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The Brandt module of ternary quadratic lattices

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The Brandt module of ternary quadratic lattices

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This work grew from the love and support of my wife Jimena and my kids Camila and Agustín. All of my accomplishments are dedicated to them, for being with me in this adventure.

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The Brandt module of ternary quadratic lattices

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Given a positive definite ternary quadratic lattice Λ, we construct a free module $\mathcal{M}(\Lambda)$ with Hecke action, together with a family of Hecke-linear maps ϑ_l from $\mathcal{M}(\Lambda)$ to certain spaces of modular forms of half integral weight. The latter are given explicitly by a new kind of generalized theta series, which we prove to be modular with level independent of l. Key to our work is the introduction of a refinement to the classic notion of proper equivalence of lattices.

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Notation

Symbol	**Meaning**	**Page**
$(\Lambda : \Gamma)$	for $\Gamma \subseteq \Lambda$, the index	31
$(\boldsymbol{v}, \boldsymbol{u})_l$	the l-symbol	49
(x, y, z)	a vector in $\mathbb{Q}^3$ by its coordinates	27
$\left(\frac{\cdot}{p}\right)$	quadratic character modulo p	21
$\| \ \|$	norm in a quadratic space	9
$\|V\|$, $\|\Lambda\|$	image of the norm map	15, 19
$\langle \, , \rangle$	bilinear form in a quadratic space	9
$\langle \Lambda, \Gamma \rangle$	image of the bilinear form map	31
$\sharp\, S$	cardinality of S	38
$\{\boldsymbol{v}\}^{\perp}$	orthogonal complement of $\boldsymbol{v}$	11
$\sim$	proper equivalence	18
$\simeq$	Θ-equivalence	21
$[\Lambda]$	Θ-class of Λ	21
$[\Gamma]^s$, $\Gamma^{(s)}$	action of $s \in \theta(V)$ on $\mathscr{C}(\Lambda)$	22, 37
$\left\langle\!\left\langle [\Gamma], [\Gamma'] \right\rangle\!\right\rangle$	inner product in $\mathcal{M}_{\mathbb{R}}(\Lambda)$	38

Symbol	**Meaning**	**Page**
1_V	identity autometry of V	13
$\mathbb{C}$	the field of complex numbers	46
$\mathscr{C}(\Lambda)$	Θ-classes properly equivalent to Λ	21
$\det V$	determinant of the quadratic space V	10
$\det(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$	determinant of a basis	10
$\det \sigma$	determinant of the autometry σ	10
$\operatorname{disc} \Lambda$, D	discriminant of Λ	19, 48
$e(z)$	$e^{2\pi i z}$	57
$\mathbb{F}_p$	finite field of p elements	53
$\mathfrak{H}$	upper half plane	61
I	an integral domain in k	17
$I^\times$	group of units of I	19
$j_n(\gamma, z)$	automorphy factor of weight $n/2$	68
k	a field of characteristic $\neq 2$	9
$k^\times$	multiplicative group of k	10
l	an odd squarefree integer	47, 57
$\mathcal{M}(\Lambda)$	the Brandt module of Λ	38
$\mathcal{M}_{\mathbb{R}}(\Lambda)$	the Brandt module with coefficients in $\mathbb{R}$	38
$M_{n/2}(N, \chi)$	a space of modular forms of weight $n/2$	48, 68
$N(\Lambda)$	level of Λ	47, 58
n	dimension of V	9
$O(V)$, $O(\Lambda)$	orthogonal group	10, 18
$O^+(V)$, $O^+(\Lambda)$	proper orthogonal group	11, 18
P	a spherical function of order ν	61

Symbol	Meaning	Page
p	a prime number	23
$\mathbb{Q}$	field of rational numbers	16
$\mathbb{Q}^\times$	multiplicative group of non-zero rationals	16
$\mathbb{Q}_p$	field of p-adic numbers	23
$\mathbb{Q}_{>0}$	positive rational numbers	16
q	$e^{2\pi i z}$	47
$\mathbb{R}$	field of real numbers	38
$r_l(\Gamma, m)$	weighted count of vectors of norm ml	47
$SL_2(\mathbb{Z})$	the modular group	64
$T(p^2)$	the Hecke operators on $M_{3/2}$	53
t_p	the Hecke operators on $\mathcal{M}(\Lambda)$	38
$\mathscr{U}(V_p)$	proper autometries of unit spinor norm	24
V	a quadratic space	9
V_p	localization of V	23
$\mathbb{Z}$	domain of rational integers	19
$\mathbb{Z}_p$	domain of p-adic integers	19
$\mathbb{Z}_p^\times$	group of p-adic units	20
z	variable in $\mathfrak{H}$	61
$\Gamma_0(N)$	congruence subgroup of level N	68
Δ	determinant of Λ	61
ε_d	1 or i according as $d \equiv 1$ or $3 \pmod 4$	68
$\Theta(V)$	proper autometries of spinor norm 1	14
$\theta(\Lambda, \Gamma)$	θ-distance between Λ and Γ	23
$\theta(\sigma)$	spinor norm of the autometry σ	14

Symbol	Meaning	Page
$\theta(V)$, $\theta(\Lambda)$	image of the spinor norm map	15, 18
$\vartheta(z;\Lambda,\mathbf{h})$	usual theta series of Λ	61
$\vartheta_{\mathfrak{l}}$	generalized theta series map	48
$\vartheta_{\mathfrak{l}}(z;\Lambda,\omega,\mathbf{h})$	generalized theta series of Λ	47, 61
Λ, Γ	quadratic lattices in V	17
$\Lambda^{\#}$	dual lattice of Λ	32
Λ_p, Γ_p	$\mathbb{Z}_p$-lattices, localization of Λ, Γ	20, 23
$\Lambda_{\boldsymbol{v}}$	neighboring lattice of Λ	32
ν	the order (degree) of P	61
σ, ρ	autometries of V	10
$\tau_{\boldsymbol{v}}$	The symmetry in the direction of $\boldsymbol{v}$	11
χ_Λ	character of Λ	47, 67
ω	an $\mathfrak{l}$-function	46
$\widehat{\omega}$	the dual $\mathfrak{l}$-function	57
ω_Λ	a weight function on Λ	46

Introduction

The "Basis Problem" for spaces of modular forms, as formulated by Eichler (1973, p. 77), seeks "to give bases of linearly independent forms of these spaces which are arithmetically distinguished and whose Fourier series are known or easy to obtain." For spaces of cusp forms of integral weight $k \geq 2$, the problem was first solved by Eichler in many cases; the most complete solution is given in Hijikata, Pizer, and Shemanske (1989).

The nice result of Shimura (1973) on the correspondence of modular forms of half integral weight with those of even weight, leads naturally to consider the Basis Problem for spaces of cusp forms of half integral weight $(k+1)/2 \geq 3/2$. The formula of Waldspurger (1981) relates the Fourier coefficients of cusp forms g of half integral weight with the central values of a certain family of modular L-series, namely those associated to (quadratic twists of) the even weight cusp form in Shimura correspondence with g.

To emphasize the geometric and arithmetic importance of said central values, the Conjecture of Birch and Swinnerton-Dyer should be mentioned; a wonderful example of its application is the work of Tunnell (1983) on the congruent number problem. Thus, constructions that make explicit the *inverse* Shimura correspondence are of enormous interest, from both theoretical and computational points of view. For instance, there are some conjectures on the frequency of vanishing at the center of such families of L-series predicted by random matrix theory (Conrey, Keating, Rubinstein, and Snaith, 2002). In this context, the computational aspects are quite important.

Essentially, to obtain the simplest of Eichler's results on the Basis Problem (Eichler, 1955), one has to study the arithmetic of a maximal order R in the quaternion algebra ramified precisely at an odd prime p and ∞. Let $\mathcal{M}(R)$ be the free module with basis the left R-ideal classes. This module has a Hecke action given by the Brandt matrices, and the eigenvectors for them are in one to one correspondence with the eigenforms of weight 2 and level p.

A key idea of Gross (1987) is to consider a linear map ϑ from $\mathcal{M}(R)$ to a certain space of ternary theta series—modular forms of weight $3/2$ and level $4p$—by mapping a left R-ideal class to a ternary lattice corresponding to its *right* order. Through this map the Hecke operators in weight $3/2$ act by the Brandt matrices, and thus this is a concrete realization of the inverse Shimura correspondence.

However, this map ϑ is not in general injective: precisely when the central value $L(f, 1) = 0$ for an eigenform f, the corresponding eigenvector in $\mathcal{M}(R)$ is in the kernel of ϑ. This problem has been completely overcome with the aid of certain *weight functions* for maximal orders, by constructing a family of maps ϑ_l with no common kernel (Mao, Rodriguez-Villegas, and Tornaría, 2004). The analogues of these weight functions in the context of ternary lattices will play an important role in the present work.

Generalizations of Gross's construction have been obtained by Böcherer and Schulze-Pillot (1990, 1994) for odd squarefree levels and for higher weights $\equiv 2 \pmod 4$, and more recently by Pacetti and Tornaría (2004) for level p^2; an example of weight 4 has been constructed by Rosson and Tornaría (2005). A general framework has been developed by Pacetti and Tornaría (2005) that permits, in principle, to extend Gross's construction to any quaternion order, provided one knows how to compute representatives for the ideal classes. The case of level p^2 already exhibits the difficulties of applying this method to non-squarefree levels.

A different construction is discussed by Birch (1991); see also Ponomarev (1981) and Lehman (1997). For a positive integer D, he denotes by $\mathcal{M}(D)$ the free module with basis the set of classes of positive definite ternary quadratic forms with integral coefficients and discriminant D. The method of neighboring lattices of Kneser (1957) provides a Hecke action on $\mathcal{M}(D)$, and the natural map ϑ from $\mathcal{M}(D)$ to ternary theta series turns out to be Hecke-linear.

The similarities between the approaches are clear; they all rely in the explicit construction of certain *Brandt modules*: free modules with a Hecke action, together with one—or more—Hecke-linear maps ϑ from $\mathcal{M}$ to certain spaces of modular forms of half integral weight, given explicitly by theta series.

In view of the correspondence between ternary quadratic forms and quaternion orders (Brandt, 1943; Llorente, 2000), one expects the two kinds of constructions to be related. Indeed it is clear that Gross's map factors through the latter construction. However, the usual equivalence of quadratic forms is too coarse; classes of ternary forms correspond to types of orders, while the Brandt module $\mathcal{M}(R)$ has classes of ideals as a basis. For instance, the eigenvectors in $\mathcal{M}(p)$ are in one to one correspondence with those eigenforms of weight 2 and level p *whose* L*-series have a* $+$ *sign in their functional equation.* Besides, the maps ϑ_l mentioned above are not well defined for classes of ternary quadratic forms.

Our program is to give an explicit construction of a new Brandt module for positive definite ternary *lattices*, together with a family of maps ϑ_ι to *generalized theta series*. To overcome the limitations of the above construction, we start by refining the notion of equivalence of quadratic lattices to that of Θ-equivalence. I believe this is, in the case of ternary lattices, akin to the introduction of proper equivalence in the case of binary quadratic forms by Gauss (1801), which is *essential* in his beautiful theory of composition. In that respect, it might even lead to a theory of composition of ternary lattices.

We now give an outline of the content of this thesis:

Chapter 1 is preliminary. In Section 1.1 we set up the notation for quadratic spaces that will be used throughout the thesis. In Section 1.2 we discuss symmetries of a quadratic space, which we use in Section 1.3 to recall the definition of the spinor norm of an autometry. Everything here is classic and can be found e.g. in Cassels (1978).

Chapter 2 introduces the notion of Θ-equivalence, which is a refinement to the classic notion of proper equivalence. Despite its simplicity, it is the key ingredient that enables our theories of weight functions and generalized theta series discussed in Chapters 4 and 5.

Section 2.1 is introductory. In Section 2.2 we give the definition of Θ-equivalence, and that of θ-distance between properly equivalent lattices. In Section 2.3 we develop the localized notions of Θ-genus and $\mathscr{U}$-genus. While the number of Θ-classes in a class is infinite, we prove (Proposition 2.7) the finiteness of the number of Θ-classes in a given $\mathscr{U}$-genus (cf. the diagram in page 24). On the other hand, in dimension at least 3, a $\mathscr{U}$-genus has representatives for every proper class in its genus (Proposition 2.8).

Section 2.4 provides examples to illustrate the similarities and differences between the "classes in genus" and "Θ-classes in $\mathscr{U}$-genus" approaches. For instance, if p is an odd prime, there is exactly one genus of ternary lattices of discriminant p. It is well known that the number of classes in said genus equals the dimension of the space $M_2^+(p)$ of weight 2 modular forms of level p and trivial character whose L-series have a $+$ sign in their functional equation. The number of Θ-classes on a $\mathscr{U}$-genus in this genus, on the other hand, will equal the dimension of the whole space $M_2(p)$.

Chapter 3 introduces the Brandt module of ternary lattices. The novelty here relies on two observations: first, that neighboring lattices are defined for actual lattices, not classes, and so we can work with Θ-classes. Second, that with a slight modification, neighboring lattices are in the same $\mathscr{U}$-genus. In the case of ternary lattices of discriminant p, the Brandt module thus constructed will be isomorphic, as Hecke modules, to the whole of $M_2(p)$.

Section 3.1 reviews Kneser's construction of neighboring lattices. The presentation is ours, and unifies the case of $p = 2$ and $p \mid \operatorname{disc} \Lambda$, which are usually discussed separately or not discussed at all. Compare our concise Theorem 3.5 with assertions (vi) and (vii) in Schulze-Pillot (1991, p. 135)—note that all the complexity in Schulze-Pillot treatment arises for the case $p = 2$, and for the most part he assumes $p \nmid \operatorname{disc} \Lambda$.

Section 3.2 starts by computing the local θ-distance between neighboring lattices, the fairly important Proposition 3.6. As a corollary, neighboring lattices are in the same genus; this result is usually stated with the proviso that $p^2 \nmid \operatorname{disc} \Lambda$ and $p \neq 2$. More important for our purposes, up to acting by an autometry with spinor norm p, they are in the same $\mathscr{U}$-genus. We then define $\mathcal{M}(\Lambda)$ and the operators t_p, and easily conclude that $\mathcal{M}(\Lambda)$ has a basis of simultaneous eigenvectors for the t_p (Proposition 3.8).

Section 3.3 finishes the examples started in Section 2.4, by computing a basis of eigenvectors and comparing to modular forms of weight 2.

Chapter 4 relates the Brandt module with modular forms of half-integral weight and the Shimura correspondence. I learnt about weight functions from Zhengyu Mao, in the context of maximal orders in quaternion algebras and the Weil representation. The construction in the context of ternary lattices, as well as the definition of the l-symbol, are mine, inspired in the above. The relation with the spinor norm (Proposition 4.5) is what led to my definition of Θ-equivalence.

Sections 4.1 and 4.2 define weight functions on a lattice. We use weight functions to define the generalized theta series $\vartheta_l(\Lambda)$. For $l = 1$ this is the usual theta series, which depends only on the class of Λ. For general l it depends on the Θ-class of Λ. We show how to *transport* a given weight function to other lattices in its $\mathscr{U}$-genus. Thus, by fixing a weight function of conductor l on Λ we obtain a linear map $\vartheta_l : \mathcal{M}(\Lambda) \to M_{n/2}(N, \chi)$, where N is the level of Λ and χ is the quadratic character corresponding to the discriminant of Λ.

Sections 4.3 and 4.4 are restricted to Λ of dimension 3 and l relatively prime to $\operatorname{disc} \Lambda$. In Section 4.3 we define the l-symbol for l-vectors in a ternary lattice Λ, prove its properties, and use it to construct nontrivial weight functions on Λ of conductor l (Corollary 4.6). This is done for odd primes $l \nmid \operatorname{disc} \Lambda$, but it is easy to extend to any odd squarefree l relatively prime to $\operatorname{disc} \Lambda$.

In Section 4.4 we prove that the map $\vartheta_l : \mathcal{M}(\Lambda) \to M_{3/2}(N, \chi)$ preserves the action of the Hecke operators (Theorem 4.8). In other words, since $\mathcal{M}(\Lambda)$ corresponds to modular forms of weight 2, these maps give explicit inverses for the Shimura correspondence.

Chapter 5 is independent of the rest. There we prove that the generalized theta series $\vartheta_l(\Lambda)$ are modular form of weight $n/2$, with level and character independent of l (Theorem 5.11). The proof mimics that given by Shimura (1973), with extra complications arising from the use of weight functions.

1. The spinor norm

In this preliminary chapter we will set up the notation for quadratic spaces. The focus is on the orthogonal group; the main objective is to recall the definition of the spinor norm.

1.1 Quadratic spaces

Let k be a field of characteristic $\neq 2$. A *quadratic space* over k is a vector space V of finite dimension $n \geq 1$ together with a mapping $\| \ \| : V \to k$ such that

1. $\|x\boldsymbol{v}\| = x^2 \|\boldsymbol{v}\|$ for $x \in k$, $\boldsymbol{v} \in V$,
2. $\langle \boldsymbol{v}, \boldsymbol{v}' \rangle := \|\boldsymbol{v} + \boldsymbol{v}'\| - \|\boldsymbol{v}\| - \|\boldsymbol{v}'\|$ is a symmetric bilinear form.

Note that

$$\|\boldsymbol{v}\| := \tfrac{1}{2}\langle \boldsymbol{v}, \boldsymbol{v} \rangle,$$

so that $\| \ \|$ is determined by the bilinear form. We will always assume that V is *regular*, i.e. the bilinear form is non-degenerate.

If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a basis of V, its *determinant* is defined by

$$\det(\mathbf{v}_1, \ldots, \mathbf{v}_n) := \det\Big(\langle \mathbf{v}_i, \mathbf{v}_j \rangle\Big)_{\substack{1 \le i \le n \\ 1 \le j \le n}}.$$

Since V is regular, this determinant is nonzero. If $\mathbf{v}'_1, \ldots, \mathbf{v}'_n$ is another basis of V, then

$$\det(\mathbf{v}'_1, \ldots, \mathbf{v}'_n) = x^2 \det(\mathbf{v}_1, \ldots, \mathbf{v}_n),$$

where $x \in k^\times$ is the determinant of the matrix of change of basis. Therefore, the determinant of any basis of V is a well defined element of $k^\times/(k^\times)^2$, which we call the *determinant* of V, and denote by $\det V$.

An *autometry* of V is a linear map $\sigma : V \to V$ which preserves the quadratic structure of V, i.e.

$$\|\sigma \mathbf{v}\| = \|\mathbf{v}\| \qquad \text{for } \mathbf{v} \in V.$$

The *orthogonal group* $O(V)$ is the group of all autometries of V under composition.

Lemma 1.1. *If* $\sigma \in O(V)$ *then* $\det \sigma = \pm 1$.

Proof. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be a basis of V. Note that

$$\det(\mathbf{v}_1, \ldots, \mathbf{v}_n) = \det(\sigma \mathbf{v}_1, \ldots, \sigma \mathbf{v}_n) = (\det \sigma)^2 \det(\mathbf{v}_1, \ldots, \mathbf{v}_n).$$

Since $\det(\mathbf{v}_1, \ldots, \mathbf{v}_n) \neq 0$ it follows that $(\det \sigma)^2 = 1$. $\square$

We say that an autometry σ is *proper* if $\det \sigma = +1$, and *improper* if $\det \sigma = -1$. The proper autometries form a subgroup of $O(V)$, the *proper orthogonal group*, which we denote $O^+(V)$. In the next section we will see that there are improper autometries. Thus the map

$$\det : O(V) \to \pm 1$$

is a surjective group homomorphism with kernel $O^+(V)$; hence $O^+(V)$ is a normal subgroup of $O(V)$ of index 2.

1.2 Symmetries

A vector $\mathbf{v} \in V$ is called *anisotropic* if $\|\mathbf{v}\| \neq 0$. There are anisotropic vectors in V, since otherwise $\langle\, , \rangle$ would be identically 0 and V would not be regular. If $\mathbf{v}$ is anisotropic, there is an orthogonal decomposition

$$V = k\mathbf{v} + \{\mathbf{v}\}^{\perp},$$

where $\{\mathbf{v}\}^{\perp} := \{\mathbf{u} \in V \ : \ \langle \mathbf{u}, \mathbf{v} \rangle = 0\}$. Hence, there is a unique autometry $\tau_{\mathbf{v}} \in O(V)$ such that

$$\begin{aligned} \tau_{\mathbf{v}}\, \mathbf{v} &= -\mathbf{v}, \\ \tau_{\mathbf{v}}\, \mathbf{u} &= \mathbf{u} \qquad \text{if } \langle \mathbf{u}, \mathbf{v} \rangle = 0. \end{aligned}$$

This is called a *symmetry* of V; it is given explicitly by

$$\tau_{\mathbf{v}}\, \mathbf{u} := \mathbf{u} - \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{v}\|}\, \mathbf{v}.$$

It is clear that $\det \tau_{\mathbf{v}} = -1$.

Lemma 1.2. *Let* $\mathbf{u}, \mathbf{v} \in V$ *such that* $\|\mathbf{u}\| = \|\mathbf{v}\|$ *and* $\|\mathbf{u} - \mathbf{v}\| \neq 0$. *Then*

$$\tau_{\mathbf{u}-\mathbf{v}}\, \mathbf{u} = \mathbf{v}.$$

Proof. Note that

$$\begin{aligned} \|\mathbf{u} - \mathbf{v}\| &= \|\mathbf{u}\| + \|\mathbf{v}\| - \langle \mathbf{u}, \mathbf{v} \rangle = 2\,\|\mathbf{u}\| - \langle \mathbf{u}, \mathbf{v} \rangle \\ &= \langle \mathbf{u}, \mathbf{u} \rangle - \langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{u} - \mathbf{v} \rangle. \end{aligned}$$

Thus, using the definition of $\tau_{\mathbf{u}-\mathbf{v}}$ we get

$$\tau_{\mathbf{u}-\mathbf{v}}\, \mathbf{u} = \mathbf{u} - \frac{\langle \mathbf{u}, \mathbf{u} - \mathbf{v} \rangle}{\|\mathbf{u} - \mathbf{v}\|}\,(\mathbf{u} - \mathbf{v}) = \mathbf{u} - (\mathbf{u} - \mathbf{v}) = \mathbf{v},$$

as claimed. □

Proposition 1.3. *If* $\mathbf{u}, \mathbf{v} \in V$ *are such that* $\|\mathbf{u}\| = \|\mathbf{v}\| \neq 0$, *then there is* $\sigma \in O(V)$ *such that* $\sigma\mathbf{u} = \mathbf{v}$. *Moreover, if* $n > 1$ *we can take* σ *to be the product of exactly two symmetries.*

Proof. Note that $\|\mathbf{u} + \mathbf{v}\| + \|\mathbf{u} - \mathbf{v}\| = 2\,\|\mathbf{u}\| + 2\,\|\mathbf{v}\| = 4\,\|\mathbf{u}\| \neq 0$. If $\|\mathbf{u} + \mathbf{v}\| \neq 0$, take $\sigma = \tau_{\mathbf{v}}\,\tau_{\mathbf{u}+\mathbf{v}}$. Otherwise, we must have $\|\mathbf{u} - \mathbf{v}\| \neq 0$, and we can take $\sigma = \tau_{\mathbf{u}-\mathbf{v}}$. If $n > 1$ there must be some $\mathbf{w} \in V$ orthogonal to $\mathbf{u}$ with $\|\mathbf{w}\| \neq 0$, and we can take $\sigma = \tau_{\mathbf{u}-\mathbf{v}}\,\tau_{\mathbf{w}}$. □

Proposition 1.4. *Every autometry of V is product of symmetries.*

Proof. Let $\sigma \in O(V)$ and let $\mathbf{v} \in V$ such that $\|\mathbf{v}\| \neq 0$. By Proposition 1.3, there is a product of symmetries ρ such that $\rho\sigma\mathbf{v} = \mathbf{v}$. But then $\rho\sigma$ is an autometry of $\{\mathbf{v}\}^{\perp}$, and the result follows by induction on the dimension of V. □

Remark. The above proves that σ is product of at most $2n$ symmetries. It is possible to prove that σ is the product of at most n symmetries; see Example 8 in Cassels (1978, p. 30).

Lemma 1.5. *Let $\mathbf{v}_1, \dots, \mathbf{v}_s$ be anisotropic vectors in V. If*

$$\tau_{\mathbf{v}_1} \cdots \tau_{\mathbf{v}_s} = 1_V$$

then $\|\mathbf{v}_1\| \cdots \|\mathbf{v}_s\| \in (k^\times)^2$.

Proof. Since $\det 1_V = +1$, we have s even. In the Clifford algebra of V, the action of the autometry $\tau_{\mathbf{v}_1} \cdots \tau_{\mathbf{v}_s}$ is given by conjugation by $\mathbf{u} = \mathbf{v}_1 \cdots \mathbf{v}_s$. Thus $\mathbf{u}$ is an even element in the center of the Clifford algebra, hence $\mathbf{u} \in k^\times$ and it follows that $\|\mathbf{u}\| = \|\mathbf{v}_1\| \cdots \|\mathbf{v}_s\| \in (k^\times)^2$. For details see Chapter 10 in Cassels (1978). □

1.3 The spinor norm

Consider an autometry $\sigma \in O(V)$. By Proposition 1.4 we can write σ as a product of symmetries

$$\sigma = \tau_{\mathbf{v}_1} \cdots \tau_{\mathbf{v}_s}.$$

The *spinor norm* of σ is defined by

$$\theta(\sigma) := \|\mathbf{v}_1\| \cdots \|\mathbf{v}_s\|$$

Proposition 1.6. *The map $\theta : O(V) \to k^\times/(k^\times)^2$ is a well defined homomorphism of groups.*

Proof. Suppose that also

$$\sigma = \tau_{\mathbf{u}_1} \cdots \tau_{\mathbf{u}_r},$$

then

$$\tau_{\mathbf{v}_1} \cdots \tau_{\mathbf{v}_s} \tau_{\mathbf{u}_r} \cdots \tau_{\mathbf{u}_1} = \sigma\,\sigma^{-1} = 1_V,$$

and it follows from Lemma 1.5 that

$$\|\mathbf{v}_1\| \cdots \|\mathbf{v}_s\|\,(k^\times)^2 = \|\mathbf{u}_1\| \cdots \|\mathbf{u}_r\|\,(k^\times)^2.$$

It is clear that θ is an homomorphism. □

We are interested in the restriction

$$\theta : O^+(V) \to k^\times/(k^\times)^2.$$

The kernel of this restriction will be denoted by $\Theta(V)$, that is

$$\Theta(V) := \{\sigma \in O^+(V) \;:\; \theta(\sigma) = 1\}.$$

It is clear that $\Theta(V)$ contains the commutator subgroup of $O(V)$, since θ maps to an Abelian group.

Proposition 1.7. *If $n \leq 3$, $\Theta(V)$ is the commutator subgroup of $O(V)$.*

Proof. We prove that any $\sigma \in \Theta(V)$ is a commutator. By Proposition 1.4, and since $\sigma \in O^+(V)$, we can write $\sigma = \tau_{\mathbf{u}}\tau_{\mathbf{v}}$ with $\mathbf{u}, \mathbf{v} \in V$ anisotropic. Now, since $\theta(\sigma) = 1$, it follows that $\|\mathbf{u}\|\,\|\mathbf{v}\| \in (k^\times)^2$, and we can assume after scaling $\mathbf{v}$ that $\|\mathbf{u}\| = \|\mathbf{v}\|$. By Proposition 1.3 there is some autometry $\rho \in O(V)$ such that $\rho\mathbf{u} = \mathbf{v}$, hence $\sigma = \tau_{\mathbf{u}}\,\rho\,\tau_{\mathbf{u}}\,\rho^{-1} = \tau_{\mathbf{u}}\,\rho\,\tau_{\mathbf{u}}^{-1}\,\rho^{-1}$ is a commutator. □

We denote the image of $O^+(V)$ under θ by $\theta(V)$, i.e.

$$\theta(V) := \{\theta(\sigma) \;:\; \sigma \in O^+(V)\} \subseteq k^\times/(k^\times)^2.$$

When k is an ordered field, we will say that V is *definite* if $\theta(V) > 0$. This definition is related to the usual one in view of

Lemma 1.8. *The following are equivalent*

1. *V is definite.*
2. *Either $\|V\| \geq 0$ or $\|V\| \leq 0$.*

Proof. Suppose first that $\theta(V) > 0$. Then $\|\mathbf{v}_1\|\,\|\mathbf{v}_2\| > 0$ for every pair of anisotropic vectors $\mathbf{v}_1, \mathbf{v}_2 \in V$, and it follows that either $\|V\| \geq 0$ or $\|V\| \leq 0$.

Conversely, assume that all anisotropic vectors in V have norm with the same sign, and let $\sigma = \tau_{v_1} \cdots \tau_{v_s} \in O^+(V)$. Since s is even, $\theta(\sigma) = \|v_1\| \cdots \|v_s\| > 0$, hence $\theta(V) > 0$. □

For $k = \mathbb{Q}$, we have the following

Proposition 1.9. *Let V be a quadratic space over $\mathbb{Q}$ of dimension at least 3. Then*

$$\theta(V) = \begin{cases} \mathbb{Q}^\times & \text{if } V \text{ is indefinite,} \\ \mathbb{Q}_{>0} & \text{if } V \text{ is definite.} \end{cases}$$

Proof. This is Lemma 3.2 in Cassels (1978, p. 207). □

2. Θ-classes and $\mathscr{U}$-genera

In this chapter we introduce the notion of Θ-equivalence of quadratic lattices, which is a refinement to the classic notion of proper equivalence.

In Section 2.3 we develop the local notion of Θ-genus. This is not quite what we need; we will replace it by the slightly coarser notion of $\mathscr{U}$-genus of a quadratic lattice. Then we will prove that the number of Θ-classes in a given $\mathscr{U}$-genus is finite, and that a $\mathscr{U}$-genus of lattices of dimension at least 3 has representatives for every proper class in its genus.

2.1 Quadratic lattices

Let I be an integral domain contained in k. A (quadratic) I-*lattice* Λ in a quadratic space V over k is a finitely generated I-submodule of V such that $\Lambda \otimes_{\mathrm{I}} \mathrm{k} = \mathrm{V}$.

Two lattices Λ and Γ in V are *equivalent* if

$$\sigma\Lambda = \Gamma, \qquad \text{some } \sigma \in O(V).$$

The lattices Λ and Γ are said to be *properly equivalent*, denoted $\Lambda \sim \Gamma$, if the above holds with some $\sigma \in O^+(V)$. This refinement to the notion of equivalence is due to Gauss (1801, §157), and is key to enable his beautiful theory of composition of binary quadratic forms.

An *autometry* of Λ is $\sigma \in O(V)$ such that $\sigma\Lambda = \Lambda$. The set of autometries of Λ is a subgroup of $O(V)$, which will be denoted $O(\Lambda)$, that is

$$O(\Lambda) := \{\sigma \in O(V) \ : \ \sigma\Lambda = \Lambda\}.$$

The *proper autometries* of Λ form a subgroup

$$O^+(\Lambda) := O(\Lambda) \cap O^+(V) = \{\sigma \in O^+(V) \ : \ \sigma\Lambda = \Lambda\},$$

of index 1 or 2 in $O(\Lambda)$ according to whether $O(\Lambda) \subseteq O^+(V)$ or not. When $O(\Lambda)$ contains an improper autometry, we say that Λ is *ambiguous*. In that case, any lattice equivalent to Λ will be properly equivalent to Λ. For example, when the dimension of V is odd every lattice is ambiguous, since $-1_V \in O(\Lambda)$ is improper, hence the notions of equivalence and proper equivalence coincide.

We will also need

$$\theta(\Lambda) := \{\theta(\sigma) \ : \ \sigma \in O^+(\Lambda)\} \subseteq k^\times/(k^\times)^2.$$

Lemma 2.1. *If Λ and Γ are equivalent, then*

$$\theta(\Lambda) = \theta(\Gamma).$$

Proof. If $\sigma\Lambda = \Gamma$, then

$$O^{+}(\Lambda) = \left\{\sigma^{-1}\rho\sigma \ : \ \rho \in O^{+}(\Gamma)\right\},$$

and $\theta(\sigma^{-1}\rho\sigma) = \theta(\rho)$. $\square$

An I-lattice Λ is *free* if there is a basis $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ of V such that

$$\Lambda = I\boldsymbol{v}_1 + \cdots + I\boldsymbol{v}_n.$$

In this case we say that $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ is a basis of Λ, and define the *discriminant* of Λ to be

$$\operatorname{disc}\Lambda := \begin{cases} (-1)^{n/2}\det(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n) & \text{if } n \text{ is even,} \\ \frac{1}{2}\det(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n) & \text{if } n \text{ is odd.} \end{cases}$$

Note that this is a well defined element of $k^{\times}/(I^{\times})^2$, since the matrix that changes this basis to another basis of Λ will be an invertible matrix with coefficients from I, and so its determinant will be in $I^{\times}$.

An I-lattice Λ is said to be *integral* if $\|\Lambda\| \subseteq I$. If Λ is integral and free, we have $\operatorname{disc}\Lambda \in I$. If Λ is integral and $\operatorname{disc}\Lambda \in I^{\times}$, we say that Λ is *unimodular*.

Note that when I is a principal ideal domain (e.g. $\mathbb{Z}$ or $\mathbb{Z}_p$), the structure theorem for I-modules implies that every I-lattice is free, since a vector space has no torsion.

Lemma 2.2. *Let Λ_p be an integral $\mathbb{Z}_p$-lattice of dimension n, and assume*

$$p_0^{\frac{n(n-1)}{2}} \nmid \operatorname{disc} \Lambda_p, \qquad \textit{with } p_0 = \begin{cases} p & \textit{if } p \textit{ odd,} \\ 4 & \textit{if } p = 2. \end{cases}$$

Then

$$\theta(\Lambda_p) \supseteq \mathbb{Z}_p^\times (\mathbb{Q}_p^\times)^2.$$

Proof. This is Theorem 20 (a) and (c) in Conway and Sloane (1993). Note that the proof does not require Λ_p to be indefinite for those parts. □

Lemma 2.3 ($p \neq 2$)**.** *If Λ_p is a unimodular $\mathbb{Z}_p$-lattice, then every autometry $\sigma \in O(\Lambda_p)$ is a product of symmetries $\tau_{\boldsymbol{v}}$ where $\boldsymbol{v} \in \Lambda_p$ and $\|\boldsymbol{v}\| \in \mathbb{Z}_p^\times$.*

Proof. This is Corollary 2 in Cassels (1978, p. 115). □

Lemma 2.4. *Let Λ_p be a unimodular $\mathbb{Z}_p$-lattice of dimension at least 2. Then*

$$\theta(\Lambda_p) = \mathbb{Z}_p^\times (\mathbb{Q}_p^\times)^2.$$

Proof. Lemma 2.2 applies, so that $\theta(\Lambda_p) \supseteq \mathbb{Z}_p^\times (\mathbb{Q}_p^\times)^2$. The other inclusion, for $p \neq 2$, follows directly from Lemma 2.3. In general, it can be proved using Theorems 55 and 56 in Watson (1960, pp. 90–93); see also Theorem 18 in Conway and Sloane (1993) and the algorithm following it. □

Lemma 2.5 ($p \neq 2$). *Let Λ_p be a unimodular $\mathbb{Z}_p$-lattice of dimension at least 3. If $v \in \Lambda_p$ is such that $\|v\| \in \mathbb{Z}_p^\times$, then there is some $\sigma \in O^+(\Lambda_p)$ such that $\sigma v = v$ and $\left(\frac{\theta(\sigma)}{p}\right) = -1$.*

Proof. Let $\widetilde{\Lambda_p} := \{u \in \Lambda_p : \langle u, v\rangle = 0\}$. It is not hard to see that, since $p \nmid \langle v, v\rangle$, there is an orthogonal decomposition

$$\Lambda_p = \mathbb{Z}_p v + \widetilde{\Lambda_p}.$$

Hence $\widetilde{\Lambda_p}$ is unimodular of dimension at least 2. By Lemma 2.4 there is $\sigma \in O^+(\widetilde{\Lambda_p})$ such that $\left(\frac{\theta(\sigma)}{p}\right) = -1$. We now extend σ to Λ_p by setting $\sigma v := v$. $\square$

2.2 Θ-equivalence

We will further refine the notion of equivalence of lattices; we say that Λ and Γ are Θ-*equivalent*, denoted $\Lambda \simeq \Gamma$, provided

$$\sigma\Lambda = \Gamma, \qquad \text{some } \sigma \in \Theta(V).$$

This is clearly an equivalence relation: the Θ-*class* of Λ will be denoted by $[\Lambda]$, namely

$$[\Lambda] := \{\Gamma : \Gamma \simeq \Lambda\} = \{\sigma\Lambda : \sigma \in \Theta(V)\}.$$

To understand the extent of this refinement, we are led to consider the set

$$\mathscr{C}(\Lambda) := \big\{[\Gamma] : \Gamma \sim \Lambda\big\},$$

consisting of the Θ-classes of lattices properly equivalent to Λ. There is a transitive action of $\theta(V)$ on $\mathscr{C}(\Lambda)$ given by

$$[\Gamma]^{\theta(\sigma)} := [\sigma\Gamma]$$

for $\sigma \in O^+(V)$. This is well defined: if $\theta(\rho) = \theta(\sigma)$ for another $\rho \in O^+(V)$, then

$$\rho\Gamma = (\rho\sigma^{-1})\sigma\Gamma \simeq \sigma\Gamma,$$

because $\rho\sigma^{-1} \in \Theta(V)$. Similarly one sees that if $\Gamma' \simeq \Gamma$, then $\sigma\Gamma' \simeq \sigma\Gamma$.

Proposition 2.6. *The set* $\mathscr{C}(\Lambda)$ *is a principal homogeneous space (or torsor) for* $\theta(V)/\theta(\Lambda)$.

Proof. We have already shown that $\theta(V)$ acts transitively on $\mathscr{C}(\Lambda)$. We prove that for $\Gamma \sim \Lambda$ we have $[\Gamma]^s = [\Gamma]$ if and only if $s \in \theta(\Lambda)$. By Lemma 2.1, $\theta(\Lambda) = \theta(\Gamma)$. If $s = \theta(\sigma)$, with $\sigma \in O^+(\Gamma)$, then

$$[\Gamma]^s = [\sigma\Gamma] = [\Gamma].$$

Conversely, if $[\Gamma]^s = [\Gamma]$, with $s = \theta(\sigma) \in \theta(V)$, it follows that $\rho\sigma\Gamma = \Gamma$ for some $\rho \in \Theta(V)$. But then $\rho\sigma \in O^+(\Gamma)$, and thus

$$s = \theta(\sigma) = \theta(\rho\sigma) \in \theta(\Gamma),$$

proving the claim. $\square$

If Λ and Γ are two lattices in V, we define the θ-*distance* between them by

$$\theta(\Lambda, \Gamma) := \begin{cases} \infty & \text{if } \Lambda \nsim \Gamma, \\ \theta(\sigma) \in \theta(V)/\theta(\Lambda) & \text{if } \sigma\Lambda = \Gamma. \end{cases}$$

Thus, $\Lambda \sim \Gamma$ if and only if $\theta(\Lambda, \Gamma) = s \neq \infty$, in which case $[\Lambda]^s = [\Gamma]$, and $\Lambda \simeq \Gamma$ if and only if $\theta(\Lambda, \Gamma) = 1$. That is, $\theta(\Lambda, \Gamma) =$ "$[\Gamma]/[\Lambda]$", in the sense of torsor.

2.3 Localization

For simplicity, we will only work here with quadratic spaces over $\mathbb{Q}$ and $\mathbb{Z}$-lattices, and their localizations. Let V be a quadratic space over $\mathbb{Q}$. If p is a prime number, we will use a subscript p to denote localization with respect to p. For instance

$$V_p := V \otimes \mathbb{Q}_p,$$

is a quadratic space over $\mathbb{Q}_p$, and for a $\mathbb{Z}$-lattice Λ in V, we have

$$\Lambda_p := \Lambda \otimes \mathbb{Z}_p,$$

a $\mathbb{Z}_p$-lattice in V_p. Note that, for a given $\mathbb{Z}$-lattice Λ, the localizations Λ_p are unimodular for almost all p.

Two $\mathbb{Z}$-lattices Λ, Γ in V are in the same *genus* if

$$\Lambda_p \sim \Gamma_p \qquad \text{all } p.$$

As we did with the equivalence classes, we will further subdivide the genera. We will say that Λ and Γ are in the same Θ-*genus* if

$$\Lambda_p \simeq \Gamma_p \qquad \text{all } p.$$

Note that a Θ-genus is not closed by the action of $O^+(V)$. The closure of a Θ-genus by this action is, by definition, a *spinor genus*.

In general, a genus contains several spinor genera; the notion of Θ-genus is too fine and has to be modified. Consider

$$\mathscr{U}(V_p) := \left\{\sigma \in O^+(V_p) \;:\; \theta(\sigma) \in \mathbb{Z}_p^\times\right\}.$$

We will say that Λ and Γ are in the same $\mathscr{U}$-*genus* if

$$\sigma_p \Lambda_p = \Gamma_p, \qquad \text{some } \sigma_p \in \mathscr{U}(V_p), \quad \text{for all } p.$$

What is the same, $\theta(\Lambda_p, \Gamma_p) \in \mathbb{Z}_p^\times \theta(\Lambda_p)$. Assume the dimension of V is at least 2, since the case of dimension 1 is trivial. By Lemma 2.2 we have $\mathbb{Z}_p^\times \subseteq \theta(\Lambda_p)$ for almost all p, and so the definition of Θ-genus and $\mathscr{U}$-genus differ only at finitely many primes.

We have the following inclusions:

$$\begin{tikzcd}
 & \text{genus} \arrow[d, dash] \\
\mathscr{U}\text{-genus} \arrow[ur, dashed, dash] \arrow[d, dash] & \text{spinor genus} \arrow[d, dash] \\
\Theta\text{-genus} \arrow[ur, dashed, dash] \arrow[d, dash] & \text{class} \\
\Theta\text{-class} \arrow[ur, dashed, dash] &
\end{tikzcd}$$

Proposition 2.7. *There is a finite number of Θ-classes in a given $\mathscr{U}$-genus.*

Proof. It is well known that a genus contains a finite number of classes. Hence, it will suffice to show that each class contains a finite number of Θ-classes in a given $\mathscr{U}$-genus.

Let Λ be a $\mathbb{Z}$-lattice of dimension at least 2—the case of dimension 1 is trivial—and let S be a finite set of primes such that

$$\theta(\Lambda_p) \subseteq \mathbb{Z}_p^\times (\mathbb{Q}_p^\times)^2, \qquad \text{for } p \notin S.$$

For instance, by Lemma 2.4 we can take S to be the set of primes p such that Λ_p is not unimodular, a finite set.

Suppose $\Lambda \sim \Gamma$, with $\theta(\Lambda, \Gamma) = s\theta(\Lambda)$, with s a squarefree integer. If they are in the same $\mathscr{U}$-genus, then

$$\theta(\Lambda_p, \Gamma_p) = s\theta(\Lambda_p) \in \mathbb{Z}_p^\times \theta(\Lambda_p).$$

But for $p \notin S$, we have $\theta(\Lambda_p) \subseteq \mathbb{Z}_p^\times(\mathbb{Q}_p^\times)^2$; it follows that $s \in \mathbb{Z}_p^\times$ for $p \notin S$, and so $[\Gamma] = [\Lambda]^s$ with a finite number of choices for s. $\square$

Remark. Indeed, when Λ is integral, it follows that S consists of primes dividing $\operatorname{disc} \Lambda$, and so the Θ-classes properly equivalent to Λ in the $\mathscr{U}$-genus of Λ are all given by

$$[\Lambda]^s, \qquad s \mid \operatorname{disc} \Lambda,$$

with $[\Lambda]^s = [\Lambda]^{s'}$ if and only if $ss' \in \theta(\Lambda)$.

Proposition 2.8. *Let Λ be a $\mathbb{Z}$-lattice in V of dimension at least 3. The closure of the $\mathscr{U}$-genus of Λ by the action of $O^+(V)$ is the genus of Λ.*

Proof. Let Γ be a lattice in the genus of Λ. We prove that Γ is properly equivalent to a lattice in the $\mathscr{U}$-genus of Λ. Consider the matrix that changes a basis of Λ into a basis of Γ. This is an invertible matrix with rational coefficients, hence an invertible matrix over $\mathbb{Z}_p$ for almost all p. It follows that

$$\Lambda_p = \Gamma_p, \qquad \text{almost all } p.$$

In particular $\theta(\Lambda_p, \Gamma_p) = 1$ for almost all p, and so there exists some $x \in \mathbb{Q}_{>0}$ such that

$$x \in \mathbb{Z}_p^\times \theta(\Lambda_p, \Gamma_p), \qquad \text{all } p.$$

By Proposition 1.9, there is some $\sigma \in O^+(V)$ such that $\theta(\sigma) = x$. It follows that $\sigma\Gamma$ is in the $\mathscr{U}$-genus of Λ. $\square$

2.4 Examples

We will give here a few examples of ternary $\mathbb{Z}$-lattices, discussing the classification in Θ-classes, Θ-genera and $\mathscr{U}$-genera. Note that, in view of Lemma 2.2, the Θ-genera and $\mathscr{U}$-genera are the same when the discriminant is cubefree.

Discriminant 11. Let $V := \mathbb{Q}^3$, with norm

$$\|(x, y, z)\| := x^2 + y^2 + 3z^2 - xz.$$

The lattice $\Lambda_1 := \mathbb{Z}^3 \subseteq V$ has discriminant 11. Another lattice in the genus of Λ_1 is given by $\Lambda_2 := \mathbb{Z}\boldsymbol{v}_1 + \mathbb{Z}\boldsymbol{v}_2 + \mathbb{Z}\boldsymbol{v}_3$, where

$$\boldsymbol{v}_1 = \frac{(0, -1, -1)}{2}, \quad \boldsymbol{v}_2 = \frac{(0, 1, -1)}{2}, \quad \boldsymbol{v}_3 = (2, 0, 0),$$

with corresponding quadratic form

$$\|x\boldsymbol{v}_1 + y\boldsymbol{v}_2 + z\boldsymbol{v}_3\| = x^2 + y^2 + 4z^2 + yz + xz + xy.$$

An easy calculation involving ternary quadratic forms shows that there are two proper equivalence classes of lattices in the given genus, and Λ_1 and Λ_2 are a complete set representatives. Alternatively, a table of proper equivalence classes of ternary quadratic forms is available online (Tornaría, 2004).

As in the proof of Proposition 2.8, one verifies that $[\Lambda_1]$ and $[\Lambda_2]^2$ are in the same $\mathscr{U}$-genus. By the remark following Proposition 2.7, one can see that the Θ-classes in the $\mathscr{U}$-genus of Λ_1 will be among

$$[\Lambda_1], \qquad [\Lambda_1]^{11}, \qquad [\Lambda_2]^2, \quad \text{and} \quad [\Lambda_2]^{2\cdot 11},$$

with certain identifications given by the autometries of Λ_1 and Λ_2 of nontrivial spinor norm. Indeed, Λ_1 has 4 proper autometries, among which $(x, y, z) \mapsto (-x, y, -z)$ has spinor norm 11, and Λ_2 has 6 proper autometries, among which $(x, y, z) \mapsto (-y, -x, -z)$ has spinor norm 11. It follows that

$$[\Lambda_1] = [\Lambda_1]^{11}, \quad \text{and} \quad [\Lambda_2] = [\Lambda_2]^{11};$$

thus we have two Θ-classes in the given $\mathscr{U}$-genus. There is really nothing new in the Θ-class approach in this example.

Discriminant 37. Let $V := \mathbb{Q}^3$, with norm

$$\|(x, y, z)\| := x^2 + 2y^2 + 5z^2 - yz - xz.$$

The genus of the lattice $\Lambda_1 := \mathbb{Z}^3 \subseteq V$, of discriminant 37, contains two proper equivalence classes; a lattice not equivalent to Λ_1 is given by $\Lambda_2 := \mathbb{Z}\boldsymbol{v}_1 + \mathbb{Z}\boldsymbol{v}_2 + \mathbb{Z}\boldsymbol{v}_3$, where

$$\boldsymbol{v}_1 = \frac{(0, -1, 1)}{2}, \quad \boldsymbol{v}_2 = \frac{(2, 1, 1)}{2}, \quad \boldsymbol{v}_3 = \frac{(-2, 1, 1)}{2},$$

with corresponding quadratic form

$$\|x\boldsymbol{v}_1 + y\boldsymbol{v}_2 + z\boldsymbol{v}_3\| = 2x^2 + 2y^2 + 3z^2 + yz + 2xz + xy.$$

The autometry of Λ_1 given by $(x, y, z) \mapsto (x - z, -y, -z)$ has spinor norm 37, and so $[\Lambda_1] = [\Lambda_1]^{37}$. On the other hand, Λ_2 has no non-trivial autometries, thus $[\Lambda_2] \neq [\Lambda_2]^{37}$. As in the previous example, $[\Lambda_1]$ and $[\Lambda_2]^2$ are in the same $\mathscr{U}$-genus, and we conclude that

$$[\Lambda_1], \qquad [\Lambda_2]^2, \quad \text{and} \quad [\Lambda_2]^{2\cdot 37},$$

are the three Θ-classes in the given $\mathscr{U}$-genus. We point out that the space $M_2(37)$ of weight 2 modular forms of level 37 and trivial character has dimension 3.

Discriminant 7^3. Let $V := \mathbb{Q}^3$, with norm

$$\|(x, y, z)\| := x^2 + 2y^2 + 49z^2 - xy.$$

The lattice $\Lambda_1 := \mathbb{Z}^3 \subseteq V$ has discriminant 7^3. There are three proper equivalence classes in the genus of Λ_1; a set of representatives is given by Λ_1, $\Lambda_2 = \mathbb{Z}\boldsymbol{v}_1 + \mathbb{Z}\boldsymbol{v}_2 + \mathbb{Z}\boldsymbol{v}_3$, and $\Lambda_3 = \mathbb{Z}\boldsymbol{w}_1 + \mathbb{Z}\boldsymbol{w}_2 + \mathbb{Z}\boldsymbol{w}_3$, where

$$\boldsymbol{v}_1 = (-1, 0, 0), \quad \boldsymbol{v}_2 = (1, 2, 0), \quad \boldsymbol{v}_3 = \frac{(-1, -2, -1)}{2},$$

and

$$\boldsymbol{w}_1 = (1, 1, 0), \quad \boldsymbol{w}_2 = \frac{(-3, 1, 1)}{3}, \quad \boldsymbol{w}_3 = (2, -1, 0),$$

with corresponding quadratic forms

$$\|x\boldsymbol{v}_1 + y\boldsymbol{v}_2 + z\boldsymbol{v}_3\| = x^2 + 7y^2 + 14z^2 - 7yz,$$

and

$$\|x\boldsymbol{w}_1 + y\boldsymbol{w}_2 + z\boldsymbol{w}_3\| = 2x^2 + 7y^2 + 8z^2 - 7yz - xz.$$

In this case one checks that $[\Lambda_1]$, $[\Lambda_2]^2$, and $[\Lambda_3]^3$ are in the same $\mathscr{U}$-genus. On the other hand $\left(\frac{3}{7}\right) = -1$, and one verifies that $[\Lambda_3]^3$ is not in the same Θ-genus as $[\Lambda_1]$ or $[\Lambda_2]^2$.

The autometries $(x,y,z) \mapsto (-x,-y,z)$ of Λ_1, and $(x,y,z) \mapsto (x,-y,-z)$ of Λ_2, have spinor norm 7. The third lattice Λ_3 has one nontrivial autometry, but with trivial spinor norm.

We conclude that the $\mathscr{U}$-genus of Λ_1 consists of two Θ-genera, with two Θ-classes in each, namely

$$\left[\Lambda_1\right], \qquad \left[\Lambda_2\right]^2,$$

in one of the Θ-genera, and

$$\left[\Lambda_3\right]^3, \qquad \left[\Lambda_3\right]^{3\cdot 7},$$

in the other.

3. The Brandt module

In this chapter we introduce the Brandt module of ternary quadratic lattices. In order to define the Hecke operators, we need to review the construction of p-neighboring lattices. With a suitable modification, the lattices thus obtained will be in the same $\mathscr{U}$-genus.

In Section 3.3 we complete the examples from the last chapter, by showing a few cases of the p-neighbor construction, computing some Hecke operators, and diagonalizing the Brandt modules.

3.1 Neighboring lattices

Let Λ and Γ be two integral $\mathbb{Z}$-lattices in a quadratic space V over $\mathbb{Q}$, and fix a prime p. We say that Λ and Γ are p-*neighbors* if

1. $(\Lambda : \Lambda \cap \Gamma) = (\Gamma : \Lambda \cap \Gamma) = p$,
2. $\langle \Lambda, \Gamma \rangle \not\subseteq \mathbb{Z}$.

By the second condition we mean that there are vectors $\boldsymbol{v} \in \Lambda$ and $\boldsymbol{w} \in \Gamma$ such that $\langle \boldsymbol{v}, \boldsymbol{w} \rangle \notin \mathbb{Z}$. This is equivalent to the lattice $\Lambda + \Gamma$ not being integral.

Remark. Note that two p-neighbors Λ and Γ are equal outside p, and they are $\mathbb{Q}_p$-equivalent with the same discriminant. When p is odd and $p^2 \nmid \operatorname{disc} \Lambda$, this already implies $\mathbb{Z}_p$-equivalence, hence Λ and Γ are in the same genus. This is true in general, and we prove in Corollary 3.7 a stronger assertion.

Fix an integral $\mathbb{Z}$-lattice Λ. We will describe now an explicit construction that yields all the p-neighboring lattices of Λ, and gives a characterization of them. For each $\boldsymbol{v} \in \Lambda$, let

$$\begin{aligned} \Lambda_{\boldsymbol{v}}^0 &:= \{\boldsymbol{u} \in \Lambda \ : \ \langle \boldsymbol{v}, \boldsymbol{u} \rangle \equiv 0 \pmod p\}, \\ \Lambda_{\boldsymbol{v}} &:= \mathbb{Z}\tfrac{\boldsymbol{v}}{p} + \Lambda_{\boldsymbol{v}}^0. \end{aligned}$$

These lattices fit together as in the following diagram

$$\begin{tikzcd} \Lambda \arrow[dr, dash] & & \Lambda_{\boldsymbol{v}} \arrow[dl, dash] \\ & \Lambda_{\boldsymbol{v}}^0 & \end{tikzcd}$$

where the lines denote inclusions.

The *dual lattice* of Λ is

$$\Lambda^{\#} := \{\boldsymbol{v} \in V \ : \ \langle x, \Lambda \rangle \subseteq \mathbb{Z}\}.$$

This is another $\mathbb{Z}$-lattice in V. Since Λ is integral, we have $\Lambda \subseteq \Lambda^{\#}$.

Lemma 3.1. *The indices of the inclusions in the above diagram are as follows:*

$_{1}\diagdown\diagup_{1}$	$_{1}\diagdown\diagup_{p}$	$_{p}\diagdown\diagup_{p^2}$	$_{p}\diagdown\diagup_{p}$
$\boldsymbol{v} \in p\Lambda$	$\boldsymbol{v} \in p\Lambda^{\#}$ $\boldsymbol{v} \notin p\Lambda$	$\boldsymbol{v} \notin p\Lambda^{\#}$ $p \nmid \langle \boldsymbol{v}, \boldsymbol{v} \rangle$	$\boldsymbol{v} \notin p\Lambda^{\#}$ $p \mid \langle \boldsymbol{v}, \boldsymbol{v} \rangle$

Proof. Consider first the case where $\boldsymbol{v} \in p\Lambda^{\#}$. This clearly implies that $\langle \boldsymbol{v}, \Lambda \rangle \subseteq p\mathbb{Z}$, hence $\Lambda_{\boldsymbol{v}}^{0} = \Lambda$. Moreover, $\Lambda_{\boldsymbol{v}}^{0} = \Lambda_{\boldsymbol{v}}$ if and only if $\boldsymbol{v} \in p\Lambda$, and $(\Lambda_{\boldsymbol{v}} : \Lambda_{\boldsymbol{v}}^{0}) = p$ otherwise.

Assume now $\boldsymbol{v} \notin p\Lambda^{\#}$. Then there is a basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots\}$ of Λ such that $\langle \boldsymbol{v}, \boldsymbol{v}_1 \rangle \notin p\mathbb{Z}$, and by changing $\boldsymbol{v}_i$ to $\boldsymbol{v}_i + \alpha_i \boldsymbol{v}_1$ if necessary, for convenient $\alpha_i \in \mathbb{Z}$, we can assume also that $\langle \boldsymbol{v}, \boldsymbol{v}_i \rangle \in p\mathbb{Z}$ for $i > 1$. But then $\{p\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots\}$ is a basis of $\Lambda_{\boldsymbol{v}}^{0}$, and it follows that $(\Lambda : \Lambda_{\boldsymbol{v}}^{0}) = p$. Finally, $\frac{\boldsymbol{v}}{p}$ is clearly not in $\Lambda_{\boldsymbol{v}}^{0}$, and $\boldsymbol{v} \in \Lambda_{\boldsymbol{v}}^{0}$ if and only if $p \mid \langle \boldsymbol{v}, \boldsymbol{v} \rangle$; in this case $(\Lambda_{\boldsymbol{v}} : \Lambda_{\boldsymbol{v}}^{0}) = p$, and otherwise $(\Lambda_{\boldsymbol{v}} : \Lambda_{\boldsymbol{v}}^{0}) = p^2$. □

Lemma 3.2. $\Lambda_{\boldsymbol{v}}$ *is integral if and only if* $p^2 \mid \|\boldsymbol{v}\|$*.*

Proof. The condition is clearly necessary, since $\frac{\boldsymbol{v}}{p} \in \Lambda_{\boldsymbol{v}}$. For the sufficiency, note that $\frac{\langle \boldsymbol{v}, \boldsymbol{u} \rangle}{p}$ is an integer for any $\boldsymbol{u} \in \Lambda_{\boldsymbol{v}}^{0}$, by definition, hence

$$\left\| x\frac{\boldsymbol{v}}{p} + \boldsymbol{u} \right\| = x^2 \frac{\|\boldsymbol{v}\|}{p^2} + x\frac{\langle \boldsymbol{v}, \boldsymbol{u} \rangle}{p} + \|\boldsymbol{u}\|$$

is an integer, provided $p^2 \mid \|\boldsymbol{v}\|$. □

Proposition 3.3. *If $\boldsymbol{v} \in \Lambda$, a necessary and sufficient condition for Λ and $\Lambda_{\boldsymbol{v}}$ to be* p*-neighbors is that $\boldsymbol{v} \notin \mathrm{p}\Lambda^{\#}$ and $\mathrm{p}^2 \mid \|\boldsymbol{v}\|$.*

Proof. The necessity follows from Lemmas 3.1 and 3.2. Conversely, if the condition is met, Lemma 3.2 proves that $\Lambda_{\boldsymbol{v}}$ is integral, and Lemma 3.1 proves that $\Lambda \cap \Lambda_{\boldsymbol{v}} = \Lambda^0_{\boldsymbol{v}}$ has index p in both Λ and $\Lambda_{\boldsymbol{v}}$. Now consider $\frac{\boldsymbol{v}}{\mathrm{p}} \in \Lambda_{\boldsymbol{v}}$, and let $\boldsymbol{w} \in \Lambda$ such that $\boldsymbol{w} \notin \Lambda^0_{\boldsymbol{v}}$. Then $\left\langle \frac{\boldsymbol{v}}{\mathrm{p}}, \boldsymbol{w} \right\rangle \notin \mathbb{Z}$, and hence Λ and $\Lambda_{\boldsymbol{v}}$ are p-neighbors. □

Proposition 3.4. *If Γ is any* p*-neighboring lattice of Λ, there is some $\boldsymbol{v} \in \Lambda$ such that $\Gamma = \Lambda_{\boldsymbol{v}}$.*

Proof. There is a vector $\boldsymbol{w} \in \Gamma$ such that $\langle \Lambda, \boldsymbol{w} \rangle \not\subseteq \mathbb{Z}$, i.e. $\boldsymbol{w} \notin \Lambda^{\#}$, and thus $\boldsymbol{v} := \mathrm{p}\boldsymbol{w} \notin \mathrm{p}\Lambda^{\#}$. On the other hand, since $(\Gamma : \Lambda \cap \Gamma) = \mathrm{p}$, it follows that $\boldsymbol{v} \in \Lambda$, and clearly $\mathrm{p}^2 \mid \|\boldsymbol{v}\|$. Now it is easy to check that $\Gamma = \Lambda_{\boldsymbol{v}}$. □

Theorem 3.5. *The map $\boldsymbol{v} \mapsto \Lambda_{\boldsymbol{v}}$, for $\boldsymbol{v} \in \Lambda$ in the conditions of the Proposition 3.3, induces a one-to-one correspondence between the set of non-singular projective solutions of*

$$\|\boldsymbol{v}\| \equiv 0 \pmod{\mathrm{p}}, \qquad \boldsymbol{v} \in \Lambda, \tag{3.1}$$

and the p*-neighboring lattices of Λ. Moreover, if $\Lambda_1 \neq \Lambda_2$ are two* p*-neighbors, then $\Lambda_1 \cap \Lambda_2 \subseteq \Lambda$.*

Proof. Let $\boldsymbol{v}$ be a non-singular solution of (3.1). By Hensel Lemma we can assume $\mathrm{p}^2 \mid \|\boldsymbol{v}\|$. Indeed, $\boldsymbol{v}$ being non-singular is equivalent to

$v \notin p\Lambda^\#$, and so there is a $u \in \Lambda$ such that $p \nmid \langle v, u\rangle$. Then we can choose $\alpha \in \mathbb{Z}$ such that $\|v + p\alpha u\| \equiv \|v\| + p\alpha\langle v, u\rangle \equiv 0 \pmod{p^2}$, and $v + p\alpha u$ corresponds to the same projective solution.

Now suppose that v_1 and v_2 are vectors in Λ, with $p^2 \mid \|v_1\|$ and $p^2 \mid \|v_2\|$. If v_1 and v_2 correspond to the same solution of (3.1), we have $v_2 = xv_1 + pu$ for some $x \not\equiv 0 \pmod p$ and some $u \in \Lambda$. Taking norms we conclude that $0 \equiv xp\langle v_1, u\rangle \pmod{p^2}$, and so $u \in \Lambda^0_{v_1}$. But then $\frac{v_2}{p} = x\frac{v_1}{p} + u \in \Lambda_{v_1}$, hence $\Lambda_{v_2} = \Lambda_{v_1}$.

Conversely, if v_1 and v_2 correspond to different solutions, we have $\frac{v_1}{p} \notin \frac{v_2}{p} + \Lambda$. Since $\Lambda_{v_2} \subseteq \frac{v_2}{p} + \Lambda$, it follows that $\frac{v_1}{p} \notin \Lambda_{v_2}$, but $\frac{v_1}{p} \in \Lambda_{v_1}$, so $\Lambda_{v_1} \neq \Lambda_{v_2}$. In addition, we have proved that $\frac{v_1}{p} \notin \Lambda_{v_1} \cap \Lambda_{v_2} \subseteq \frac{v_1}{p} + \Lambda$, and so $\Lambda_{v_1} \cap \Lambda_{v_2} \subseteq \Lambda$.

The surjectivity follows from Propositions 3.4 and 3.3. $\square$

3.2 Hecke operators

As we've already remarked, if Λ and Γ are p-neighbors, then $\Gamma_q = \Lambda_q$ for all primes $q \neq p$. The following Proposition gives us information about the localization at p:

Proposition 3.6. *Suppose Λ and Γ are p-neighbors. Then we have $\Lambda_p \sim \Gamma_p$ and $\theta(\Lambda_p, \Gamma_p) = p$, with $\theta(\Lambda_p) \supseteq \mathbb{Z}_p^\times$.*

Proof. Since Λ and Γ are p-neighbors, there are vectors $v \in \Lambda$ and $w \in \Gamma$ such that $\langle v, w\rangle \notin \mathbb{Z}$. Let $\Lambda^0 := \Lambda \cap \Gamma$, so that $\Lambda = \Lambda^0 + \mathbb{Z}v$

and $\Gamma = \Lambda^0 + \mathbb{Z}\boldsymbol{w}$. Localizing we get

$$\Lambda_p = \Lambda_p^0 + \mathbb{Z}_p\boldsymbol{v}, \qquad \Gamma_p = \Lambda_p^0 + \mathbb{Z}_p\boldsymbol{w}, \qquad p\langle \boldsymbol{v}, \boldsymbol{w}\rangle \in \mathbb{Z}_p^\times.$$

Since $p\boldsymbol{w} \in \Lambda_p^0$, these relations are unchanged if we replace $\boldsymbol{v}$ with $\boldsymbol{v} + xp\boldsymbol{w}$ for $x \in \mathbb{Z}_p$, and similarly if $\boldsymbol{w}$ is replaced with $\boldsymbol{w} + yp\boldsymbol{v}$ for $y \in \mathbb{Z}_p$. Now note that

$$\|\boldsymbol{v} + xp\boldsymbol{w}\| = \|\boldsymbol{v}\| + xp\langle \boldsymbol{v}, \boldsymbol{w}\rangle + x^2p^2\,\|\boldsymbol{w}\|$$

can take any value modulo p^2; a fortiori it can take any value in $\mathbb{Z}_p$, by Hensel's Lemma, because $p\langle \boldsymbol{v}, \boldsymbol{w}\rangle \in \mathbb{Z}_p^\times$ (even for $p = 2$), and the same for $\|\boldsymbol{w} + yp\boldsymbol{v}\|$. Thus we can assume that $\|\boldsymbol{v}\| = \|\boldsymbol{w}\| \in \mathbb{Z}_p^\times$.

We claim that

$$\tau_{\boldsymbol{v}-\boldsymbol{w}}\Lambda_p = \Gamma_p.$$

Indeed, if $\boldsymbol{u} \in \Lambda_p^0$, then $\langle \boldsymbol{u}, \boldsymbol{v} - \boldsymbol{w}\rangle \in \mathbb{Z}_p$ and so

$$\frac{\langle \boldsymbol{u}, \boldsymbol{v} - \boldsymbol{w}\rangle}{\|\boldsymbol{v} - \boldsymbol{w}\|} \in p\mathbb{Z}_p,$$

since $\|\boldsymbol{v} - \boldsymbol{w}\| = \|\boldsymbol{v}\| + \|\boldsymbol{w}\| - \langle \boldsymbol{v}, \boldsymbol{w}\rangle \notin \mathbb{Z}_p$. Thus

$$\tau_{\boldsymbol{v}-\boldsymbol{w}}\boldsymbol{u} = \boldsymbol{u} - \frac{\langle \boldsymbol{u}, \boldsymbol{v} - \boldsymbol{w}\rangle}{\|\boldsymbol{v} - \boldsymbol{w}\|}(\boldsymbol{v} - \boldsymbol{w}) \in \Lambda_p^0,$$

because $p(\boldsymbol{v} - \boldsymbol{w}) \in \Lambda_p^0$. We have proved that $\tau_{\boldsymbol{v}-\boldsymbol{w}}\Lambda_p^0 = \Lambda_p^0$, but $\tau_{\boldsymbol{v}-\boldsymbol{w}}\boldsymbol{v} = \boldsymbol{w}$ by Lemma 1.2, so that $\tau_{\boldsymbol{v}-\boldsymbol{w}}\Lambda_p = \Gamma_p$.

To finish the proof we note that, since $\|\boldsymbol{v}\| \in \mathbb{Z}_p^\times$, we have $\tau_{\boldsymbol{v}} \in O(\Lambda_p)$, and so

$$\sigma_p = \tau_{\boldsymbol{v}-\boldsymbol{w}}\tau_{\boldsymbol{v}} \in O^+(V_p)$$

will be a proper equivalence between Λ_p and Γ_p, with

$$\theta(\sigma_p) = \|\boldsymbol{v} - \boldsymbol{w}\| \, \|\boldsymbol{v}\| \in p\mathbb{Z}_p^\times (\mathbb{Q}_p^\times)^2.$$

The last statement completes the proof; we have already seen that the vectors $\boldsymbol{u} = \boldsymbol{v} + zp\boldsymbol{w} \in \Lambda$, for $z \in \mathbb{Z}_p$, have arbitrary $\|\boldsymbol{u}\| \in \mathbb{Z}_p^\times$, hence $\tau_{\boldsymbol{v}}\tau_{\boldsymbol{u}} \in O^+(\Lambda_p)$ has arbitrary $\theta(\tau_{\boldsymbol{v}}\tau_{\boldsymbol{u}}) \in \mathbb{Z}_p^\times$. $\square$

With a certain abuse of notation we will denote by $\Gamma^{(p)}$ a lattice such that $\left[\Gamma^{(p)}\right] = \left[\Gamma\right]^p$, i.e. $\Gamma^{(p)} = \sigma\Gamma$ for some $\sigma \in O^+(V)$ with $\theta(\sigma) = p$. Such a lattice exists by Proposition 1.9, but it is only defined up to Θ-equivalence.

Corollary 3.7. *If Λ and Γ are p-neighbors, then Λ and Γ are in the same genus. Moreover, Λ and $\Gamma^{(p)}$ are in the same $\mathscr{U}$-genus.*

Proof. The first part follows immediately from the proposition. For the second part, note that

$$\theta(\Lambda_q, \Gamma_q^{(p)}) = p\theta(\Lambda_q, \Gamma_q) \in \mathbb{Z}_q^\times \theta(\Lambda_q)$$

trivially for $q \neq p$, and by the proposition for $q = p$. $\square$

Remark. More is true if we assume that

$$p \in \theta(\Lambda_q), \qquad \text{all } q \neq p.$$

Note that the assumption is always true if $q^{\frac{n(n-1)}{2}} \nmid \operatorname{disc} \Lambda$, since Lemma 2.2 implies $\theta(\Lambda_q) \supseteq \mathbb{Z}_q^\times (\mathbb{Q}_q^\times)^2 \ni p$. In this case

$$\theta(\Gamma_q^{(p)}, \Lambda_q) \in \theta(\Lambda_q),$$

by assumption for $q \neq p$, and by the proposition for $q = p$. That is, $\Gamma^{(p)}$ and Λ are in the same Θ-genus. For instance, this is the case when p is a square modulo $\operatorname{disc}\Lambda$.

Definition. Let $\mathcal{M}(\Lambda)$ be the free $\mathbb{Z}$-module with basis the Θ-classes in the $\mathscr{U}$-genus of Λ. When Λ is a definite lattice of dimension 3, we call $\mathcal{M}(\Lambda)$ the *Brandt module* of the ternary quadratic lattice Λ. The *Hecke operators* $t_p : \mathcal{M}(\Lambda) \to \mathcal{M}(\Lambda)$ are linear operators given in the basis by

$$t_p[\Gamma] := \sum_i [\Gamma_i]^p,$$

where the sum is over all the p-neighbors of Γ.

Clearly $\mathcal{M}(\Lambda)$ depends only on the $\mathscr{U}$-genus of Λ. On the other hand, if $\sigma \in O^+(V)$, we have an isomorphism

$$\mathcal{M}(\Lambda) \to \mathcal{M}(\sigma\Lambda)$$

given by $[\Gamma] \mapsto [\Gamma]^{\theta(\sigma)}$, which preserves the action of the Hecke operators. Hence, by Proposition 2.8, $\mathcal{M}(\Lambda)$ really depends only, up to a Hecke-linear isomorphism, on the genus of Λ.

From now on we will assume that V is a definite quadratic space. This implies that $O(\Gamma)$ is finite for any lattice in V. We thus define an inner product in $\mathcal{M}_{\mathbb{R}}(\Lambda) := \mathcal{M}(\Lambda) \otimes \mathbb{R}$ by

$$\left\langle\!\!\left\langle [\Gamma], [\Gamma'] \right\rangle\!\!\right\rangle := \sharp\{\sigma \in \Theta(V) \;:\; \sigma\Gamma = \Gamma'\} = \begin{cases} \sharp\,\Theta(\Gamma) & \text{if } [\Gamma] = [\Gamma'], \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 3.8. *The Hecke operators* t_p *generate a commutative algebra of self-adjoint operators. Hence* $\mathcal{M}_{\mathbb{R}}(\Lambda)$ *has an orthogonal basis of simultaneous eigenvectors for the* t_p.

Proof. It is immediate that t_p and t_q commute for $p \neq q$. To prove that t_p is self-adjoint, note that

$$\begin{aligned}
&\left\langle\!\!\left\langle t_p[\Gamma], [\Gamma'] \right\rangle\!\!\right\rangle \\
&\qquad\qquad = \sharp\{\sigma \in O^+(V) \,:\, \theta(\sigma) = p,\ \sigma\Gamma' \text{ is a } p\text{-neighbor of } \Gamma\} \\
&\qquad\qquad = \sharp\left\{\sigma \in O^+(V) \,:\, \theta(\sigma) = p,\ \Gamma' \text{ is a } p\text{-neighbor of } \sigma^{-1}\Gamma\right\} \\
&\qquad\qquad = \left\langle\!\!\left\langle [\Gamma], t_p[\Gamma'] \right\rangle\!\!\right\rangle,
\end{aligned}$$

since $\theta(\sigma^{-1}) = \theta(\sigma)$. The last statement follows from this by the spectral theorem. □

3.3 Examples

We continue here with the examples from Chapter 2. We show how to compute p-neighbors in a few examples, compute the matrices (Brandt matrices) for the Hecke operators t_p for a few p, compute eigenvectors for the Brandt modules, and compare their eigenvalues with those of eigenforms of weight 2.

The reader should not get the false impression that the eigenvectors and eigenvalues of a Brandt module are always rational. This is a convenience for our examples, but it is not at all true in general.

Discriminant 11. Let Λ_1 and Λ_2 the lattices of discriminant 11 given in Section 2.4. We show how to find the 2-neighbors of Λ_1. Representatives for the 3 projective solutions of

$$x^2 + y^2 + 3z^2 - xz \equiv 0 \pmod 2$$

are $(0,1,1)$, $(1,1,1)$, and $(1,1,0)$. The first two have norm 4, but the last one has norm 2; we must replace it by the equivalent $(1,1,2)$ of norm 12 (divisible by 4).

Let $\boldsymbol{v} = (0,1,1)$. We compute $\langle \boldsymbol{v}, (x,y,z)\rangle \equiv x \pmod 2$, and thus $\Lambda_{\boldsymbol{v}}^0 = \{(x,y,z) \in \mathbb{Z}^3 : x \equiv 0 \pmod 2\}$. To obtain $\Lambda_{\boldsymbol{v}}$ we need to add $\frac{\boldsymbol{v}}{2}$; we easily find that $\Lambda_{\boldsymbol{v}}$ is indeed Λ_2. In the same way compute $\Lambda_{(1,1,1)}$ and $\Lambda_{(1,1,2)}$, and by reducing the corresponding quadratic forms, we find out that $\Lambda_{(1,1,1)} \sim \Lambda_2$ and $\Lambda_{(1,1,2)} \sim \Lambda_1$. We summarize this information by

$$t_2[\Lambda_1] = [\Lambda_1] + 2[\Lambda_2]^2;$$

note that there is no need to keep track of the Θ-equivalence classes in this example.

Computing the 2-neighboring lattices of Λ_2, we see that

$$t_2[\Lambda_2]^2 = 3[\Lambda_1],$$

which concludes the computation of the Hecke operator $t_2 = \left(\begin{smallmatrix} 1 & 3 \\ 2 & 0 \end{smallmatrix}\right)$. Since the eigenvalues of t_2 have multiplicity 1, this is enough to find a basis of eigenvectors, namely

$$E_0 := 3\left[\Lambda_1\right] + 2\left[\Lambda_2\right]^2, \qquad E_1 := \left[\Lambda_1\right] - \left[\Lambda_2\right]^2,$$

of eigenvalues 3 and -2 for t_2, respectively.

By computing p-neighbors for other primes $p \neq 11$, one can easily compute more Hecke operators:

$$t_3 = \left(\begin{smallmatrix} 2 & 3 \\ 2 & 1 \end{smallmatrix}\right), \ t_5 = \left(\begin{smallmatrix} 4 & 3 \\ 2 & 3 \end{smallmatrix}\right), \ t_7 = \left(\begin{smallmatrix} 4 & 6 \\ 4 & 2 \end{smallmatrix}\right), \ t_{13} = \left(\begin{smallmatrix} 10 & 6 \\ 4 & 8 \end{smallmatrix}\right), \ t_{17} = \left(\begin{smallmatrix} 10 & 12 \\ 8 & 6 \end{smallmatrix}\right), \ \ldots$$

The eigenvalues for E_0 are $4, 6, 8, 14, 18, \ldots$, respectively; they correspond to the Eisenstein series of weight 2 and level 11. The eigenvalues for E_1 are $-1, 1, -2, 4, -2, \ldots$, respectively, corresponding to the modular form 11A of weight 2 and level 11.

Discriminant 37. Let Λ_1 and Λ_2 be the lattices of discriminant 37 given in Section 2.4. We compute the 2-neighbors of Λ_1 as before. Representatives for the 3 projective solutions of

$$x^2 + 2y^2 + 5z^2 - yz - xz \equiv 0 \pmod 2$$

are $(0, 1, 3)$, $(1, 3, 1)$, and $(0, 1, 2)$, all with norm divisible by 4. A simple computation as before shows that $\Lambda_{(0,1,3)} = \Lambda_2$; again, by reducing ternary quadratic forms, we can see that $\Lambda_{(1,3,1)} \sim \Lambda_2$ and $\Lambda_{(0,1,2)} \sim \Lambda_1$. However, we need to be more specific about $\Lambda_{(1,3,1)}$. Is it Θ-equivalent to $\left[\Lambda_2\right]$, or to $\left[\Lambda_2\right]^{37}$? From the reduction matrices one obtains a proper autometry σ such that $\sigma\Lambda_{(1,3,1)} = \Lambda_2$; the spinor norm is $\theta(\sigma) = 37$. Thus,

$$t_2[\Lambda_1] = [\Lambda_1] + [\Lambda_2]^2 + [\Lambda_2]^{2\cdot 37}.$$

Similarly, one computes the 2-neighbors of Λ_2, and concludes

$$\begin{aligned} t_2[\Lambda_2]^2 \quad &= [\Lambda_1] + 2[\Lambda_2]^{2\cdot 37}, \\ t_2[\Lambda_2]^{2\cdot 37} &= [\Lambda_1] + 2[\Lambda_2]^2; \end{aligned}$$

hence $t_2 = \left(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{smallmatrix}\right)$.

By computing p-neighbors for other primes $p \neq 37$, one computes more Hecke operators:

$$t_3 = \left(\begin{smallmatrix} 2 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 3 & 0 \end{smallmatrix}\right),\ t_5 = \left(\begin{smallmatrix} 2 & 2 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \end{smallmatrix}\right),\ t_7 = \left(\begin{smallmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{smallmatrix}\right),\ t_{11} = \left(\begin{smallmatrix} 6 & 3 & 3 \\ 3 & 2 & 6 \\ 3 & 7 & 2 \end{smallmatrix}\right),\ \ldots$$

The eigenvectors in this Brandt module are

$$E_0 := [\Lambda_1] + [\Lambda_2]^2 + [\Lambda_2]^{2\cdot 37},$$

with eigenvalues $3, 4, 6, 8, 12, \ldots$, corresponding to the Eisenstein series of weight 2 and level 37;

$$E_1 := 2[\Lambda_1] - [\Lambda_2]^2 - [\Lambda_2]^{2\cdot 37},$$

with eigenvalues $0, 1, 0, -1, 3, \ldots$, corresponding to the modular form 37B of weight 2, level 37, and sign $+$ in the functional equation; and

$$E_2 := [\Lambda_2]^2 - [\Lambda_2]^{2\cdot 37},$$

with eigenvalues $-2, -3, -2, -1, -5, \ldots$, corresponding to the modular form 37A of weight 2, level 37, and sign $-$ in the functional equation.

Note that if we identify proper equivalence classes of lattices, E_0 and E_1 are still well defined; however, E_2 would become trivial. Indeed, computing usual theta series for E_0 and E_1 makes perfect sense, but computing a usual theta series for E_2 would trivially yield zero.

Discriminant 7^3. Let Λ_1, Λ_2 and Λ_3 be the lattices of discriminant 7^3 given in Section 2.4. By computing p-neighbors of these lattices, for primes $p \neq 7$ one is able to compute

$$t_2 = \begin{pmatrix} 2&1&0&0\\1&2&0&0\\0&0&2&1\\0&0&1&2 \end{pmatrix},\ t_3 = \begin{pmatrix} 0&0&2&2\\0&0&2&2\\2&2&0&0\\2&2&0&0 \end{pmatrix},\ t_5 = \begin{pmatrix} 0&0&3&3\\0&0&3&3\\3&3&0&0\\3&3&0&0 \end{pmatrix},$$
$$t_{11} = \begin{pmatrix} 8&4&0&0\\4&8&0&0\\0&0&8&4\\0&0&4&8 \end{pmatrix},\ t_{13} = \begin{pmatrix} 0&0&7&7\\0&0&7&7\\7&7&0&0\\7&7&0&0 \end{pmatrix},\ t_{17} = \begin{pmatrix} 0&0&9&9\\0&0&9&9\\9&9&0&0\\9&9&0&0 \end{pmatrix},\ \ldots$$

Notice how the Hecke operator t_p either preserves or permutes the two Θ-genera, depending on the sign of $\left(\frac{p}{7}\right)$. Besides this, the action on the two Θ-genera is identical.

The eigenvectors in this Brandt module are

$$E_0 := [\Lambda_1] + [\Lambda_2]^2 + [\Lambda_3]^3 + [\Lambda_3]^{3\cdot 7},$$

with eigenvalues $3, 4, 6, 12, 14, 18, \ldots$, which corresponds to the Eisenstein series of weight 2 and level 7^2;

$$E_0' := [\Lambda_1] + [\Lambda_2]^2 - [\Lambda_3]^3 - [\Lambda_3]^{3\cdot 7},$$

with eigenvalues $3, -4, -6, 12, -14, -18, \ldots$, which corresponds to a twist of the Eisenstein series of weight 2 and level 7^2 (where the sign of the eigenvalues is given by the quadratic character modulo 7);

$$E_1 := [\Lambda_1] - [\Lambda_2]^2,$$

with eigenvalues $1, 0, 0, 4, 0, 0, \ldots$, which corresponds to the modular form 49A of weight 2 and level 7^2; and

$$E_2 := [\Lambda_3]^3 - [\Lambda_3]^{3 \cdot 7},$$

with the same eigenvalues $1, 0, 0, 4, 0, 0, \ldots$

4. Weight functions and Hecke operators

In this chapter we define weight functions for a lattice Λ, and use them to construct generalized theta series $\vartheta_l(\Lambda)$, which will be shown to be modular forms in the next chapter.

We will then give an explicit construction of nontrivial weight functions for ternary lattices, and prove that ϑ_l preserves the action of the Hecke operators. We remark that this is not true for lattices other than ternary.

We restrict at first to the case of odd prime l, and for the most part assume that $l \nmid \operatorname{disc} \Lambda$; however, it is not hard to extend the definition and the construction in the case of ternary lattices to odd squarefree l relatively prime to $\operatorname{disc} \Lambda$, by multiplying together the weight functions with the prime factors of l as conductors.

As a special case we allow $l = 1$, with constant weight functions, which yields the usual theta series of a quadratic form.

4.1 Weight functions of prime conductor

Let Λ be an integral $\mathbb{Z}$-lattice of dimension n, and let $l > 0$ be an odd prime. An l-*function* on Λ is a function $\omega : \Lambda_l/l\Lambda_l \to \mathbb{C}$. Note that the natural inclusion $\Lambda \subseteq \Lambda_l$ induces a group isomorphism

$$\Lambda/l\Lambda \xrightarrow{\sim} \Lambda_l/l\Lambda_l,$$

so that an l-function on Λ is determined by its values on Λ. We say that

- ω is *normal*:

  if $\omega(\boldsymbol{v}) = 0$ for $\boldsymbol{v} \in \Lambda$ such that $l \nmid \|\boldsymbol{v}\|$,

- ω is *homogeneous*:

  if $\omega(d\boldsymbol{v}) = \left(\frac{d}{l}\right)^n \omega(\boldsymbol{v})$ for $\boldsymbol{v} \in \Lambda$ and $d \in \mathbb{Z}_l^\times$,

- ω is *transportable*:

  if $\omega(\sigma\boldsymbol{v}) = \left(\frac{\theta(\sigma)}{l}\right) \omega(\boldsymbol{v})$ for $\boldsymbol{v} \in \Lambda$ and $\sigma \in O^+(\Lambda_l)$.

If all the above conditions are met, we say that ω is a *weight function* on Λ of conductor l.

If ω_Λ is a weight function on Λ of conductor l, we *transport* it to any lattice Γ in the $\mathscr{U}$-genus of Λ as follows. Let $\sigma_l \in O(V_l)$ with $\theta(\sigma_l) \in \mathbb{Z}_l^\times$ such that $\Gamma_l = \sigma_l\Lambda_l$. We set

$$\omega_\Gamma(\sigma_l\boldsymbol{v}) := \left(\frac{\theta(\sigma_l)}{l}\right) \omega_\Lambda(\boldsymbol{v}).$$

It is clear that ω_Γ is a weight function on Γ of conductor l. Moreover, ω_Γ is independent of the choice of σ_l, because ω_Λ is transportable.

4.2 More weight functions, and theta series

Let now $l > 0$ be an odd squarefree integer, and let $l = l_1 \cdots l_s$ be its factorization. A weight function on Λ of conductor l is a product $\omega = \omega_1 \cdots \omega_s$, where ω_i is a weight function of conductor l_i on Λ. In the special case $l = 1$, a weight function is just a constant.

Note that a weight function defines a function on $\Lambda/l\Lambda$, which is itself normal and homogeneous. The values of ω on Λ determine the factors ω_i up to multiplication by a nonzero constant.

Fix a weight function ω_Λ on Λ of conductor l. If Γ is a lattice in the $\mathscr{U}$-genus of Λ, we let ω_Γ be the weight function on Γ that is obtained by transporting each of the factors of ω_Λ, as explained in the previous section.

We let, for $m \in \mathbb{Z}$,

$$r_l(\Gamma, m) := \sum_{\substack{v \in \Gamma \\ \|v\| = ml}} \omega_\Gamma(v),$$

be the count, with multiplicity given by ω_Γ, of vectors of norm ml in Γ. When $m \notin \mathbb{Z}$ we have $r_l(\Gamma, m) = 0$, since ω_Γ is normal. We set

$$\vartheta_l(\Gamma) := \sum_{m \geq 0} r_l(\Gamma, m)\, q^m,$$

where $q = e^{2\pi i z}$ as usual. Note that, when n is odd, $\vartheta_l(\Gamma) = 0$ unless $l \equiv 1 \pmod 4$, since ω_Γ is an odd function otherwise.

In Chapter 5 we will prove that this theta series is a modular form of weight $n/2$, whose level $N = N(\Lambda)$ and character $\chi = \chi_\Lambda$ are *independent of* l. Combine this with the fact that $r_l(\Gamma, m)$ depends only on the Θ-class of Γ; hence ϑ_l extends to a linear map

$$\vartheta_l : \mathcal{M}(\Lambda) \to M_{n/2}(N, \chi),$$

where $M_{n/2}(N, \chi)$ is the space of (holomorphic) modular forms of weight $n/2$, level N, and character χ (see Section 5.3).

Consider an eigenvector E in the Brandt module $\mathcal{M}(\Lambda)$, with Λ ternary. We expect an explicit version of Waldspurger's formula, in the spirit of Gross (1987), to hold; cf. Mao, Rodriguez-Villegas, and Tornaría (2004) for the case of prime level. A consequence of such a formula would be that the vanishing of $\vartheta_l(E)$ is related to the vanishing of the central value of certain modular L-series with a twist by the quadratic character of discriminant l. The following very plausible Conjecture would then follow by an application of a result of Bump, Friedberg, and Hoffstein (1990).

Conjecture 4.1. *Let* E *be an eigenvector in the Brandt module* $\mathcal{M}(\Lambda)$. *There is an odd squarefree integer* l, *relatively prime to any given integer, such that* $\vartheta_l(E) \neq 0$.

4.3 The l-symbol for ternary lattices

Consider an integral $\mathbb{Z}$-lattice Λ of dimension 3 and discriminant D, and fix an odd prime $l \nmid D$, so that Λ_l is unimodular. An l-*vector* in Λ is a vector $\boldsymbol{v} \in \Lambda_l$ such that $\|\boldsymbol{v}\| \equiv 0 \pmod{l}$, but $v \notin l\Lambda_l$.

Lemma 4.2. *Let $\mathbf{v}, \mathbf{u}$ be two l-vectors in Λ. If $\langle \mathbf{v}, \mathbf{u} \rangle \equiv 0 \pmod{l}$, then there is some $k \in \mathbb{Z}_l^\times$ such that $\mathbf{v} \equiv k\mathbf{u} \pmod{l\Lambda}$.*

Proof. $\Lambda/l\Lambda$ is a regular quadratic space of dimension 3 over $\mathbb{Z}/l\mathbb{Z}$, and every vector in the subspace spanned by $\mathbf{v}$ and $\mathbf{u}$ is isotropic, since $\|\mathbf{v}\| \equiv \|\mathbf{u}\| \equiv \langle \mathbf{v}, \mathbf{u} \rangle \equiv 0 \pmod{l}$. But an isotropic subspace has dimension at most 1, or else $\Lambda/l\Lambda$ would not be regular. Hence $\mathbf{v} \equiv k\mathbf{u} \pmod{l\Lambda}$ for some $k \in \mathbb{Z}_l^\times$. ☐

Definition. The l-*symbol* for l-vectors $\mathbf{v}$ and $\mathbf{u}$ in Λ, is defined by

$$(\mathbf{v}, \mathbf{u})_l := \begin{cases} \left(\frac{D\langle \mathbf{v}, \mathbf{u} \rangle}{l}\right) & \text{if } \langle \mathbf{v}, \mathbf{u} \rangle \not\equiv 0 \pmod{l}, \\ \left(\frac{k}{l}\right) & \text{if } \mathbf{v} \equiv k\mathbf{u} \pmod{l\Lambda}. \end{cases}$$

In addition we set $(\mathbf{v}, \mathbf{u})_l := 0$ whenever one or both of $\mathbf{v}, \mathbf{u} \in \Lambda$ are not l-vectors.

Remark. Lemma 4.2 ensures that the l-symbol is well defined.

Lemma 4.3. *The l-symbol has the following properties:*

1. $(\mathbf{v}, \mathbf{v})_l = \begin{cases} 1 & \textit{if } \mathbf{v} \textit{ is an } l\textit{-vector,} \\ 0 & \textit{otherwise.} \end{cases}$
2. $(\mathbf{v}, \mathbf{u})_l = (\mathbf{v}', \mathbf{u})_l$ *provided* $\mathbf{v} \equiv \mathbf{v}' \pmod{l\Lambda}$.
3. $(\mathbf{v}, \mathbf{u})_l = (\mathbf{u}, \mathbf{v})_l$.
4. $(d\mathbf{v}, \mathbf{u})_l = (\mathbf{v}, d\mathbf{u})_l = \left(\frac{d}{l}\right)(\mathbf{v}, \mathbf{u})_l$ *for* $d \in \mathbb{Z}_l$.
5. $(\mathbf{v}_1, \mathbf{v}_2)_l\,(\mathbf{v}_2, \mathbf{v}_3)_l = (\mathbf{v}_1, \mathbf{v}_3)_l$, *provided* $\mathbf{v}_2$ *is an* l*-vector.*

Proof. Statements 1–4 are immediate from the definition. To prove 5, we may assume that the $\mathbf{v}_i$ are all l-vectors, and also that

$$a_{ij} := \langle \mathbf{v}_i, \mathbf{v}_j \rangle \not\equiv 0 \pmod{l} \qquad i \neq j,$$

so that $(\mathbf{v}_i, \mathbf{v}_j)_l = \left(\frac{D\, a_{ij}}{l}\right)$. Hence the $\mathbb{Z}$-lattice Λ' spanned by the $\mathbf{v}_i$ has discriminant

$$\operatorname{disc} \Lambda' \equiv a_{23}\, a_{31}\, a_{12} \not\equiv 0 \pmod{l}.$$

Thus

$$\left(\frac{D}{l}\right) = \left(\frac{a_{23}\, a_{31}\, a_{12}}{l}\right),$$

since Λ and Λ' are lattices in the same quadratic space, and so

$$\left(\frac{D\, a_{12}}{l}\right)\left(\frac{D\, a_{23}}{l}\right) = \left(\frac{D\, a_{31}}{l}\right),$$

which is the claimed statement. □

Lemma 4.4. *Let $\mathbf{u} \in \Lambda_l$ such that $\|\mathbf{u}\| \in \mathbb{Z}_l^\times$. If $\mathbf{v}$ is an l-vector in Λ, then $\tau_{\mathbf{u}} \mathbf{v}$ is also an l-vector in Λ, and*

$$(\mathbf{v}, \tau_{\mathbf{u}} \mathbf{v})_l = \left(\frac{-D\, \|\mathbf{u}\|}{l}\right).$$

Proof. Let $\mathbf{v}' := \tau_{\mathbf{u}} \mathbf{v}$, and note that $\|\mathbf{v}'\| = \|\mathbf{v}\|$. Consider first the case when $\langle \mathbf{v}, \mathbf{u} \rangle \not\equiv 0 \pmod{l}$. Since

$$\langle \mathbf{v}, \mathbf{v}' \rangle = 2\, \|\mathbf{v}\| - \frac{\langle \mathbf{v}, \mathbf{u} \rangle^2}{\|\mathbf{u}\|} \equiv -\, \|\mathbf{u}\| \left(\frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\|\mathbf{u}\|}\right)^2 \not\equiv 0 \pmod{l},$$

it follows that $\mathbf{v}'$ is an l-vector and, moreover,

$$(\mathbf{v}, \mathbf{v}')_l = \left(\frac{D \langle \mathbf{v}, \mathbf{v}' \rangle}{l}\right) = \left(\frac{-D\, \|\mathbf{u}\|}{l}\right)$$

If, on the other hand, $\langle \mathbf{v}, \mathbf{u} \rangle \equiv 0 \pmod{l}$, we conclude that $\mathbf{v}' \equiv \mathbf{v} \pmod{l\Lambda}$, so that $(\mathbf{v}', \mathbf{v})_l = 1$. Now, since $\mathbf{v} \notin l\Lambda$, and $l \nmid \operatorname{disc} \Lambda$, there must be $\mathbf{w} \in \Lambda$ such that $\langle \mathbf{v}, \mathbf{w} \rangle \not\equiv 0 \pmod{l}$. The $\mathbb{Z}$-lattice Λ' spanned by $\mathbf{u}, \mathbf{v}, \mathbf{w}$ has discriminant

$$\operatorname{disc} \Lambda' \equiv -\langle \mathbf{v}, \mathbf{w} \rangle^2 \|\mathbf{u}\| \not\equiv 0 \pmod{l},$$

hence $\left(\frac{D}{l}\right) = \left(\frac{-\|\mathbf{u}\|}{l}\right)$, and it follows that $\left(\frac{-D\|\mathbf{u}\|}{l}\right) = 1$, as required. □

Proposition 4.5. *Let $\sigma \in O^+(\Lambda_l)$. If $\mathbf{v}$ is an l-vector in Λ, then $\sigma\mathbf{v}$ is also an l-vector in Λ, and*

$$(\mathbf{v}, \sigma\mathbf{v})_l = \left(\frac{\theta(\sigma)}{l}\right).$$

Proof. By Lemma 2.3, σ is a product of (an even number of) symmetries $\tau_{\mathbf{u}}$ with $\mathbf{u} \in \Lambda_l$ and $\|\mathbf{u}\| \in \mathbb{Z}_l^\times$. The result then follows from Lemma 4.4, together with property 5 of Lemma 4.3. □

Corollary 4.6. *Fix an l-vector $\mathbf{v}_0$ in Λ. Then*

$$\omega_\Lambda(\mathbf{v}) := (\mathbf{v}, \mathbf{v}_0)_l$$

is a weight function on Λ of conductor l.

Proof. Immediate by Lemma 4.3 and Proposition 4.5. □

4.4 Ternary lattices and Hecke operators

Let Λ be an integral $\mathbb{Z}$-lattice of dimension 3 and discriminant D, as in the previous section. Fix an odd squarefree integer l relatively prime to D, and a weight function ω_Λ on Λ of conductor l.

Let p be a prime such that $p \nmid D$ Recall that in this case

$$\|v\| \equiv 0 \pmod{p}$$

is the equation of a non-degenerate conic modulo p, which has $p+1$ projective solutions, all of them non-singular. We let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{p+1}$ be representatives of the different solutions, such that $p^2 \mid \|\boldsymbol{v}_i\|$, and set $\Lambda_i := \Lambda_{\boldsymbol{v}_i}$. Thus $\Lambda_1, \ldots, \Lambda_{p+1}$ are all the distinct p-neighboring lattices of Λ, by Theorem 3.5.

Lemma 4.7. *Let $\boldsymbol{v} \in \Lambda$. The number of p-neighbors Λ_i of Λ such that $\boldsymbol{v} \in \Lambda_i$ is $p+1$ if $\boldsymbol{v} \in p\Lambda$, and*

$$1 + \left(\frac{-D\,\|\boldsymbol{v}\|}{p}\right)$$

otherwise.

Proof. If $\boldsymbol{v} \in p\Lambda$, then $\boldsymbol{v} \in \Lambda_i$ for all i; as remarked above, the number of p-neighbors of Λ is exactly $p+1$.

Consider now the case when $\boldsymbol{v} \notin p\Lambda$. Note that $\boldsymbol{v} \in \Lambda_i$ if and only if $p \mid \langle \boldsymbol{v}_i, \boldsymbol{v}\rangle$. Thus, we need to count the number of projective solutions of

$$\|\boldsymbol{w}\| \equiv 0 \pmod{p}, \qquad \boldsymbol{w} \in \widetilde{\Lambda},$$

where

$$\widetilde{\Lambda} := \{ \boldsymbol{w} \in \Lambda \ : \ \langle \boldsymbol{w}, \boldsymbol{v} \rangle \equiv 0 \pmod{p} \}$$

is a quadratic space over $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$ of dimension 2. These will be given by the solutions of a quadratic equation of discriminant $\operatorname{disc} \widetilde{\Lambda}$. But over $\mathbb{F}_p$ we have an orthogonal decomposition

$$\Lambda = \mathbb{F}_p \, \boldsymbol{v} + \widetilde{\Lambda},$$

hence $\operatorname{disc} \widetilde{\Lambda} \equiv -D \, \|\boldsymbol{v}\| \pmod{p}$, and the result follows. □

Theorem 4.8. *Suppose* $p \nmid lD$. *Then*

$$\sum_i r_l(\Lambda_i^{(p)}, m) = r_l(\Lambda, p^2 m) + \left(\frac{-Dm}{p} \right) r_l(\Lambda, m) + p \, r_l(\Lambda, m/p^2),$$

where the sum is over all p*-neighbors* Λ_i *of* Λ. *In other words,*

$$\vartheta_l(t_p \Lambda) = \vartheta_l(\Lambda)_{|T(p^2)},$$

i.e. ϑ_l *is a Hecke-linear map (outside* l*).*

Remark. We do not prove linearity of ϑ_l for the Hecke operators t_p with $p \mid l$. We believe that an elementary proof in the same lines, but with additional complications, should be possible. Nevertheless, we prove it below under the assumption of Conjecture 4.1.

Proof. We assume $l \equiv 1 \pmod{4}$ since, as it was already remarked, both sides are trivially 0 otherwise. The conic given by (3.1) has no singular solutions, and thus

$$\left\{ \boldsymbol{v} \in \Lambda - p\Lambda \ : \ p^2 \mid \|\boldsymbol{v}\| \right\} = \bigcup_i (p\Lambda_i - p\Lambda),$$

where the union is over all p-neighbors of Λ. By the last statement of Theorem 3.5, this union is disjoint. Counting vectors of norm p^2ml with weight ω_Λ, and noting that

$$r_l(p\Lambda, p^2m) = \left(\frac{p}{l}\right) r_l(\Lambda, m) = r_l(\Lambda^{(p)}, m),$$

we conclude

$$r_l(\Lambda, p^2m) - \left(\frac{p}{l}\right) r_l(\Lambda, m) = \sum_i r_l(\Lambda_i^{(p)}, m) - \left(\frac{p}{l}\right) \sum_i r_l(\Lambda_i \cap \Lambda, m).$$

It remains only to evaluate

$$\sum_i r_l(\Lambda_i \cap \Lambda, m) = \sum_i r_l(\Lambda_i \cap p\Lambda, m) + \sum_i r_l(\Lambda_i \cap (\Lambda - p\Lambda))$$

By Lemma 4.7, we have

$$\begin{aligned}\sum_i r_l(\Lambda_i \cap p\Lambda, m) &= (p+1)\, r_l(p\Lambda, m) \\ &= (p+1) \left(\frac{p}{l}\right) r_l(\Lambda, m/p^2),\end{aligned}$$

and

$$\begin{aligned}&\sum_i r_l(\Lambda_i \cap (\Lambda - p\Lambda), m) \\ &\qquad = \left(1 + \left(\frac{-Dml}{p}\right)\right) r_l(\Lambda - p\Lambda, m) \\ &\qquad = \left(1 + \left(\frac{l}{p}\right) \left(\frac{-Dm}{p}\right)\right) r_l(\Lambda, m) - \left(\frac{p}{l}\right) r_l(\Lambda, m/p^2).\end{aligned}$$

Since we are assuming $l \equiv 1 \pmod 4$, we have $\left(\frac{p}{l}\right)\left(\frac{l}{p}\right) = 1$, and the claimed formula follows.

For the last statement, write the formula as

$$r_l(t_p\Lambda, m) = r_l(\Lambda, p^2 m) + \chi_1(p)\left(\frac{m}{p}\right) r_l(\Lambda, m) + p\, r_l(\Lambda, m/p^2),$$

where

$$\chi_1(p) := \left(\frac{-D}{p}\right) = \chi_\Lambda(p)\left(\frac{-1}{p}\right),$$

and compare this formula with the action of the Hecke operators on modular forms of weight 3/2, as given in terms of their Fourier coefficients by Theorem 1.7 in Shimura (1973, p. 450). □

Proposition 4.9. *Assume Conjecture 4.1 holds. Then* ϑ_l *is linear for the Hecke operators* t_p *with* $p \mid l$ *as well.*

Proof. Let E be an eigenvector in the Brandt module $\mathcal{M}_{\mathbb{R}}(\Lambda)$. We prove that $\vartheta_l(t_p E) = \vartheta_l(E)_{|T(p^2)}$ for all $p \nmid N$; the result follows from this by Proposition 3.8. Assume $\vartheta_l(E) \neq 0$, as the claim is trivial otherwise. Let l' be an odd squarefree integer relatively prime to l and such that $\vartheta_{l'}(E) \neq 0$, as in the Conjecture.

By Theorem 4.8, we know that $\vartheta_l(E)$ and $\vartheta_{l'}(E)$ are eigenforms of $T(p^2)$ with the same eigenvalues as E for all $p \nmid lD$ and for all $p \nmid l'D$, respectively. The next Lemma implies that $\vartheta_l(E)$ and $\vartheta_{l'}(E)$ are indeed eigenforms of $T(p^2)$ with the same eigenvalues for all $p \nmid N$. Since l and l' are relatively prime, these eigenvalues are equal to those of E for all $p \nmid N$, and it follows that $\vartheta_l(t_p E) = \vartheta_l(E)_{|T(p^2)}$ for all $p \nmid N$, as claimed. □

Lemma 4.10. *Let* f, g *be modular forms of weight* $3/2$ *and level* N*, and assume* f *and* g *are common eigenforms of* $T(p^2)$ *with the same eigenvalue for all prime numbers* $p \nmid AN$*, where* A *is a given integer. Then* f *and* g *are indeed eigenforms of* $T(p^2)$ *with the same eigenvalue for all prime numbers* $p \nmid N$*.*

Proof. Let V be the subspace of modular forms of weight $3/2$ and level N which are common eigenforms of $T(p^2)$ with the same eigenvalue as f and g for all $p \nmid AN$. Suppose $h_1, h_2 \in V$ are two common eigenforms of $T(p^2)$ for all $p \nmid N$, and let H_1 and H_2 be their images under the Shimura correspondence. Then H_1 and H_2 are a modular forms of weight 2 and level dividing N which are common eigenforms of $T(p)$ for all $p \nmid N$, with same eigenvalues as h_1 and h_2, respectively. In particular, their eigenvalues of h_1 and h_2 are the same for all $p \nmid AN$. But then the strong multiplicity one Theorem for modular forms of integral weight implies that the eigenvalues are indeed the same for all $p \nmid N$, and thus the same is true for the eigenvalues of h_1 and h_2. It follows that V is a common eigenspace of $T(p^2)$ for all $p \nmid N$, and the result follows since $f, g \in V$. $\square$

5. Generalized theta series

Throughout this chapter, Λ will be a positive definite integral $\mathbb{Z}$-lattice of dimension n. The aim is to prove transformation formulas for the generalized theta series $\vartheta_l(\Lambda)$. In particular we show that they are modular forms of level and character independent of l.

5.1 Dual l-functions

Let l be an odd squarefree integer, and consider an l-function ω on Λ. The *dual* l-function $\widehat{\omega}$ on $\Lambda^{\#}$ is defined by

$$\widehat{\omega}(\mathbf{h}) := l^{-n/2} \sum_{\mathbf{u}\in\Lambda/l\Lambda} \omega(\mathbf{u})\, \mathrm{e}\left(\frac{\langle \mathbf{u}, \mathbf{h}\rangle}{l}\right), \tag{5.1}$$

for $\mathbf{h} \in \Lambda^{\#}$, where

$$\mathrm{e}\,(z) := e^{2\pi i z}.$$

Note that $\omega \mapsto \widehat{\omega}$ is the Fourier transform on the finite group $\Lambda/l\Lambda$, and so it is natural to expect the following Lemma to hold.

Lemma 5.1. *For* $\boldsymbol{v} \in \Lambda$ *we have* $\widehat{\widehat{\omega}}(\boldsymbol{v}) = \omega(-\boldsymbol{v})$.

Proof.

$$\begin{aligned}
\widehat{\widehat{\omega}}(\boldsymbol{v}) &= l^{-n/2} \sum_{\mathbf{h} \in \Lambda^{\#}/l\Lambda^{\#}} \widehat{\omega}(\mathbf{h})\, \mathrm{e}\left(\frac{\langle \mathbf{h}, \boldsymbol{v}\rangle}{l}\right) \\
&= l^{-n} \sum_{\substack{\mathbf{h} \in \Lambda^{\#}/l\Lambda^{\#} \\ \mathbf{u} \in \Lambda/l\Lambda}} \omega(\mathbf{u})\, \mathrm{e}\left(\frac{\langle \mathbf{h}, \mathbf{u} + \boldsymbol{v}\rangle}{l}\right) \\
&= l^{-n} \sum_{\mathbf{u} \in \Lambda/l\Lambda} \omega(\mathbf{u}) \sum_{\mathbf{h} \in \Lambda^{\#}/l\Lambda^{\#}} \mathrm{e}\left(\frac{\langle \mathbf{h}, \mathbf{u} + \boldsymbol{v}\rangle}{l}\right) \\
&= \omega(-\boldsymbol{v}),
\end{aligned}$$

since

$$\sum_{\mathbf{h} \in \Lambda^{\#}/l\Lambda^{\#}} \mathrm{e}\left(\frac{\langle \mathbf{h}, \mathbf{u} + \boldsymbol{v}\rangle}{l}\right) = \begin{cases} l^n & \text{if } \mathbf{u} + \boldsymbol{v} \in l\Lambda, \\ 0 & \text{otherwise.} \end{cases} \qquad \square$$

In what follows we will assume that l is odd and relatively prime to $\operatorname{disc} \Lambda$. This way $\Lambda \subseteq \Lambda^{\#}$ induces a group isomorphism

$$\Lambda/l\Lambda \xrightarrow{\sim} \Lambda^{\#}/l\Lambda^{\#},$$

so that l-functions on Λ and $\Lambda^{\#}$ are really the same. We remark that the definition of $\widehat{\omega}$ is independent of whether we consider ω as an l-function on Λ or as an l-function on $\Lambda^{\#}$; we can assume that the summation in (5.1) is always over $\Lambda/l\Lambda$.

Let $N = N(\Lambda)$ be the *level* of Λ, the smallest positive integer such that

$$N \,\|\mathbf{h}\| \in \mathbb{Z}, \qquad \text{all } \mathbf{h} \in \Lambda^{\#}.$$

Note that $N\Lambda^{\#} \subseteq \Lambda$, $\operatorname{disc} \Lambda \mid N^n$, and $N \mid 4 \operatorname{disc} \Lambda$.

Lemma 5.2. *Let $\mathbf{h}, \mathbf{k} \in \Lambda^{\#}$, and let d be an integer relatively prime to l. The following generalization of (5.1) holds:*

$$\sum_{\mathbf{v}\in\Lambda/l\Lambda} \omega(\mathbf{h}+d\mathbf{v})\,\mathrm{e}\left(\frac{\langle \mathbf{h}+d\mathbf{v},\mathbf{k}\rangle}{dl}\right) = l^{n/2}\,\mathrm{e}\left(\frac{y\langle \mathbf{h},\mathbf{k}\rangle}{d}\right)\widehat{\omega_d}(\mathbf{k}), \tag{5.2}$$

where y is an integer such that $yl \equiv 1 \pmod{dN}$, and where $\omega_d(\mathbf{v}) := \omega(d\mathbf{v})$.

Proof. Note that $(1-yl)\mathbf{h} \in d\Lambda$, hence

$$\begin{aligned}
&\sum_{\mathbf{v}\in\Lambda/l\Lambda} \omega(\mathbf{h}+d\mathbf{v})\,\mathrm{e}\left(\frac{\langle \mathbf{h}+d\mathbf{v},\mathbf{k}\rangle}{dl}\right)\\
&= \mathrm{e}\left(\frac{y\langle \mathbf{h},\mathbf{k}\rangle}{d}\right)\sum_{\mathbf{v}\in\Lambda/l\Lambda} \omega((1-yl)\mathbf{h}+d\mathbf{v})\;\mathrm{e}\left(\frac{\langle (1-yl)\mathbf{h}+d\mathbf{v},\mathbf{k}\rangle}{dl}\right)\\
&= \mathrm{e}\left(\frac{y\langle \mathbf{h},\mathbf{k}\rangle}{d}\right)\sum_{\mathbf{v}\in\Lambda/l\Lambda} \omega_d(\mathbf{v})\;\mathrm{e}\left(\frac{\langle \mathbf{v},\mathbf{k}\rangle}{l}\right)\\
&\overset{(5.1)}{=} l^{n/2}\,\mathrm{e}\left(\frac{y\langle \mathbf{h},\mathbf{k}\rangle}{d}\right)\widehat{\omega_d}(\mathbf{k})
\end{aligned}$$

□

Lemma 5.3. *Let $\mathbf{h} \in \Lambda^{\#}$, and let b and d be integers such that d is relatively prime to l. If ω is a normal l-function on Λ, then*

$$\omega(\mathbf{h})\,\mathrm{e}\left(\frac{b\,\|\mathbf{h}\|}{dl}\right) = \omega(\mathbf{h})\,\mathrm{e}\left(\frac{by\,\|\mathbf{h}\|}{d}\right), \tag{5.3}$$

where y is an integer such that $yl \equiv 1 \pmod{dN}$

Proof.

$$\begin{aligned}
\omega(\mathbf{h})\,\mathrm{e}\left(\frac{b\,\|\mathbf{h}\|}{dl}\right) &= \omega(\mathbf{h})\,\mathrm{e}\left(\frac{b(1-yl)\,\|\mathbf{h}\|}{dl}\right)\mathrm{e}\left(\frac{by\,\|\mathbf{h}\|}{d}\right)\\
&= \omega(\mathbf{h})\,\mathrm{e}\left(\frac{by\,\|\mathbf{h}\|}{d}\right),
\end{aligned}$$

since $\omega(\mathbf{h}) = 0$ unless $b\frac{1-yl}{d}\,\|\mathbf{h}\| \in \mathbb{Z}$ is a multiple of l. □

Lemma 5.4. *If ω is a transportable l-function on Λ of dimension at least 3, then $\widehat{\omega}$ is normal.*

Proof. We assume l is prime. Let $\boldsymbol{v} \in \Lambda$ such that $l \nmid \|\boldsymbol{v}\|$; Lemma 2.5 gives some $\sigma \in O^+(\Lambda_l)$ with $\sigma\boldsymbol{v} = \boldsymbol{v}$ and $\theta(\sigma) = -1$. Then

$$\begin{aligned} l^{n/2}\,\widehat{\omega}(\boldsymbol{v}) &= \sum_{\boldsymbol{u}\in\Lambda/l\Lambda} \omega(\boldsymbol{u})\,\mathrm{e}\left(\frac{\langle\boldsymbol{u},\boldsymbol{v}\rangle}{l}\right) \\ &= \sum_{\boldsymbol{u}\in\Lambda/l\Lambda} \omega(\sigma\boldsymbol{u})\,\mathrm{e}\left(\frac{\langle\sigma\boldsymbol{u},\boldsymbol{v}\rangle}{l}\right) \\ &= \left(\frac{\theta(\sigma)}{l}\right) \sum_{\boldsymbol{u}\in\Lambda/l\Lambda} \omega(\boldsymbol{u})\,\mathrm{e}\left(\frac{\langle\boldsymbol{u},\boldsymbol{v}\rangle}{l}\right) \\ &= -l^{n/2}\,\widehat{\omega}(\boldsymbol{v}), \end{aligned}$$

and so $\widehat{\omega}(\boldsymbol{v}) = 0$. □

5.2 Generalized theta series

We prove now a general transformation formula for generalized theta series. Recall that a spherical function of order $\nu \geq 0$ is an homogeneous polynomial of degree ν (with complex coefficients) satisfying the Laplace differential equation.

Although we don't use theta series with spherical polynomials in this thesis, we include them here for the sake of generality, since the proofs remain almost the same. The extra generality will undoubtfully find applications, as it has been exemplified by Rosson and Tornaría (2005).

We do note, however, that our weight functions could be regarded, in a sense, as the non-archimedean analogues of spherical functions.

Definition. Let Λ be a positive definite integral $\mathbb{Z}$-lattice, ω an l-function on $\Lambda^{\#}$, and P a spherical function of order ν. For $\mathbf{h} \in \Lambda^{\#}$ we define a *generalized theta series* with variable $z \in \mathfrak{H}$:

$$\vartheta_l(z; \Lambda, \omega, \mathbf{h}) := \sum_{\boldsymbol{\nu} \in \Lambda} P(\mathbf{h} + \boldsymbol{\nu}) \omega(\mathbf{h} + \boldsymbol{\nu}) \; e\left(\frac{z \, \|\mathbf{h} + \boldsymbol{\nu}\|}{l} \right).$$

We do not include P in the notation, since P will always be the same.

Note that $\vartheta_l(z; \Lambda, \omega, \mathbf{h})$ really depends only on $\mathbf{h}$ modulo Λ. When $\mathbf{h} \in \Lambda$, we will drop the $\mathbf{h}$ and write

$$\vartheta_l(z; \Lambda, \omega) := \sum_{\boldsymbol{\nu} \in \Lambda} P(\boldsymbol{\nu}) \omega(\boldsymbol{\nu}) \; e\left(\frac{z \, \|\boldsymbol{\nu}\|}{l} \right),$$

If $P = 1$, this agrees with the definition given in the last chapter.

In particular, for $l = 1$ and $\omega = 1$, we have the usual theta series

$$\vartheta(z; \Lambda, \mathbf{h}) := \vartheta_1(z; \Lambda, 1, \mathbf{h}) = \sum_{\boldsymbol{\nu} \in \Lambda} e\left(z \, \|\mathbf{h} + \boldsymbol{\nu}\| \right),$$

for which we recall the following

Lemma 5.5. *Let Λ be a positive definite integral $\mathbb{Z}$-lattice of dimension n and determinant Δ. For $\mathbf{h} \in \Lambda^{\#}$, the following transformation formula holds:*

$$(5.4) \quad \vartheta\left(\tfrac{-1}{z}; \Lambda, \mathbf{h} \right) = \Delta^{-1/2} \left(-iz \right)^{n/2} \left(-z \right)^{\nu} \sum_{\mathbf{k} \in \Lambda^{\#}/\Lambda} e\left(\langle \mathbf{h}, \mathbf{k} \rangle \right) \vartheta(z; \Lambda, \mathbf{k}).$$

Proof. This is a classical result; it follows by applying the Poisson summation formula for the lattice Λ and its dual $\Lambda^\#$ to the function

$$\boldsymbol{\nu} \mapsto P(\mathbf{h}+\boldsymbol{\nu})\, e\left(\tfrac{-1}{z}\,\|\mathbf{h}+\boldsymbol{\nu}\|\right),$$

defined in $\Lambda \otimes \mathbb{R}$, whose Fourier transform is

$$\mathbf{k} \mapsto P(-z\mathbf{k})(-iz)^{n/2}\, e\left(\langle \mathbf{h}, \mathbf{k}\rangle\right) e\left(z\,\|\mathbf{k}\|\right),$$

and by noting that $e\left(\langle \mathbf{h}, \mathbf{k}\rangle\right)$ depends on $\mathbf{k}$ modulo Λ. □

Proposition 5.6. *Let Λ be a positive definite integral $\mathbb{Z}$-lattice of dimension n and determinant Δ; let $l > 0$ be an integer relatively prime to Δ, and let ω be a normal l-function on Λ. For $\mathbf{h} \in \Lambda^\#$, and for $b, d \in \mathbb{Z}$ with $d > 0$ and relatively prime to l, the following transformation formulas hold:*

$$(5.5)\quad \vartheta_l\left(\frac{-1}{d^2 z}; d\Lambda, \omega, \mathbf{h}\right) = \Delta^{-1/2}\,(-iz)^{n/2}\,(-dz)^{\nu} \sum_{\mathbf{k}\in\Lambda^\#/d\Lambda} e\left(\frac{y\langle \mathbf{h}, \mathbf{k}\rangle}{d}\right) \vartheta_l(z; d\Lambda, \widehat{\omega_d}, \mathbf{k}),$$

$$(5.6)\quad \vartheta_l\left(z + \frac{b}{d}; \Lambda, \omega, \mathbf{h}\right) = \sum_{\mathbf{u}\in\Lambda/d\Lambda} e\left(\frac{by\,\|\mathbf{h}+\mathbf{u}\|}{d}\right) \vartheta_l(z; d\Lambda, \omega, \mathbf{h}+\mathbf{u}),$$

where y is an integer such that $yl \equiv 1 \pmod{dN}$.

Proof. Note that

$$\vartheta_l(z; \Lambda, \omega, \mathbf{h}) = \sum_{\boldsymbol{\nu}\in\Lambda/l\Lambda} \omega(\mathbf{h}+\boldsymbol{\nu})\, \vartheta\left(\frac{z}{l}; l\Lambda, \mathbf{h}+\boldsymbol{\nu}\right).$$

We will obtain the first formula as an application of the lemma to the lattice $dl\Lambda$, of determinant $(dl)^{2n}\Delta$, whose dual lattice is given by $(dl\Lambda)^\# = (dl)^{-1}\Lambda^\#$:

$$\begin{aligned}
\vartheta_l\left(\frac{-1}{d^2 z}; d\Lambda, \omega, \mathbf{h}\right) &= \sum_{\boldsymbol{v}\in\Lambda/l\Lambda} \omega(\mathbf{h}+d\boldsymbol{v})\,\vartheta_l\left(\frac{-1}{d^2 l z}; dl\Lambda, \mathbf{h}+d\boldsymbol{v}\right) \\
&\overset{(5.4)}{=} l^{-n}\,\Delta^{-1/2}\,(-ilz)^{n/2}\,(-d^2lz)^{\nu} \quad \cdot \\
&\qquad \sum_{\substack{\boldsymbol{v}\in\Lambda/l\Lambda\\ \mathbf{k}\in\Lambda^\#/d^2l^2\Lambda}} \omega(\mathbf{h}+d\boldsymbol{v})\ \overbrace{\mathsf{e}\left(\left\langle \mathbf{h}+d\boldsymbol{v}, \frac{\mathbf{k}}{dl}\right\rangle\right)}^{\text{depends on } \mathbf{k} \text{ modulo } dl\Lambda}\ \underbrace{\vartheta_l(d^2lz; dl\Lambda, \tfrac{\mathbf{k}}{dl})}_{\vartheta_l(\frac{z}{l}; d^2l^2\Lambda, \mathbf{k})\cdot(dl)^{-\nu}} \\
&= l^{-n/2}\,\Delta^{-1/2}\,(-iz)^{n/2}\,(-dz)^{\nu} \quad \cdot \\
&\qquad \sum_{\substack{\boldsymbol{v}\in\Lambda/l\Lambda\\ \mathbf{k}\in\Lambda^\#/dl\Lambda}} \omega(\mathbf{h}+d\boldsymbol{v})\ \mathsf{e}\left(\frac{\langle \mathbf{h}+d\boldsymbol{v}, \mathbf{k}\rangle}{dl}\right)\ \vartheta_l(\tfrac{z}{l}; dl\Lambda, \mathbf{k}) \\
&\overset{(5.2)}{=} \Delta^{-1/2}(-iz)^{n/2}\,(-dz)^{\nu} \sum_{\mathbf{k}\in\Lambda^\#/dl\Lambda} \mathsf{e}\left(\frac{y\langle \mathbf{h}, \mathbf{k}\rangle}{d}\right)\ \widehat{\omega_d}(\mathbf{k})\,\vartheta_l(\tfrac{z}{l}; dl\Lambda, \mathbf{k}) \\
&= \Delta^{-1/2}(-iz)^{n/2}\,(-dz)^{\nu} \sum_{\mathbf{k}\in\Lambda^\#/d\Lambda} \mathsf{e}\left(\frac{y\langle \mathbf{h}, \mathbf{k}\rangle}{d}\right)\ \vartheta_l(z; d\Lambda, \widehat{\omega_d}, \mathbf{k})
\end{aligned}$$

For the second formula, denoting $P\omega(\boldsymbol{v}) := P(\boldsymbol{v})\omega(\boldsymbol{v})$,

$$\begin{aligned}
\vartheta_l\left(z+\tfrac{b}{d}; \Lambda, \omega, \mathbf{h}\right) &= \sum_{\boldsymbol{v}\in\Lambda} P\omega(\mathbf{h}+\boldsymbol{v})\,\mathsf{e}\left(\frac{b\,\|\mathbf{h}+\boldsymbol{v}\|}{dl}\right)\mathsf{e}\left(\frac{z\,\|\mathbf{h}+\boldsymbol{v}\|}{l}\right) \\
&\overset{(5.3)}{=} \sum_{\boldsymbol{v}\in\Lambda} P\omega(\mathbf{h}+\boldsymbol{v})\,\mathsf{e}\left(\frac{by\,\|\mathbf{h}+\boldsymbol{v}\|}{d}\right)\mathsf{e}\left(\frac{z\,\|\mathbf{h}+\boldsymbol{v}\|}{l}\right) \\
&= \sum_{\mathbf{u}\in\Lambda/d\Lambda} \mathsf{e}\left(\frac{by\,\|\mathbf{h}+\mathbf{u}\|}{d}\right) \sum_{\boldsymbol{v}\in\mathbf{u}+d\Lambda} P\omega(\mathbf{h}+\boldsymbol{v})\,\mathsf{e}\left(\frac{z\,\|\mathbf{h}+\boldsymbol{v}\|}{l}\right) \\
&= \sum_{\mathbf{u}\in\Lambda/d\Lambda} \mathsf{e}\left(\frac{by\,\|\mathbf{h}+\mathbf{u}\|}{d}\right)\,\vartheta_l(z; d\Lambda, \omega, \mathbf{h}+\mathbf{u}),
\end{aligned}$$

since $e\left(\frac{by\|\mathbf{h}+\boldsymbol{\nu}\|}{d}\right)$ depends only on $\boldsymbol{\nu}$ modulo $d\Lambda$, because $\mathbf{h} \in \Lambda^{\#}$. $\square$

Corollary 5.7. *With the same notation as in the proposition,*

$$\vartheta_l\left(z+\frac{b}{d}; d\Lambda, \omega, \mathbf{h}\right) = e\left(\frac{by\,\|\mathbf{h}\|}{d}\right)\vartheta_l(z; d\Lambda, \omega, \mathbf{h}). \tag{5.7}$$

Proof. Note that the right hand side depends only on y modulo dN, so we can assume $yl \equiv 1 \pmod{d^{2n+1}N}$. Since $\mathbf{h} \in \Lambda^{\#} \subseteq (d\Lambda)^{\#}$, we can then apply (5.6) to the lattice $d\Lambda$:

$$\begin{aligned}
&\vartheta_l\left(z+\frac{b}{d}; d\Lambda, \omega, \mathbf{h}\right)\\
&\qquad\overset{(5.6)}{=} \sum_{\mathbf{u}\in d\Lambda/d^2\Lambda} e\left(\frac{by\,\|\mathbf{h}+\mathbf{u}\|}{d}\right)\vartheta_l(z; d^2\Lambda, \omega, \mathbf{h}+\mathbf{u})\\
&\qquad= e\left(\frac{by\,\|\mathbf{h}\|}{d}\right)\sum_{\mathbf{u}\in d\Lambda/d^2\Lambda}\vartheta_l(z; d^2\Lambda, \omega, \mathbf{h}+\mathbf{u})\\
&\qquad= e\left(\frac{by\,\|\mathbf{h}\|}{d}\right)\vartheta_l(z; d\Lambda, \omega, \mathbf{h}),
\end{aligned}$$

since $e\left(\frac{by\|\mathbf{h}+\mathbf{u}\|}{d}\right) = e\left(\frac{by\|\mathbf{h}\|}{d}\right)$ for $\mathbf{u} \in d\Lambda$. $\square$

Recall the usual action of the modular group $SL_2(\mathbb{Z})$ on the upper-half plane; if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, then $\gamma(z) := \frac{az+b}{cz+d}$.

Proposition 5.8. *Let* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ *such that* $c \geq 0$, *and* $d > 0$ *is relatively prime to* l. *Assume* ω *is an* l*-function such that both* ω *and* $\widehat{\omega}$ *are normal. For* $\mathbf{h} \in \Lambda^{\#}$ *we have*

$$\vartheta_l(\gamma(z); \Lambda, \omega, \mathbf{h}) = (cz+d)^{n/2+\nu}\sum_{\mathbf{g}\in\Lambda^{\#}/\Lambda}\Psi^l_{\Lambda,\gamma}(\mathbf{h},\mathbf{g})\,\vartheta_l(z; \Lambda, \omega_d, \mathbf{g}),$$

where

$$\Psi^{l}_{\Lambda,\gamma}(\mathbf{h},\mathbf{g}) := \Delta^{-1}\sum_{\mathbf{k}\in\Lambda^{\#}/\Lambda}\Phi^{l}_{\Lambda,\gamma}(\mathbf{h},\mathbf{k})\,\mathrm{e}\left(-y\langle\mathbf{k},\mathbf{g}\rangle\right),$$

$$\Phi^{l}_{\Lambda,\gamma}(\mathbf{h},\mathbf{k}) := d^{-n/2}\sum_{\mathbf{u}\in\Lambda/d\Lambda}\mathrm{e}\left(\frac{by\,\|\mathbf{h}+\mathbf{u}\|+y\langle\mathbf{k},\mathbf{h}+\mathbf{u}\rangle-cy\,\|\mathbf{k}\|}{d}\right).$$

As before y is an integer such that $yl\equiv 1 \pmod{dN}$.

Proof. First we note that $\widehat{\omega_d}(\boldsymbol{\nu})=\widehat{\omega}(x\boldsymbol{\nu})$, where x is an integer such that $xd\equiv 1 \pmod{l}$, hence $\widehat{\omega}$ normal implies that $\widehat{\omega_d}$ is normal too.

We can write

$$\gamma(z)=\frac{b}{d}+\frac{z}{d(cz+d)},$$

and thus

$$\begin{aligned}
&\vartheta_l(\gamma(z);\Lambda,\omega,\mathbf{h})\\
&\overset{(5.6)}{=}\sum_{\mathbf{u}\in\Lambda/d\Lambda}\mathrm{e}\left(\frac{by\,\|\mathbf{h}+\mathbf{u}\|}{d}\right)\vartheta_l\left(\frac{z}{d(cz+d)};d\Lambda,\omega,\mathbf{h}+\mathbf{u}\right)\\
&\overset{(5.5)}{=}\Delta^{-1/2}\left(\frac{i(cz+d)}{dz}\right)^{n/2}\left(\frac{cz+d}{z}\right)^{\nu}\cdot\\
&\qquad\sum_{\substack{\mathbf{u}\in\Lambda/d\Lambda\\ \mathbf{k}\in\Lambda^{\#}/d\Lambda}}\mathrm{e}\left(\frac{by\,\|\mathbf{h}+\mathbf{u}\|+y\langle\mathbf{k},\mathbf{h}+\mathbf{u}\rangle}{d}\right)\vartheta_l\left(-\frac{c}{d}-\frac{1}{z};d\Lambda,\widehat{\omega_d},\mathbf{k}\right)\\
&\overset{(5.7)}{=}\Delta^{-1/2}\left(\frac{i(cz+d)}{z}\right)^{n/2}\left(\frac{cz+d}{z}\right)^{\nu}\cdot\\
&\qquad\sum_{\mathbf{k}\in\Lambda^{\#}/d\Lambda}\Phi^{l}_{\Lambda,\gamma}(\mathbf{h},\mathbf{k})\,\vartheta_l\left(\frac{-1}{z};d\Lambda,\widehat{\omega_d},\mathbf{k}\right).
\end{aligned}$$

It will follow from Lemma 5.9 below that $\Phi^{l}_{\Lambda,\gamma}(\mathbf{h},\mathbf{k})$ depends only on $\mathbf{k}$ modulo Λ, and so we can group terms modulo Λ.

The expression then becomes

$$\Delta^{-1/2}\left(\frac{i(cz+d)}{z}\right)^{n/2}\left(\frac{cz+d}{z}\right)^{\nu}\cdot \sum_{k\in\Lambda^{\#}/\Lambda}\Phi^{l}_{\Lambda,\gamma}(\mathbf{h},\mathbf{k})\,\vartheta_l\left(\frac{-1}{z};\Lambda,\widehat{\omega_d},\mathbf{k}\right).$$

We are now ready to apply (5.5) again; before doing it we note that

$$\left(\frac{i(cz+d)}{z}\right)^{n/2}(-iz)^{n/2}=(cz+d)^{n/2},$$

since $c\geq 0$ and $d>0$. Hence

$$\begin{aligned}
&\vartheta_l(\gamma(z);\Lambda,\omega,\mathbf{h})\\
&\qquad=(-1)^{\nu}\Delta^{-1}(cz+d)^{n/2+\nu}\cdot \sum_{\substack{\mathbf{g}\in\Lambda^{\#}/\Lambda\\ \mathbf{k}\in\Lambda^{\#}/\Lambda}}\Phi^{l}_{\Lambda,\gamma}(\mathbf{h},\mathbf{k})\;\mathrm{e}\left(y\langle\mathbf{k},\mathbf{g}\rangle\right)\,\vartheta_l(z;\Lambda,\widehat{\widehat{\omega_d}},\mathbf{g})\\
&\qquad=(cz+d)^{n/2+\nu}\sum_{\mathbf{g}\in\Lambda^{\#}/\Lambda}\Psi^{l}_{\Lambda,\gamma}(\mathbf{h},\mathbf{g})\,\vartheta_l(z;\Lambda,\omega_d,\mathbf{g}),
\end{aligned}$$

because $\vartheta_l(z;\Lambda,\widehat{\widehat{\omega_d}},\mathbf{g})=(-1)^{\nu}\vartheta_l(z;\Lambda,\omega_d,-\mathbf{g})$ by Lemma 5.1. □

Lemma 5.9. *For $\mathbf{h},\mathbf{k}\in\Lambda^{\#}$ we have*

$$\Phi^{l}_{\Lambda,\gamma}(\mathbf{h},\mathbf{k})=\mathrm{e}\left(ay\langle\mathbf{h},\mathbf{k}\rangle-acy\,\|\mathbf{k}\|\right)\Phi^{l}_{\Lambda,\gamma}(\mathbf{h}-c\mathbf{k},0).$$

In particular, $\Phi^{l}_{\Lambda,\gamma}(\mathbf{h},\mathbf{k})$ depends only on $\mathbf{h}$ and $\mathbf{k}$ modulo Λ.

Proof. It's clear that $\Phi^{l}_{\Lambda,\gamma}(\mathbf{h},\mathbf{k})$ depends on $\mathbf{h}$ modulo Λ, hence it will follow from the claimed formula that it also depends on $\mathbf{k}$ modulo Λ.

Since $1 = ad - bc$, we get

$$\begin{aligned}
\Phi^{l}_{\Lambda,\gamma}(\mathbf{h},\mathbf{k}) &= d^{-n/2} \sum_{\mathbf{u}\in\Lambda/d\Lambda} e\left(\frac{by\,\|\mathbf{h}+\mathbf{u}\| + y\langle \mathbf{k}, \mathbf{h}+\mathbf{u}\rangle - cy\,\|\mathbf{k}\|}{d}\right) \\
&= d^{-n/2} \sum_{\mathbf{u}\in\Lambda/d\Lambda} e\left(\frac{by\,\|\mathbf{h}+\mathbf{u}\| - bcy\langle \mathbf{k}, \mathbf{h}+\mathbf{u}\rangle + bc^2y\,\|\mathbf{k}\|}{d}\right) \cdot \\
&\qquad\qquad\qquad\qquad e\left(ay\langle \mathbf{k}, \mathbf{h}+\mathbf{u}\rangle - acy\,\|\mathbf{k}\|\right) \\
&= e\left(ay\langle \mathbf{k}, \mathbf{h}\rangle - acy\,\|\mathbf{k}\|\right)\, d^{-n/2} \sum_{\mathbf{u}\in\Lambda/d\Lambda} e\left(by\,\|\mathbf{h}+\mathbf{u}-c\mathbf{k}\|\right) \\
&= e\left(ay\langle \mathbf{k}, \mathbf{h}\rangle - acy\,\|\mathbf{k}\|\right) \Phi^{l}_{\Lambda,\gamma}(\mathbf{h}-c\mathbf{k}, 0),
\end{aligned}$$

as stated. □

5.3 Weight functions and modularity

We will prove now that if ω is a weight function on Λ of conductor l, the generalized theta series

$$\vartheta_l(z;\Lambda,\omega) := \vartheta_l(z;\Lambda,\omega,0)$$

is a modular form of weight $n/2+\nu$, level $N = N(\Lambda)$, and character

$$\chi(d) = \chi_{\Lambda,\nu}(d) := \left(\frac{(-1)^{(n+1)\nu}\operatorname{disc}\Lambda}{d}\right).$$

We remark that $P\omega$ is either even or odd, and $\vartheta_l(z;\Lambda,\omega) = 0$ unless $P\omega$ is even. If we set $s := \left(\frac{-1}{l}\right)^n(-1)^\nu$, then $P\omega$ is even if $s = 1$, and odd if $s = -1$.

We briefly recall now the definition of modular forms of integral and half-integral weight, and refer the reader to Shimura (1973) for details. For a positive integer N we let

$$\Gamma_0(N) := \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) \ : \ c \equiv 0 \pmod{N}\},$$

and define automorphy factors

$$j_n(\gamma, z) := (cz+d)^{n/2} \qquad\qquad \gamma \in \Gamma_0(1),\ n \text{ even},$$

and

$$j_n(\gamma, z) := \left(\frac{c}{d}\right) \varepsilon_d^{-n} (cz+d)^{n/2} \qquad \gamma \in \Gamma_0(4),\ n \text{ odd},$$

where ε_d is 1 or i according as $d \equiv 1$ or $3 \pmod 4$.

A *(holomorphic) modular form* of weight $n/2$, level N, and character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}$ is a holomorphic function $f : \mathfrak{H} \to \mathbb{C}$ such that

$$f(\gamma(z)) = \chi(d)\, j_n(\gamma, z)\, f(z), \qquad \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N),$$

and such that f is holomorphic also at the cusps. If, in addition, f vanishes at the cusps, we say that f is a *cusp form*. Implicitly we are requiring that N be divisible by 4 when n is odd.

Lemma 5.10. *Let* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$*, and assume* $d > 0$ *is an odd prime such that* $d \nmid l \operatorname{disc} \Lambda$*. Then*

$$(cz+d)^{n/2+\nu}\, \Phi^l_{\Lambda,\gamma}(0,0) \left(\tfrac{d}{l}\right)^n = \left(\frac{s}{d}\right) \chi_{\Lambda,\nu}(d)\, j_{n+2\nu}(\gamma, z),$$

where $s = \left(\frac{-1}{l}\right)^n (-1)^\nu$.

Proof. Since d is an odd prime, $\Lambda/d\Lambda$ is a quadratic space over the finite field $\mathbb{Z}/d\mathbb{Z}$, and therefore it has an orthogonal basis. We can lift such a basis to a basis $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ of Λ, so that

$$\langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle \equiv 0 \pmod{d} \qquad i \neq j.$$

Setting $\alpha_i := \|\boldsymbol{v}_i\|$, we have

$$\begin{aligned}
\Phi^l_{\Lambda,\gamma}(0,0) &= d^{-n/2} \sum_{\boldsymbol{u} \in \Lambda/d\Lambda} e\left(\frac{by\,\|u\|}{d}\right) \\
&= d^{-n/2} \prod_{i=1}^{n} \left(\sum_{x \in \mathbb{Z}/d\mathbb{Z}} e\left(\frac{by\,\alpha_i x^2}{d}\right)\right) \\
&= \varepsilon_d^n \left(\frac{(2by)^n \Delta}{d}\right),
\end{aligned}$$

where Δ is the determinant of Λ.

If n is even, $\varepsilon_d^n = \left(\frac{-1}{d}\right)^{n/2}$, hence

$$\Phi^l_{\Lambda,\gamma}(0,0) = \left(\frac{(-1)^{n/2}\Delta}{d}\right) = \left(\frac{s}{d}\right)\chi_{\Lambda,\nu}(d),$$

since $\operatorname{disc}\Lambda = (-1)^{n/2}\Delta$ and $s = (-1)^{\nu}$.

When n is odd, $\varepsilon_d^n = \left(\frac{(-1)^{\nu+1}}{d}\right)\varepsilon_d^{-(n+2\nu)}$. If we now use that $bc \equiv -1 \pmod{d}$ and $yl \equiv 1 \pmod{d}$, we get

$$\begin{aligned}
\Phi^l_{\Lambda,\gamma}(0,0)\left(\frac{d}{l}\right) &= \left(\frac{(-1)^{\nu+1}}{d}\right)\varepsilon_d^{-(n+2\nu)}\left(\frac{-cl}{d}\right)\left(\frac{2\Delta}{d}\right)\left(\frac{d}{l}\right) \\
&= \left(\frac{s}{d}\right)\chi_{\Lambda,\nu}(d)\left(\frac{c}{d}\right)\varepsilon_d^{-(n+2\nu)},
\end{aligned}$$

since $\operatorname{disc}\Lambda = \frac{1}{2}\Delta$, and $\left(\frac{(-1)^{\nu}}{d}\right)\left(\frac{l}{d}\right)\left(\frac{d}{l}\right) = \left(\frac{s}{d}\right)$ by the quadratic reciprocity law. $\square$

Theorem 5.11. *Let Λ be a positive definite integral $\mathbb{Z}$-lattice, and suppose l is an odd squarefree integer relatively prime to* $\operatorname{disc}\Lambda$. *Let ω be a weight function on Λ of conductor l, and assume $\widehat{\omega}$ is normal. Then*

$$\vartheta_l(\gamma(z); \Lambda, \omega, \mathbf{h}) = \left(\frac{s}{d}\right) \chi_{\Lambda,\nu}(d) j_{n+2\nu}(\gamma, z)\, \mathrm{e}\left(aby \|\mathbf{h}\|\right) \vartheta_l(z, \Lambda, \omega, a\mathbf{h})$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, where $N = N(\Lambda)$ is the level of Λ. Here n is the dimension, ν is the order of the (implicit) spherical function, y is an integer such that $yl \equiv 1 \pmod N$, and $s = \left(\frac{-1}{l}\right)^n (-1)^\nu$.

In particular $\vartheta_l(z; \Lambda, \omega)$ is a modular form of weight $n/2+\nu$, level N, and character $\chi_{\Lambda,\nu}$.

Remark. When the dimension of Λ is at least 3, the hypothesis on $\widehat{\omega}$ is redundant, by Lemma 5.4.

Proof. The formula is invariant under $\gamma \mapsto -\gamma$, so we will assume $c \geq 0$. Indeed, $\chi_{\Lambda,\nu}(d)$ and $j_{n+2\nu}(\gamma, z)$ are even functions of γ except in the case $n+2\nu \equiv 2 \pmod 4$, when they are both odd. On the other hand, the parity of $\left(\frac{s}{d}\right)$ depends on the sign of s which, as already remarked, corresponds to the parity of $P\omega$, and hence to the parity of $\vartheta_l(z, \Lambda, \omega, a\mathbf{h})$ as a function of a. Thus, the right hand side of the equation is an even function of γ; but so is the left hand side.

Note also that, for $m \in \mathbb{Z}$,

$$j_{n+2\nu}(\gamma, z+m) = j_{n+2\nu}(\gamma_m, z)$$

where $\gamma_m(z) := \gamma(z+m) = \frac{az+(b+am)}{cz+(d+cm)}$. Moreover,

$$e\left(aby\,\|\mathbf{h}\|\right)\vartheta_l(z+m,\Lambda,\omega,a\mathbf{h}) = e\left(a(b+am)y\,\|\mathbf{h}\|\right)\vartheta_l(z,\Lambda,\omega,a\mathbf{h})$$

by (5.6), and thus the formula is also invariant under $\gamma \mapsto \gamma_m$. We may then, by Dirichlet Theorem of primes in arithmetic progressions, assume that $d > 0$ is an odd prime such that $d \nmid l \operatorname{disc}\Lambda$. We will also assume, without loss of generality, that $yl \equiv 1 \pmod{dN}$.

The hypothesis implies that $c\,\|\mathbf{k}\| \in \mathbb{Z}$ and $c\mathbf{k} \in \Lambda$, and also that $\mathbf{h} = ad\mathbf{h} - bc\mathbf{h} \equiv ad\mathbf{h} \pmod{\Lambda}$. Hence

$$\Phi^l_{\Lambda,\gamma}(\mathbf{h},\mathbf{k}) = e\left(ay\langle\mathbf{h},\mathbf{k}\rangle\right)\Phi^l_{\Lambda,\gamma}(ad\mathbf{h},0).$$

by Lemma 5.9. Now notice that

$$\begin{aligned}\Phi^l_{\Lambda,\gamma}(ad\mathbf{h},0) &= e\left(aby\cdot ad\,\|\mathbf{h}\|\right)\Phi^l_{\Lambda,\gamma}(0,0)\\ &= e\left(aby\,\|\mathbf{h}\|\right)\Phi^l_{\Lambda,\gamma}(0,0),\end{aligned}$$

since $\|\mathbf{h}\| = ad\,\|\mathbf{h}\| - bc\,\|\mathbf{h}\| \equiv ad\,\|\mathbf{h}\| \pmod{\mathbb{Z}}$. Hence

$$\begin{aligned}\Psi^l_{\Lambda,\gamma}(\mathbf{h},\mathbf{g}) &= \Phi^l_{\Lambda,\gamma}(0,0)\,e\left(aby\,\|\mathbf{h}\|\right)\cdot\Delta^{-1}\sum_{\mathbf{k}\in\Lambda^\#/\Lambda} e\left(y\langle a\mathbf{h}-\mathbf{g},\mathbf{k}\rangle\right)\\ &= \Phi^l_{\Lambda,\gamma}(0,0)\,e\left(aby\,\|\mathbf{h}\|\right)\cdot\begin{cases}1 & \text{if } a\mathbf{h}-\mathbf{g}\in\Lambda,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Since $\widehat{\omega}$ is normal, Proposition 5.8 now implies

$$\begin{aligned}&\vartheta_l(\gamma(z);\Lambda,\omega,\mathbf{h})\\ &\qquad = (cz+d)^{n/2+\nu}\Phi^l_{\Lambda,\gamma}(0,0)\,e\left(aby\,\|\mathbf{h}\|\right)\vartheta_l(z,\Lambda,\omega_d,a\mathbf{h}),\end{aligned}$$

and the result follows by the lemma, since $\omega_d = \left(\frac{d}{l}\right)^n \omega$.

When $\mathbf{h} = 0$, the formula reads

$$\vartheta_l(\gamma(z); \Lambda, \omega) = \left(\frac{s}{d}\right) \chi_{\Lambda,\nu}(d) j_{n+2\nu}(\gamma, z) \vartheta_l(z, \Lambda, \omega),$$

which proves the modularity of $\vartheta_l(\gamma(z); \Lambda, \omega)$ as claimed, provided $s = 1$. Otherwise, $s = -1$, and $P\omega$ is an odd function, hence $\vartheta_l(\gamma(z); \Lambda, \omega) = 0$. $\square$

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Vita

Gonzalo Tornaría was born in Montevideo, Uruguay on March 27, 1976, the son of Eduardo Tornaría and Marta López. He is married to Jimena Rodríguez and they have two kids, Camila and Agustín.

He became seriously involved with Mathematics in 1990, at the age of 14, when he participated in a Mathematical Olympiad for the first time. In the following years he won several prizes in regional competitions, including four gold medals and two silver medals.

He received the degree of *Licenciado en Matemática* from *Universidad de la República Oriental del Uruguay* in December, 1999, after completing his thesis *El 2-subgrupo de Sylow del grupo de clases de formas cuadráticas binarias enteras* under the supervision of Dr. Pascual Llorente.

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