and

$$Y_\varepsilon(t) = \frac{1}{\alpha_{H,2}(\varepsilon)}\int_0^t \big[g(Z_\varepsilon(s))\big]ds.$$

Then, the family of random processes $\{Y_\varepsilon(t) : 0 \le t \le T\}_{0<\varepsilon<1}$, is tight in $\mathcal{C}\big([0,T],\mathbb{R}\big)$.

The next theorem yields the limit of the previous functionals, which is a Wiener process up to a multiplicative constant.

Theorem 4. *Let $0 < H \le 3/4$ and assume the hypotheses of Theorem 2. Let*

$$K_H(g,\psi) = 2\int_0^{+\infty} du \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x)g(y)p(x,y;A_\psi(u))dxdy, \tag{7}$$

where $A_\psi(u) = -\frac{1}{2\sigma_H^2}\int_{-1}^1\int_{-1}^1 \dot{\psi}(v)\dot{\psi}(u+w)|v-w|^{2H}dvdw$ and where $p(x,y;\rho)$ denotes the centered Gaussian density of a pair of random variables with variance 1 and covariance ρ. Let

$$K_H(g) = 2\int_0^{+\infty} du \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x)g(y)p(x,y;A(u))dxdy, \tag{8}$$

where $A(u) = \frac{1}{2}\Big[|u+1|^{2H} + |u-1|^{2H} - 2|u|^{2H}\Big]$, if $0 < H < 3/4$.

If $H = 3/4$ let $K_{3/4}(g) = \frac{3a_2^2}{8}$, where a_2 is the second Hermite coefficient of g.

Then,

(a) As $\varepsilon \to 0$ *the process*

$$\{S_\varepsilon(t) : t \geq 0\}$$

converges weakly in the space $\mathcal{C}\big([0,+\infty),\mathbb{R}\big)$ *to*

$$\{\sqrt{K_H(g,\psi)}W(t) : t \geq 0\},$$

where $\{W(t) : t \geq 0\}$ *is a Wiener process.*

(b) Similarly, $\{Y_\varepsilon(t) : t \geq 0\}$ *converges weakly to*

$$\{\sqrt{K_H(g)}W(t) : t \geq 0\}. \tag{9}$$

3 Proofs

The normalizing constants $\alpha_{H,2}(\varepsilon)$ have been chosen in such a way that $Var(Y_\varepsilon(t))$ has a nice limit behavior as $\varepsilon \to 0$. We start with this calculation, that will also be useful as a preparation to prove tightness.

Proposition 1. *Let us assume that the function g satisfies the hypotheses of Theorem 2.*

As $\varepsilon \to 0$ *we have:*

1. $0 < H < 3/4, \quad Var(Y_\varepsilon(t)) \to K_H(g)t.$
   The constant $K_H(g)$ *is given by formula* (8).
2. *If* $H = 3/4$ *then*
$$Var(Y_\varepsilon(t)) \to \frac{9a_2^2 t}{64}.$$
3. $3/4 < H < 1$
$$Var(Y_\varepsilon(t)) \to \frac{(2H-1)H^2 a_2^2 t^{4H-2}}{8H-6}.$$

Proof. We have:

$$Var(Y_\varepsilon(t)) = \frac{1}{\alpha_{H,2}^2(\varepsilon)} \int_0^t \int_0^t E(\eta_1\eta_2)\, dt_1\, dt_2 \tag{10}$$

where, for $i = 1, 2$ (we will be using the same notation afterwards without further reference):

$$\eta_i = g(Z_\varepsilon(t_i))$$

Consider the integral in the right-hand side of (10):

$$2\int\int_{\{0<t_1<t_2<t\}} g_\varepsilon(t_1,t_2)\ dt_1dt_2 \tag{11}$$

where

$$g_\varepsilon(t_1,t_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x)g(y)\big[p(x,y;\rho_\varepsilon(t_1,t_2)) - p(x,y;0)\big]dxdy \tag{12}$$

Let us recall that $p(x,y;\rho)$ denotes the density of a Gaussian centered pair of random variables, with variance 1 and covariance ρ and let us denote by $\rho_\varepsilon(t_1,t_2)$ the covariance of the pair $(Z_\varepsilon(t_1), Z_\varepsilon(t_2))$. An easy computation shows that

$$E\big(Z_\varepsilon(s)Z_\varepsilon(t)\big) \ = \ A\Big(\frac{t-s}{\varepsilon}\Big), \tag{13}$$

where

$$A(u) = \frac{1}{2}\Big[|u+1|^{2H} + |u-1|^{2H} - 2|u|^{2H}\Big]. \tag{14}$$

Moreover one can check that for $H \neq 1/2$

$$A(u) \sim H(2H-1)\frac{1}{u^{2(1-H)}} \tag{15}$$

when $u \to +\infty$. For the expression in brackets in the integrand of (12), we use the identity (See [7]) :

$$\frac{\partial p}{\partial \rho} = \frac{\partial^2 p}{\partial x \partial y}$$

so that

$$\frac{\partial p}{\partial \rho}(x,y;\rho) = G(x,y;\rho)p(x,y;\rho)$$

where

$$G(x,y;\rho) = \frac{(x-\rho y)(y-\rho x) + \rho(1-\rho^2)}{(1-\rho^2)^2} \tag{16}$$

and differentiating once more:

$$\frac{\partial^2 p}{\partial \rho^2} = p\big(G^2 + \frac{\partial G}{\partial \rho}\big)$$

Now, use a Taylor expansion for the bracket in (12), change variables $t_1 \rightsquigarrow u = (t_2 - t_1)/\varepsilon$ in the integral:

$$
\begin{aligned}
&2\int\int_{\{0<t_1<t_2<t\}} dt_1 dt_2 \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x)g(y)\Big[\int_0^{\rho_\varepsilon}(G^2+\frac{\partial G}{\partial \rho})p(\rho_\varepsilon-\rho)d\rho\Big]dxdy\\
&\quad = 2\varepsilon\int_0^t dt_2\int_0^{t_2/\varepsilon} du\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x)g(y)\Big[\int_0^{A(u)}(G^2+\frac{\partial G}{\partial \rho})p\big[A(u)-\rho\big]d\rho\Big]dxdy \quad (17)\\
&\quad = 2\varepsilon\int_0^t dt_2\int_0^{t_2/\varepsilon} K(u)du
\end{aligned}
$$

where:

$$K(u)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x)g(y)p(x,y;A(u))dxdy.$$

Since the function g is polynomially bounded, it follows that $|K(u)|$ is bounded by some constant K_0.

Let us consider the case $0<H<3/4$.
To prove part 1. of Proposition 1 it suffices to show that

$$\int_0^{+\infty}|K(u)|du<+\infty. \tag{18}$$

This will also imply that $K_H(g)$ in the statement is finite.

Because of (15), one can choose u_0 large enough so that $u\geq u_0$ implies $|A(u)|<1/2$. For $|\rho|<1/2$ one has a polynomial bound on

$$g(x)g(y)\Big(G^2+\frac{\partial G}{\partial \rho}\Big)$$

which does not depend on ρ. This implies that:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\Big|g(x)g(y)\big(G^2+\frac{\partial G}{\partial \rho}\big)\Big|pdxdy\leq K_1,$$

where K_1 is some constant.

Summing up, we have:

$$\int_0^{+\infty}|K(u)|\leq 2K_0u_0+K_1\int_{u_0}^{+\infty}A^2(u)du$$

which is finite, due to the behavior of $A^2(u)$ as $u\to+\infty$ when $0<H<3/4$. This finishes the proof of part 1.

Let us now turn to the proof of parts 2. and 3. of the Proposition.

First, we notice that as $u \to +\infty$, one can apply Lebesgue theorem to get the equivalent of the function $K(u)$:

$$K(u) \sim \Big[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x)g(y)\big(x^2y^2 - x^2 - y^2 + 1\big)p(x,y;0)dxdy\Big]\frac{1}{2}A^2(u)$$
$$= \frac{a_2^2\ A^2(u)}{2}.$$

To finish, replace in the right-hand side of (17), use the equivalent of $A(u)$ as $u \to +\infty$ and the definition of the normalizing constants $\alpha_{H,2}(\varepsilon)$ for $H = 3/4$ and for $3/4 < H < 1$.

□

Remark

When $3/4 < H < 1$, the above already shows tightness. In fact the same computation applies to any interval $[s,t]$ instead of $[0,t]$ and one can choose $L > 0$ large enough, so that if $t - s \geq L\varepsilon$, one has:

$$E\big([Y_\varepsilon(t) - Y_\varepsilon(s)]^2\big) \leq (const)(t-s)^{4H-2}$$

where here and in what follows, "*const*″ denotes a generic constant that may change from line to line.
If $t - s < L\varepsilon$, we have the simple bound

$$E\big([Y_\varepsilon(t) - Y_\varepsilon(s)]^2\big) \leq (const)\frac{(t-s)^2}{\varepsilon^{4(1-H)}} \leq (const)(t-s)^{4H-2}$$

Since $4H - 2 > 1$ Kolmogorov's type criterium shows tightness (see for instance [4].

Proof of Theorem 3

For $3/4 < H < 1$ this has already been proved.

Let $0 < H < 3/4$. Our aim is to obtain an inequality having the form:

$$E\big([Y_\varepsilon(t) - Y_\varepsilon(s)]^4\big) \leq (const)\ (t-s)^2 \tag{19}$$

for $0 \leq s < t \leq T$. On applying Kolmogorov's criterium, the result follows.

We have:

$$E\big([Y_\varepsilon(t) - Y_\varepsilon(s)]^4\big) = \frac{4!}{\varepsilon^2}\int_{\{s<t_1<t_2<t_3<t_4<t\}} E\big(\eta_1\eta_2\eta_3\eta_4\big)\ dt_1dt_2dt_3dt_4. \tag{20}$$

L denotes a large enough constant, that we will choose later on.

Notice that the integrand in the right-hand side of (20) is bounded. Hence, the contribution of the 4-tuples such that at least two different pairs of consecutive t_i's differ less than

$L\varepsilon$ is bounded by $(const)\ \varepsilon^2\ (t-s)^2$.

So, we need appropriate bounds for the parts of the integral corresponding to each one of the following subsets of the domain of integration:

Case A. $t_i - t_{i-1} > L\varepsilon$ for $i = 2, 3, 4$

Case B Exactly two consecutive t_i's differ less than $L\varepsilon$.

We start with **Case A**.

Let us first prove the following lemma.

Lemma 1. *Let (Y_1, Y_2, Y_3, Y_4) be a centered Gaussian vector with covariance matrix $\Sigma = (\rho_{ij})_{1\leq i,j\leq 4}$. Let g be a function of class C^∞ such that the polynomial bound (6) holds and such that the derivatives of g are polynomially bounded. Let us assume that $\rho_{ii} = 1\ \forall i = 1$ to 4, and $\rho = (\rho_{12}, \rho_{13}, ..., \rho_{34})$ be in some neighborhood of the origin of $\mathbb{R}^6$. Let $\gamma_i = g(Y_i)\ \forall i = 1$ to 4, then there exists a C^∞ function F defined on in some neighborhood of the origin of $\mathbb{R}^6$ such that*

$$E\big(\gamma_1\gamma_2\gamma_3\gamma_4\big) = F(\rho_{ij} : 1 \leq i < j \leq 4). \tag{21}$$

Moreover the Taylor expansion of F around the origin is convergent to F for $\|\rho\| \leq \delta$ for some positive δ ($\|\rho\|$ denotes here Euclidean norm in $\mathbb{R}^6$).

Proof of Lemma 1

It is easy to see that F is $\mathcal{C}^\infty$ for ρ in some neighborhood of the origin of $\mathbb{R}^6$.

We denote $p_\Sigma(x_1, x_2, x_3, x_4)$ the centered Gaussian density with covariance Σ and use (as above) the standard identity for $i < j$:

$$\frac{\partial p_\Sigma}{\partial \rho_{ij}} = \frac{\partial^2 p_\Sigma}{\partial x_i \partial x_j}$$

Then (integrate by parts to check the third equality):

$$\begin{aligned}
\frac{\partial F}{\partial \rho_{ij}} &= \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} \Big[\prod_{h=1}^{4} g(x_h)\Big] \frac{\partial p_\Sigma}{\partial \rho_{ij}} dx_1 dx_2 dx_3 dx_4 \\
&= \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} \Big[\prod_{h=1}^{4} g(x_h)\Big] \frac{\partial^2 p_\Sigma}{\partial x_i \partial x_j} dx_1 dx_2 dx_3 dx_4 \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \prod_{h\neq i,j} [g(x_h) dx_h] \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g'(x_i) g'(x_j)\ p_\Sigma(x_1, x_2, x_3, x_4) dx_i dx_j \\
&= E\Big(g'[Y_i] g'[Y_j])\Big] \prod_{h\neq i,j} g(Y_h)\Big)
\end{aligned} \tag{22}$$

where the centered Gaussian vector (Y_1, Y_2, Y_3, Y_4) has covariance Σ.

The same procedure can be used to compute the successive derivatives of F. We get:

$$\frac{\partial^m F}{\partial\rho_{12}^{m_{12}}\partial\rho_{13}^{m_{13}}...\partial\rho_{34}^{m_{34}}} = E\Big(g^{(\nu_1)}(Y_1)g^{(\nu_2)}(Y_2)g^{(\nu_3)}(Y_3)g^{(\nu_4)}(Y_4)\Big) \tag{23}$$

where

- $m_{12}, m_{13}, ..., m_{34}$ are respectively the order of differentiation with respect to the 6 variables $\rho_{12}, ..., \rho_{34}$, so that the total order of differentiation is $m = m_{12} + ... + m_{34}$.
- Y_1, Y_2, Y_3, Y_4 are as above. At the origin $\rho = 0$ these random variables are i.i.d. standard normal.
- ν_i is the number of times that the x_i variable appears under the integral sign, after differentiation. Since x_i appears once each time one differentiates with respect to a variable ρ_{kl} such that either k or l coincide with i, it turns out that ν_i is the sum of the m_{kl} such that either k or l coincide with i, that is:

$$\begin{aligned}\nu_1 &= m_{12} + m_{13} + m_{14}\\ \nu_2 &= m_{12} + m_{23} + m_{24}\\ \nu_3 &= m_{13} + m_{23} + m_{34}\\ \nu_4 &= m_{14} + m_{24} + m_{34}.\end{aligned}$$

Let us now turn to the convergence of the Taylor series of F. Consider the remainder:

$$R_m(\rho) = \sum_{\sum m_{ij}=m} \frac{1}{m_{12}!m_{13}!...m_{34}!}\rho_{12}^{m_{12}}\rho_{13}^{m_{13}}...\rho_{34}^{m_{34}}\frac{\partial^m F}{\partial\rho_{12}^{m_{12}}\partial\rho_{13}^{m_{13}}...\partial\rho_{34}^{m_{34}}}(\theta\rho) \tag{24}$$

with $0 < \theta < 1$. We want to prove that $R_m(\rho) \to 0$ as $m \to +\infty$.

The number of terms in this sum is equal to $\begin{pmatrix} m+5 \\ 5 \end{pmatrix}$.

So, it is enough to show that if $\|\rho\|$ is small enough,

$$\frac{1}{m_{12}!m_{13}!...m_{34}!}\left|\frac{\partial^m F}{\partial\rho_{12}^{m_{12}}\partial\rho_{13}^{m_{13}}...\partial\rho_{34}^{m_{34}}}(\theta\rho)\right| \le L^m \tag{25}$$

for some positive constant L. For this purpose, we need an appropriate upper-bound for the partial derivative in the left-hand side of (25).
Let us prove the following Lemma.

Lemma 2. *Let (Y_1, Y_2, Y_3, Y_4) be a centered Gaussian vector with covariance matrix $\Sigma = (\rho_{ij})_{1\le i,j\le 4}$. Let us assume that $\rho_{ii} = 1\ \forall i = 1$ to 4, and that $\rho = (\rho_{12}, \rho_{13}, ..., \rho_{34})$ be in some neighborhood $\mathcal{N}$ of the origin of $\mathbb{R}^6$. Let g be a function of class C^∞ such that*

$$|g(x)| \le K\ (|x| + 1)^M,$$

and such that the derivatives of g are polynomially bounded. Then there exists a positive constant C depending only on $\mathcal{N}$, K, M such that for all ν_1, ν_2, ν_3, ν_4 positive integers

$$\left|E\Big(g^{(\nu_1)}(Y_1)g^{(\nu_2)}(Y_2)g^{(\nu_3)}(Y_3)g^{(\nu_4)}(Y_4)\Big)\right| \le C^{\nu}\nu^{\nu/2}, \tag{26}$$

where $\nu = \nu_1 + \nu_2 + \nu_3 + \nu_4$.

Proof of Lemma 2

Let us suppose that $\mathcal{N}$ is small enough so that $\det(\Sigma) > 1/2$.

Write:

$$\begin{aligned} &E\Big(g^{(\nu_1)}(Y_1)g^{(\nu_2)}(Y_2)g^{(\nu_3)}(Y_3)g^{(\nu_4)}(Y_4)\Big) \\ &\qquad = \int_{\mathbb{R}^4} g^{(\nu_1)}(x_1)g^{(\nu_2)}(x_2)g^{(\nu_3)}(x_3)g^{(\nu_4)}(x_4)p_\Sigma(x_1,...,x_4)dx_1,...,x_4. \end{aligned} \tag{27}$$

Integrating by parts successively in (27) - integrate the derivatives of g and differentiate p_Σ - we can rewrite the right-hand side of (27) as:

$$\int_{\mathbb{R}^4} g(x_1)g(x_2)g(x_3)g(x_4)\frac{\partial^{\nu}p_\Sigma(x_1,...,x_4)}{\partial^{\nu_1}x_1\partial^{\nu_2}x_2\partial^{\nu_3}x_3\partial^{\nu_4}x_4}dx_1,...,x_4. \tag{28}$$

so that our problem is to obtain an upper-bound for the successive derivatives of the function of four variables

$$Q(x) = \exp\big[-\frac{1}{2}x^T\Sigma^{-1}x\big], \quad x \in R^4.$$

Let $A = \Sigma^{-1/2} = ((a_{ij}))_{i,j=1,2,3,4}$ be a square root of Σ^{-1}, that is $\Sigma^{-1} = A^TA$ and put $y = Ax$. We may assume that $|a_{ij}| \le a$ for some positive constant a.

Let us compute the successive derivatives of $Q(x)$ using the chain rule. The first derivatives are as follows:

$$\frac{\partial Q}{\partial x_h} = -\Big[\sum_{1\le i\le 4} a_{ih}y_i\Big]Q(x)$$

$$\frac{\partial^2 Q}{\partial x_h\partial x_{h'}} = \Big[-\sum_{1\le i\le 4} a_{ih}a_{ih'} + \sum_{1\le i,i'\le 4} a_{ih}a_{i'h'}y_iy_{i'}\Big]Q(x)$$

$$\frac{\partial^3 Q}{\partial x_h\partial x_{h'}\partial x''_h} = \Big[\sum_{1\le i,i'\le 4}\big(a_{ih}a_{i'h'}a_{i'h''}+a_{i'h}a_{ih'}a_{i'h''}+a_{i'h}a_{i'h'}a_{ih''}\big)y_i - \sum_{1\le i,i',i''\le 4} a_{ih}a_{i'h'}a_{i''h''}y_iy_{i'}y_{i''}\Big]Q(x)$$

and so on.

Consider a derivative of order n, say

$$\frac{\partial^n Q}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3} \partial x_4^{\alpha_4}}$$

with $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = n$. It has the form $P_n(y)Q(x)$ where $P_n(y)$ is a polynomial in the four variables y_1, y_2, y_3, y_4 with the following properties:

- $P_n(y) = \sum_{0 \leq k \leq n, n-k \text{ even}} P_{n,k}(y)$.
- Each $P_{n,k}(y)$ is a homogeneous polynomial of degree k which can be written in the following way:
  1. If $k = n$,
  $$P_{n,n}(y_1, ..., y_4) = (-1)^n \sum_{1 \leq i_1, ..., i_n \leq 4} a_{i_1 j_1} .... a_{i_n j_n} y_{i_1} ... y_{i_n} \tag{29}$$
  where $j_1, ..., j_n$ depend on the various orders of differentiation $\alpha_1, ..., \alpha_4$.
  2. If $0 < k < n$,
  $$P_{n,k}(y) = \sum{}^* \sum_{1 \leq i_1, ..., i_k \leq 4} a_{i'_1 j_1} .... a_{i'_n j_n} y_{i_1} ... y_{i_k} \tag{30}$$
  where $j_1, ..., j_n$ depend again on $\alpha_1, ..., \alpha_4$. $\sum{}^*$ denotes a sum with $N_{n,k}$ terms.
  3. If $k = 0$, one has:
  $$P_{n,0}(y) = \sum{}^* \; a_{i_1 j_1} .... a_{i_n j_n} \tag{31}$$
  where $\sum{}^*$ has $N_{n,0}$ terms.

(29) follows from the fact that $P_{n,n}$ is obtained on multiplying $P_{n-1,n-1}$ by the corresponding derivative of the exponent of Q, which is a linear function of y whose coefficients are entries of the matrix A.

$P_{n,0}$ is obtained on differentiating once $P_{n-1,1}$. Each term of this polynomial generates one term in the derivative, so that in (31), the number of terms satisfies

$$N_{n,0} = N_{n-1,1}$$

for each even number n.

Let us look at formula (30). If $0 < k < n+1$ and $n+1-k$ is even, the homogeneous polynomial $P_{n+1,k}$ is obtained as a sum of two parts: first, the one that comes from differentiating once $P_{n,k+1}$, which produces $k+1$ terms, and second, the product of $P_{n,k-1}$ times the derivative of the exponent of Q, which is a linear form in y with coefficients from the matrix A. This implies that:

$$N_{n+1,k} = (k+1)N_{n,k+1} + N_{n,k-1}. \tag{32}$$

From this it follows easily that:

$$N_{n,k} = \frac{n!}{2^{\frac{n-k}{2}}((n-k)/2)!k!}$$

for $0 \leq k \leq n,\ n-k$ even.
If we take A equal to the identity matrix and $\alpha_1 = n$, observe that

$$P_n(x) = (-1)^n \bar{H}_n(x_1),$$

where $\bar{H}_n$ is the the probabilistic Hermite polynomial of degree n, i.e.

$$\bar{H}_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}.$$

(see [1], for example). Hence in this special instance $P_{n,k}$ is the monomial of degree k in $(-1)^n \bar{H}_n$. By definitions (30) and (31) $N_{n,k}$ is obtained by fixing $y = (1,1,1,1)$, so this is exactly the absolute value of the coefficient of the term of degree k of the probabilistic Hermite polynomial of degree n.

So, we obtain the bound:

$$\begin{aligned}\Big|\frac{\partial^\nu Q(x_1,...,x_4)}{\partial^{\nu_1}x_1\partial^{\nu_2}x_2\partial^{\nu_3}x_3\partial^{\nu_4}x_4}\Big| &\leq (const)a^\nu\Big[\sum_{k=0,\ \nu-k \text{ even}}^{\nu} \frac{\nu!}{2^{\frac{\nu-k}{2}}\frac{\nu-k}{2}!k!}\big(\sum_{i=1}^{4}|y_i|\big)^k\Big]Q(x) \\ &\leq b^\nu\Big[\sum_{k=0,\ \nu-k \text{ even}}^{\nu} \frac{\nu!}{2^{\frac{\nu-k}{2}}\frac{\nu-k}{2}!k!}\|y\|^k\big)\Big]Q(x)\end{aligned}$$

where b is some new positive constant.

Let us use this inequality to get an upper-bound for the left-hand side of (27). Performing the change of variables $y = Ax$ in the integral (28), and using the polynomial bound (6) on the function g we can see that:

$$\Big|E\Big(g^{(\nu_1)}(Y_1)g^{(\nu_2)}(Y_2)g^{(\nu_3)}(Y_3)g^{(\nu_4)}(Y_4)\Big)\Big| \leq b_1^\nu \sum_{k=0,\ \nu-k \text{ even}}^{\nu} \frac{\nu!}{2^{\frac{\nu-k}{2}}\frac{\nu-k}{2}!k!}\ E\big(\|\xi_4\|^{k+4M}\big) \tag{33}$$

where ξ_4 is standard normal in R^4. Replacing the expectation by its value, we get for the right-hand side of (33) the bound

$$b_2^\nu \sum_{k=0,\ \nu-k \text{ even}}^{\nu} \frac{\nu!\Gamma(\frac{k}{2})}{\frac{\nu-k}{2}!k!}$$

where b_2 is a new constant (depending on M). Using Stirling's formula and an elementary maximization, we obtain the bound

$$\left|E\Big(g^{(\nu_1)}(Y_1)g^{(\nu_2)}(Y_2)g^{(\nu_3)}(Y_3)g^{(\nu_4)}(Y_4)\Big)\right| \le b_3^{\nu}\nu^{\nu/2},$$

where b_3 is a new constant.
Hence (26) and Lemma 2 are proved.

We are now ready to prove (25). We want to show that the logarithm of the left-hand side is $O(m)$. On applying again Stirling's formula, we obtain:

$$\begin{aligned}
&\log\left\{\frac{1}{m_{12}!m_{13}!...m_{34}!}\left|\frac{\partial^m F}{\partial\rho_{12}^{m_{12}}\partial\rho_{13}^{m_{13}}...\partial\rho_{34}^{m_{34}}}(\theta\rho)\right|\right\}\\
\le\ & m\log m - \sum_{1\le i<j\le 4,\ m_{ij}\ge 1} m_{ij}\log m_{ij} + O(m)\\
\le\ & \sum_{1\le i<j\le 4,\ m_{ij}\ge 1} m_{ij}\big[\log m - \log m_{ij}\big] + O(m)\\
\le\ & \sum_{1\le i<j\le 4,\ m_{ij}\ge 1} \big[m - m_{ij}\big] + O(m) = O(m)
\end{aligned}$$

where the last inequality above follows from the elementary inequality $\log(x+y)-\log(x) \le \frac{y}{x}$, valid for $x, y > 0$.

So, we can conclude that the function F can be expressed in a neighborhood of $\rho = 0$ in terms of its Taylor series, with the obvious advantage that the coefficients are given by the expectation of a product of independent random variables.

Let us recall (20) :

$$E\big([Y_\varepsilon(t) - Y_\varepsilon(s)]^4\big) = \frac{4!}{\varepsilon^2}\int_{\{s<t_1<t_2<t_3<t_4<t\}} E\big(\eta_1\eta_2\eta_3\eta_4\big)\ dt_1dt_2dt_3dt_4.$$

Let us apply Lemma 1 and consider $E\big(\eta_1\eta_2\eta_3\eta_4\big)$ as a function of the covariance matrix $\Sigma = (\rho_{ij})_{1\le i,j\le 4}$, where:

$$\rho_{ij} = E\big(Z_\varepsilon(t_i)Z_\varepsilon(t_j)\big)$$

We know that

- $\rho_{ii} = 1$
- for $i < j$, one has $\rho_{ij} = A(\frac{t_j - t_i}{\varepsilon})$.

In **Case A**, when we choose L large enough, i.e. if the crossed-covariances are small enough, ρ is in a neighborhood of the origin, and Lemma 1 can be applied.

Let us look at formula (23) when $\rho = 0$:

$$\frac{\partial^m F}{\partial\rho_{12}^{m_{12}}\partial\rho_{13}^{m_{13}}...\partial\rho_{34}^{m_{34}}}(0) = \prod_{i=1}^{i=4} E\Big(g^{(\nu_i)}(\xi)\Big) \tag{34}$$

- If some of the ν_i's is equal to 0, then the derivative in (34) is equal to 0.
- If some of the integers ν_i's is odd, then the derivative in (34) is equal to 0, since g is an even function.

  So, in the Taylor expansion of the function F, only terms with even non-zero ν_i's are to be taken into account.
- We have 3 cases of possible non-vanishing coefficients in the Taylor expansion:

  1. At least two m_{kl} are greater or equal than 2.
  2. All the m_{kl} are 0 or 1.
  3. Only one m_{kl} is greater or equal than 2 and the remaining ones smaller or equal than 1.

1. Consider the terms which are or type 1., such as, for example,

$$m_{13} \geq 2, m_{24} \geq 2 \tag{35}$$

Then, in each of the corresponding terms of the Taylor expansion, there appears the derivative of F at zero (computed by (34)), divided by a product of factorials, times the corresponding product of powers $\rho_{12}^{m_{12}} ... \rho_{34}^{m_{34}}$.

Group all the terms in the series having the property (35) and consider their sum. It appears clearly, that it is bounded by $(const)\rho_{13}^2\rho_{24}^2$. Using that the function $A(x)$ is monotone decreasing for $x > 1$, the sum of these terms can be bounded by:

$$(const)A^2(\frac{t_2 - t_1}{\varepsilon})A^2(\frac{t_4 - t_3}{\varepsilon})$$

Now, let us consider the part of the integral in (20) containing this terms and satisfying the condition $t_i - t_{i-1} > L\varepsilon$ for $i = 2, 3, 4$ (recall we are considering this part of the integral). It is bounded above by

$$(const)\int_s^{t-3L\varepsilon} dt_1 \int_{(t_1+L\varepsilon)\wedge t}^{t-2L\varepsilon} dt_2 \int_{(t_2+L\varepsilon)\wedge t}^{t-L\varepsilon} dt_3 \int_{(t_3+L\varepsilon)\wedge t}^{t} A^2((t_2 - t_1)/\varepsilon)A^2((t_4 - t_3)/\varepsilon)dt_4$$
$$\leq (const)\varepsilon^2(t - s)^2 \int_L^{+\infty} A^2(x)dx \int_L^{+\infty} A^2(y)dy$$

where the last inequality follows making the change of variables $x = \frac{t_2-t_1}{\varepsilon}, y = \frac{t_4-t_3}{\varepsilon}$.

The asymptotic (15) plus the fact that $0 < H < 3/4$, so that $4(1 - H) > 1$, imply that $\int_L^{+\infty} A^2(x)dx$ is finite, so that the part of the integral corresponding to the sum of the terms satisfying (35) is bounded by:

$$(const)\varepsilon^2(t-s)^2 \tag{36}$$

One can now easily see that if instead of (35), some other pair of m_{kl}'s are greater or equal than 2, the same computation works, in the sense that the sum of the corresponding terms of the series can be bounded by an expression having the form:

$$(const)A^2(\frac{t_\alpha - t_{\alpha-1}}{\varepsilon})A^2(\frac{t_\beta - t_{\beta-1}}{\varepsilon})$$

where $\alpha \leq \beta - 1$. Proceeding in a similar way, we get a bound with the same form for the integral of the sum of the corresponding terms in the series.

2. Let us consider now the sums of the terms having the form 2. Check that there are only three ways in which all the ν_h's are even and non-zero, i.e. equal to 2. They are the following:

$$a)\ m_{12} = m_{34} = 0, m_{13} = m_{14} = m_{23} = m_{24} = 1,$$
$$b)\ m_{13} = m_{24} = 0, m_{12} = m_{14} = m_{23} = m_{34} = 1,$$
$$c)\ m_{14} = m_{23} = 0, m_{12} = m_{13} = m_{24} = m_{34} = 1.$$

   For the term satisfying $a)$ we have the bound

$$(const)A^2(\frac{t_2 - t_1}{\varepsilon})A^2(\frac{t_3 - t_2}{\varepsilon})$$

   and for the one satisfying $c)$:

$$(const)A^2(\frac{t_2 - t_1}{\varepsilon})A^2(\frac{t_4 - t_3}{\varepsilon})$$

   Then, one can proceed further as in case 1.

   For the sum of the terms satisfying b), one needs a slight change to obtain the bound:

$$(const)A(\frac{t_2 - t_1}{\varepsilon})A(\frac{t_3 - t_2}{\varepsilon})A^2(\frac{t_4 - t_3}{\varepsilon})$$

   and after integration one finds the bound (36) again.

3. For the sum of the terms having the form 3. the procedure is similar. One has again to enumerate the cases and obtains the same bound (36).

Let us now turn to **Case B**. We will show that the bound (36) also holds true for the part of the 4-dimensional integral such that exactly 1 consecutive pair of t_i's differs less than $L\varepsilon$, while the remaining ones differ more that $L\varepsilon$. Of course, the computation for the case when all consecutive pairs differ more than $L\varepsilon$ does not apply here, since there will be a covariance which is near to the value 1 (the one corresponding to neighboring points), so that the matrix Σ becomes nearly singular.

We have three cases, according to which is the pair of consecutive $t_i's$ which differ less than $L\varepsilon$.

Let us assume first that $t_4 - t_3 < L\varepsilon$ and $t_3 - t_2 \geq L\varepsilon,\ t_2 - t_1 \geq L\varepsilon$. We perform the Gaussian regression of the pair $(Z_\varepsilon(t_3), Z_\varepsilon(t_4))$ on the pair $(Z_\varepsilon(t_1), Z_\varepsilon(t_2))$. This gives:

$$\begin{aligned} Z_\varepsilon(t_3) &= \zeta_3 + \lambda_{13} Z_\varepsilon(t_1) + \lambda_{23} Z_\varepsilon(t_2) \\ Z_\varepsilon(t_4) &= \zeta_4 + \lambda_{14} Z_\varepsilon(t_1) + \lambda_{24} Z_\varepsilon(t_2) \end{aligned} \tag{37}$$

where:

- (ζ_3, ζ_4) is independent from $(Z_\varepsilon(t_1), Z_\varepsilon(t_2))$.
- Choosing L large enough, $Var(\zeta_3)$ and $Var(\zeta_4)$ belong to a small neighborhood of 1 and for $k = 1, 2;\ l = 3, 4$:

  $$|\lambda_{kl}| \leq (const) A\big(\frac{t_3 - t_2}{\varepsilon}\big)$$

  and for $l = 3, 4$:

  $$1 - (const) A^2\big(\frac{t_3 - t_2}{\varepsilon}\big) \leq Var(\zeta_l) \leq 1.$$

We get

$$\begin{aligned} &E\big(\eta_1\eta_2\eta_3\eta_4\big) \\ &\quad = E\Big(\eta_1\eta_2\big[\bar{g}(\zeta_3) + \bar{g}'(\zeta_3)[\lambda_{13} Z_\varepsilon(t_1) + \lambda_{23} Z_\varepsilon(t_2)] + R_3\big] \\ &\qquad\qquad \big[\bar{g}(\zeta_4) + \bar{g}'(\zeta_4)[\lambda_{14} Z_\varepsilon(t_1) + \lambda_{24} Z_\varepsilon(t_2)] + R_4\big]\Big) \end{aligned} \tag{38}$$

where the remainders satisfy bounds of the form:

$$|R_3| \leq (const) A^2\big(\frac{t_3 - t_2}{\varepsilon}\big) V_3$$

$$|R_4| \leq (const) A^2\big(\frac{t_3 - t_2}{\varepsilon}\big) V_4,$$

V_3 and V_4 having bounded moments of all orders.

Now we get bounds for the various terms in (38):

- Using independence and the bound we obtained when computing order 2 moments:

  $$\big|E\Big(\eta_1\eta_2\bar{g}(\zeta_3)\bar{g}(\zeta_4)\Big)\big| \leq (const) E\Big(\eta_1\eta_2\Big) \leq (const) A^2\big(\frac{t_2 - t_1}{\varepsilon}\big)$$

  Replacing into the 4-dimensional integral and on account of $t_4 - t_3 < L\varepsilon$, we get for this part the bound $(const)\varepsilon^2(t - s)^2$.

- Next

$$E\Big(\eta_1\eta_2\bar{g}(\zeta_3)\bar{g}'(\zeta_4)[\lambda_{14}Z_\varepsilon(t_1)+\lambda_{24}Z_\varepsilon(t_2)]\Big) \\ = E(\bar{g}(\zeta_3)\bar{g}'(\zeta_4))\big[\lambda_{14}E(\eta_1\eta_2Z_\varepsilon(t_1))+\lambda_{24}E(\eta_1\eta_2Z_\varepsilon(t_2))\big].$$

  Let us look at the first term inside the brackets at the right-hand side of the last equality. To obtain a bound for

$$E(\eta_1\eta_2Z_\varepsilon(t_1))$$

  we proceed in the same form as we did for $E(\eta_1\eta_2)$ with the slight change that we have to replace in our formulas $g(x)$ by $xg(x)$. We obtain the bound:

$$\big|E(\eta_1\eta_2Z_\varepsilon(t_1))\big| \leq (const)A^2\big(\frac{t_2-t_1}{\varepsilon}\big)$$

  The second term is similar.
  Again, replacing in the 4-dimensional integral one obtains the same type of bound.

- The remaining terms can be treated in the same way (easier).

In case it is the pair t_1, t_2 which is the one that satisfies $t_2 - t_1 < L\varepsilon$, the above computation is exactly the same, mutatis mutandis. If it is t_2, t_3, there some slight differences, but everything is similar.

To finish the proof for $0 < H < 3/4$, we still have to remove the added hypothesis that the function g is C^∞ and its derivatives are polynomially bounded. If this does not hold, one can replace g by the convolution $g * \gamma$ where γ is a non-negative function of class C^∞ with total mass equal to 1, and support contained in $[-1, 1]$. One can apply the previous computations to the function $g * \gamma$, the only minor change being that instead of condition (6) we have the inequality $|(g*\gamma)(x)| \leq K\ (|x|+2)^M$. A careful analysis shows that the bounds we have found are uniform on γ, and only depend on the constants K, M, H. Hence, this allows to pass to the limit as γ approaches the Dirac measure, thus finishing the proof of Theorem 3 when $0 < H < 3/4$.

It only remains to prove the statement for $H = 3/4$. We are not going to perform the detailed computations, which are essentially the same as before. One gets the bound

$$E\Big(\big(Y_\varepsilon(t)-Y_\varepsilon(s)\big)^4\Big) \leq (const)(t-s)^2|\ln|t-s||$$

which suffices to prove tightness with Kolmogorov's criterium.□

Proof of Theorem 2.

The proof follows exactly the one of Theorem 3, excepting for the fact that we must provide an upper-bound for the covariance $E\big(\zeta_\varepsilon(s)\zeta_\varepsilon(t)\big)$ instead of the expansion of the function $E\big(Z_\varepsilon(s)Z_\varepsilon(t)\big) = A((t-s)/\varepsilon)$ that we used repeatedly in the proof of Theorem 3. Let us outline the changes. For $0 < H \leq 3/4$

$$Var(S_\varepsilon(t)) = \frac{1}{\alpha^2_{H,2}(\varepsilon)} \int_0^t \int_0^t \; E(\eta_1\eta_2)\; dt_1\; dt_2$$

where $\eta_i = g(\zeta_\varepsilon(t_i))$. If we denote by $\rho_\varepsilon(t_1,t_2) = E\left(\zeta_\varepsilon(t_1)\zeta_\varepsilon(t_2)\right)$, then (12) can still be written

$$g_\varepsilon(t_1,t_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x)g(y)\big[p(x,y;\rho_\varepsilon(t_1,t_2)) - p(x,y;0)\big]dxdy. \tag{39}$$

But (11), becomes

$$Var(S_\varepsilon(t)) = \frac{2\varepsilon}{\alpha^2_{H,2}(\varepsilon)}$$
$$\int_0^t dt_2 \int_0^{t_2/\varepsilon} du \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x)g(y)\Big[\int_0^{\rho_\varepsilon(t_2-\varepsilon u,t_2)} \big(G^2 + \frac{\partial G}{\partial\rho}\big)p\big[\rho_\varepsilon(t_2-\varepsilon u,t_2)-\rho\big]d\rho\Big]dxdy. \tag{40}$$

Please note

$$\begin{aligned}\rho_\varepsilon(t_2-\varepsilon u,t_2) &= \frac{\varepsilon^{2-2H}}{\sigma^2_H}E\left(\dot{X}^\varepsilon(0)\dot{X}^\varepsilon(\varepsilon u)\right)\\ &= \frac{\varepsilon^{-2-2H}}{\sigma^2_H}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\dot{\psi}(\frac{-v}{\varepsilon})\dot{\psi}(u-\frac{w}{\varepsilon})E\left(X_vX_w\right)dvdw\\ &= \frac{1}{\sigma^2_H}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\dot{\psi}(v)\dot{\psi}(w)E\left(X_{-v}X_{u-w}\right)dvdw\end{aligned}$$

by using self-similarity of X. Hence we get

$$\rho_\varepsilon(t_2-\varepsilon u,t_2) = A_\psi(u) \tag{41}$$

because of (1) and of $\int_{-\infty}^{+\infty}\dot{\psi}(v)dv = 0$. Then

$$Var(S_\varepsilon(t)) = \frac{2\varepsilon}{\alpha^2_{H,2}(\varepsilon)}$$
$$\int_0^t dt_2 \int_0^{t_2/\varepsilon} du \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x)g(y)\Big[\int_0^{A_\psi(u)} \big(G^2 + \frac{\partial G}{\partial\rho}\big)p\big[A_\psi(u)-\rho\big]d\rho\Big]dxdy. \tag{42}$$

Next we rely on Proposition 2.1 (e) in [2] to get the asymptotic of $A_\psi(u)$ when $u \to +\infty$ analogous to (15). Otherwise the proof of Theorem 2 is similar to that of Theorem 3.

Proof of Theorem 4.

Part (a) follows from part (b), in a similar way as Theorem 2 follows from Theorem 3. So, we only prove part (b).

Using tightness, proved in Theorem 3, it suffices to show that if ε_n is a decreasing sequence of real numbers tending to zero, and Y_{ε_n} converges weakly to the process $\mathcal{Y}^* = \{Y^*(t) : t \geq 0\}$ in the space $\mathcal{C}\big([0,+\infty),\mathbb{R}\big)$, then $\mathcal{Y}^*$ has the law of

$$\{\sqrt{K_H(g)}W(t) : t \geq 0\}.$$

To prove this, let us introduce the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ generated by the FBM $\mathcal{X}$, that is, for every $t \geq 0$, $\mathcal{F}_t$ is the σ-algebra generated by the random variables $\{X(s) : s \leq t\}$. We will prove that the process $\mathcal{Y}^*$, which has continuous paths, is an $\mathcal{F}_t$-martingale with quadratic variation $K_H(g)t$. A classical characterization of the Wiener process, due to Paul Lévy (see for example [I-W], Theorem 6.1., Chapter II), implies the theorem.

It suffices to show that if $0 < s < t$, for any choice of the positive integer k, $\tau_1, ..., \tau_k$ pairwise different parameter values strictly smaller than s, and $F : \mathbb{R}^k \to \mathbb{R}$ any bounded continuous function, we have the following equalities:

$$E\Big(\big(Y^*(t) - Y^*(s)\big)\ F\big[X(\tau_1), ..., X(\tau_k)\big]\Big) = 0 \tag{43}$$

and

$$E\Big(\big(Y^*(t) - Y^*(s)\big)^2\ F\big[X(\tau_1), ..., X(\tau_k)\big]\Big) = K_H(g)(t-s)E\Big(F\big[X(\tau_1), ..., X(\tau_k)\big]\Big). \tag{44}$$

Because of a well-known theorem due to Skorokhod (see [6] Chapter 1, Theorem 2.7), we may assume that we have chosen the probability space, so that, almost surely, convergence of $Y_{\varepsilon_n}(.)$ to $Y^*(.)$ is uniform on each compact interval of the positive axis.

We prove (43) and (44) on the basis of computations that are not far away from what we have done to compute the asymptotic variance of $Y_\varepsilon(t)$.

Let us prove (43) for $0 < H < 3/4$. It suffices to show that

$$\lim_{\varepsilon\to 0} E\Big(\big(Y_\varepsilon(t) - Y_\varepsilon(s)\big)\ F\big[X(\tau_1), ..., X(\tau_k)\big]\Big) = 0, \tag{45}$$

since the random variables under the expectation sign are bounded in L^4 of the probability space.

We have:

$$
\begin{aligned}
&E\Big(\big(Y_\varepsilon(t)-Y_\varepsilon(s)\big)\ F\big[X(\tau_1),...,X(\tau_k)\big]\Big)\\
&=\frac{1}{\sqrt{\varepsilon}}\int_s^t E\Big(g\big(Z_\varepsilon(u)\big)F\big[X(\tau_1),...,X(\tau_k)\big]\Big)du \qquad (46)\\
&=\frac{1}{\sqrt{\varepsilon}}\int_s^t du\int_{R^{k+1}} g(x)F(x_1,...,x_k)p_{\Sigma_\varepsilon(u)}(x,x_1,...,x_k)\ dxdx_1...dx_k,
\end{aligned}
$$

where p_Σ denotes the centered Gaussian density with variance matrix Σ and in our case, $\Sigma_\varepsilon(u)$ is the variance matrix of the vector $\big(Z_\varepsilon(u),X(\tau_1),...,X(\tau_k)\big)$.

As ε varies, the density in the integrand of the right-hand side of (46) is a function of the covariances

$$\rho_{j,\varepsilon}(u)=E\big(Z_\varepsilon(u)X(\tau_j)\big)\quad(\text{for } j=1,...,k)$$

since the other elements of Σ remain constant.

Let us consider the following Taylor expansion of the density $p_{\Sigma_\varepsilon(u)}(x,x_1,...,x_k)$ as a function of these k covariances, around the value 0 for all of them, namely:

$$
\begin{aligned}
p_{\Sigma_\varepsilon(u)}(x,x_1,...,x_k)=\ & p_{Z_\varepsilon(u)}(x)p_{X(\tau_1),...,X(\tau_k)}(x_1,...,x_k)\\
&+\sum_{j=1}^k\frac{\partial p_\Sigma}{\partial\rho_j}(x,x_1,...,x_k)\Big|_{\rho_j=0}\rho_{j,\varepsilon}(u) \qquad (47)\\
&+\frac{1}{2}\sum_{j,j'=1}^k\frac{\partial^2 p_\Sigma}{\partial\rho_j\partial\rho'_j}(x,x_1,...,x_k)\Big|_{\rho_j=\theta\rho_{j,\varepsilon}(u)}\rho_{j,\varepsilon}(u)\rho_{j',\varepsilon}(u),
\end{aligned}
$$

where $0<\theta<1$.

From the definition of the FBM:

$$\rho_{j,\varepsilon}(u)=\frac{1}{2\varepsilon^H}\Big[(u+\varepsilon)^{2H}-u^{2H}-(u+\varepsilon-\tau_j)^{2H}+(u-\tau_j)^{2H}\Big],$$

so that for any $u\in[s,t]$ and any $j=1,...,k$, since $\tau_1,...,\tau_k$ are strictly on the left of s, we have:

$$\big|\rho_{j,\varepsilon}(u)\big|\leq\ (const)\ \varepsilon^{1-H}, \qquad (48)$$

where the constant depends only on $H,s,t,\tau_1,...,\tau_k$.

We plug the expansion (47) into (46). Clearly:

$$\int_{R^{k+1}} g(x)F(x_1,...,x_k)p_{Z_\varepsilon(u)}(x)p_{X(\tau_1),...,X(\tau_k)}(x_1,...,x_k)dxdx_1...dx_k=0$$

given that $Z_\varepsilon(u)$ is standard normal and the conditions on g.

For the next term in the Taylor expansion, let us check that for $j = 1, ..., k$

$$\frac{\partial p_\Sigma}{\partial \rho_j}(x, x_1, ..., x_k)\Big|_{\rho_j=0}$$

is an odd function of x for fixed $x_1, ..., x_k$. For that purpose, we use the standard Gaussian identity for $j = 1, ..., k$:

$$\frac{\partial p_\Sigma}{\partial \rho_j} = \frac{\partial^2 p_\Sigma}{\partial x \partial x_j}$$

Denote $\Sigma^{-1} = ((\sigma^{ij}))_{i,j=0,1,...,k}$, where $0, 1, ..., k$ correspond respectively to the random variables $Z_\varepsilon(u), X(\tau_1), ..., X(\tau_k)$. Notice that for $j = 1, ..., k$, $\rho_j = 0$ implies $\sigma^{0j} = 0$. Then, a direct computation gives for $j = 1, ..., k$:

$$\frac{\partial^2 p_\Sigma}{\partial x \partial x_j}\Big|_{\rho_j=0} = p_{Z_\varepsilon(u)}(x) p_{X(\tau_1),...,X(\tau_k)}(x_1, ..., x_k)\; x\Big(\sum_{i=1}^{k} x_i \sigma^{ij}\Big),$$

which is an odd function of x for fixed $x_1, ..., x_k$. Since g is even, it follows that:

$$\int_{R^{k+1}} g(x) F(x_1, ..., x_k) \sum_{j=1}^{k} \frac{\partial p_\Sigma}{\partial \rho_j}(x, x_1, ..., x_k)\Big|_{\rho_j=0} dx dx_1 ... dx_k \;=\; 0$$

On account of (48), (47) and the above calculations, we get from (46):

$$\Big|E\Big(\big(Y_\varepsilon(t) - Y_\varepsilon(s)\big)\; F\big[X(\tau_1), ..., X(\tau_k)\big]\Big)\Big| \;\leq\; (const)\varepsilon^{2(1-H)-1/2}.$$

Since $0 < H < 3/4$, this implies (45). In case $H = 3/4$ the proof is similar, with only minor changes.

Let us now turn to the proof of (44). We have to prove that:

$$E\Big(\big(Y_\varepsilon(t) - Y_\varepsilon(s)\big)^2\; F\big[X(\tau_1), ..., X(\tau_k)\big]\Big) \to K_H(g)(t-s) E\Big(F\big[X(\tau_1), ..., X(\tau_k)\big]\Big).$$

This follows the same lines of the proof of (43), with minor changes. We have:

$$E\Big(\big(Y_\varepsilon(t) - Y_\varepsilon(s)\big)^2\; F\big[X(\tau_1), ..., X(\tau_k)\big]\Big) = \frac{2}{\alpha^2_{H,2}(\varepsilon)} \times$$
$$\int_s^t dt_2 \int_0^{(t_2-s)/\varepsilon} \varepsilon du \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x)g(y)F(x_1, ..., x_k) p(x, y, x_1, ..., x_k; \Gamma_\varepsilon) dx dy dx_1, ... dx_k \tag{49}$$

In (49), $p(x, y, x_1, ..., x_k; \Gamma)$ is the centered Gaussian density with covariance Γ and the $(k+2) \times (k+2)$ matrix Γ_ε is:

$$\Gamma_\varepsilon = \begin{pmatrix} C(u) & R_\varepsilon \\ R_\varepsilon & \Sigma(\tau_1, ..., \tau_k) \end{pmatrix}$$

where:

- $C(u) = \begin{pmatrix} 1 & A(u) \\ A(u) & 1 \end{pmatrix}$
- $\Sigma(\tau_1, ..., \tau_k) = \big(\big(E\big[X(\tau_i)X(\tau_j)\big]\big)\big)_{i,j=1,...,k}$
- $R_\varepsilon = \big(\big(\rho_{j,\varepsilon}(t_i)\big)\big)_{j=1,...,k}$, where for $j = 1, ..., k$ $\rho_{j,\varepsilon}(t_1) = E\big(Z_\varepsilon(t_2 + \varepsilon u)X(\tau_j)$ and $\rho_{j,\varepsilon}(t_2) = E\big(Z_\varepsilon(t_2)X(\tau_j)$.

To pass to the limit as $\varepsilon \to 0$ in (49) we apply a similar expansion to the one in the proof of Proposition 1, and use the bound (48), valid for ε small enough: where the *const* depends on $H, s, t, \tau_1, ..., \tau_k$ but not on ε. □

References

[1] Milton Abramowitz and Irene A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55 of *National Bureau of Standards Applied Mathematics Series.* For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 1964.

[2] Jean-Marc Azaïs and Mario Wschebor. Almost sure oscillation of certain random processes. *Bernoulli*, 2(3):257–270, 1996.

[3] Corinne Berzin and José R. León. Convergence in fractional models and applications. *Electron. J. Probab.*, 10: 326–370 (electronic), 2005.

[4] Patrick Billingsley. *Convergence of probability measures.* John Wiley & Sons Inc., New York, 1968.

[5] Daniel Chambers and Eric Slud. Necessary conditions for nonlinear functionals of Gaussian processes to satisfy central limit theorems. *Stochastic Process. Appl.*, 32(1):93–107, 1989.

[6] Nobuyuki Ikeda and Shinzo Watanabe. *Stochastic differential equations and diffusion processes*, volume 24 of *North-Holland Mathematical Library.* North-Holland Publishing Co., Amsterdam, 1981.

[7] R. L. Plackett. A reduction formula for normal multivariate integrals. *Biometrika*, 41:351–360, 1954.