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CENTRAL VALUES OF QUADRATIC TWISTS FOR A MODULAR FORM OF WEIGHT 4.

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1. Introduction

Let $f = f_{\mathsf{7k4A}}$ be the (cusp) modular form of weight 4 and level 7. We consider here the problem of computing the central values for the family of twisted L-functions

$$L(f, D, s) = \sum_{m=1}^{\infty} \frac{a(m)}{m^s} \left(\frac{D}{m}\right),$$

where D is a fundamental discriminant and $a(m)$ is the m-th Fourier coefficient of f.

By Waldspurger's formula [W], these central values $L(f, D, 2)$ are related to the $|D|$-th Fourier coefficient of weight 5/2 modular forms in Shimura correspondence with f. The question is, then, how to find the said modular forms, and specifically how to compute their Fourier coefficients.

A constructive version of Waldspurger's formula was proved by Gross in [G] for the case of weight 2 and prime level. This has been generalized in several ways (cf. [BSP1], [BSP2], [MRVT], [PT]). In all these constructions, the modular forms of half integral weight are obtained as linear combinations of (generalized) ternary theta series coming from the arithmetic of quaternion algebras.

For the case of higher weight as done in [BSP2] these theta series involve spherical polynomials of *even* degree, and thus apply only to the construction of modular forms of weight $\equiv 3/2 \pmod 2$ in correspondence with even weights $\equiv 2 \pmod 4$.

Indeed, to obtain modular forms of weight 5/2 from ternary quadratic forms it is necessary to utilize spherical polynomials of degree 1. However, such a theta series vanishes trivially, since such polynomials are odd functions. We will show how to solve this problem by employing the weight functions defined in [MRVT]. Although we only show here the simplest example, it is clear that a combination of the techniques of [BSP2] with those of [MRVT] should be enough to completely solve the problem in question for any modular form of even weight and prime level.

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2. Spherical polynomials and f_{7k4A}

Let $B = B(-1,-7)$ the quaternion algebra ramified at 7 and ∞. A maximal order, with class number 1, is given by

$$R = \left\langle 1, i, \frac{1+j}{2}, \frac{i+k}{2} \right\rangle.$$

The norm in the given basis is the quaternary quadratic form

$$\mathcal{N}(a,b,c,d) = a^2 + b^2 + 2c^2 + 2d^2 + ac + bd,$$

of level 7. It follows from results of Eichler on the Basis Problem [E] that f_{7k4A} can be obtained as a (generalized) theta series for $\mathcal{N}$ with some spherical polynomial of degree 2.

The group of automorphisms of $\mathcal{N}$ has order 32, and is generated by the involutions

$$\mathcal{N}(a,b,c,d) = \mathcal{N}(a,-b,c,-d) = \mathcal{N}(b,a,d,c) = \mathcal{N}(a+c,b,-c,d).$$

Let $P(a,b,c,d)$ be a spherical polynomial of degree 2. We can assume without loss of generality that P is *even* (i.e. invariant) with respect to these involutions: if P were *odd* with respect to any of the above involutions the theta series weighted by P would be zero.

We claim that such P is unique up to a constant. Indeed, P is a quadratic form in 4 variables. Since $P(a,b,c,d) = P(a,-b,c,-d)$, it follows that $P(a,b,c,d) = P_1(a,c) + P_2(b,d)$ for some quadratic forms P_1 and P_2 in 2 variables. From $P(a,b,c,d) = P(b,a,d,c)$ we conclude that $P_1 = P_2$. The last involution implies that $P_1(a,c) = P_1(a+c,-c)$, and it follows that P_1 is a linear combination of the polynomials a^2+ac and c^2. The last condition on P_1 comes from the fact that P satisfies the Laplace differential equation

$$\Delta_{\mathcal{N}}(P) = 0,$$

where $\Delta_{\mathcal{N}}$ is the Laplacian operator with respect to the quadratic form $\mathcal{N}$. Note that $\Delta_{\mathcal{N}}(P) = 2\Delta_{\mathcal{N}}(P_1)$, and since P_1 is a quadratic form we can compute

$$\Delta_{\mathcal{N}}(P_1) = \operatorname{Tr}(M(P_1)\cdot M(\mathcal{N})^{-1}).$$

Here $M(P_1)$ and $M(\mathcal{N})$ are the matrices of P_1 and $\mathcal{N}$ respectively. Hence

$$\Delta_{\mathcal{N}}\big[\alpha(a^2+ac)+\beta(c^2)\big] = \frac{6\alpha+4\beta}{7} = 0,$$

and it follows that, up to a constant, $P_1(a,c) = 2a^2+2ac-3c^2$.

Therefore we can compute the Fourier expansion of f_{7k4A} by

$$\begin{aligned} f &:= \frac{1}{4}\sum_{(a,b,c,d)\in\mathbb{Z}^4} (2a^2+2ac-3c^2)\, q^{\mathcal{N}(a,b,c,d)} \\ &= \sum_{m=1}^{\infty} a(m)\, q^m \\ &= q - q^2 - 2q^3 - 7q^4 + 16q^5 + 2q^6 - 7q^7 + O(q^8). \end{aligned}$$

Note that we have used $P_1(a,c)$ as the spherical polynomial instead of $P(a,b,c,d)$ to simplify the computations, but as explained above the resulting theta series is the same up to a constant.

Finally, the standard method to compute the central values uses the quickly convergent series

$$(1)\qquad L(f,D,2) = (1+\epsilon_D)\sum_{m=1}^{\infty}\left(1+\frac{2\pi m}{\sqrt{N_D}}\right)\exp\left(-\frac{2\pi m}{\sqrt{N_D}}\right)\left(\frac{D}{m}\right)\frac{a(m)}{m^2},$$

where ϵ_D and N_D are the sign and the level of the functional equation for $L(f,D,s)$ and are easily seen to be

$$\epsilon_D = \operatorname{sign}(D) \cdot \begin{cases} \left(\frac{D}{7}\right) & \text{if } 7 \nmid D, \\ 1 & \text{if } 7 \mid D; \end{cases} \qquad N_D = D^2 \cdot \begin{cases} 7 & \text{if } 7 \nmid D, \\ 1 & \text{if } 7 \mid D. \end{cases}$$

Although the convergence of this series is exponential, the number of terms that is required to achieve a given precision is $O(\sqrt{N_D}) = O(|D|)$. Therefore, computing $L(f,D,2)$ for $|D| \leq x$ will take time roughly proportional to x^2, with a big constant. In the next section we will show how to compute the exact value for the ratios

$$\frac{L(f,D,2)}{L(f,1,2)} \quad D > 0, \qquad \frac{L(f,D,2)}{L(f,-4,2)} \quad D < 0,$$

in time proportional to $x^{3/2}$, with a much smaller constant. Of course, the special values $L(f,1,2)$ and $L(f,-4,2)$ can be computed very quickly by the series (1), since the respective levels 7 and $7 \cdot 16$ are very small.

For instance, using the first 1000 Fourier coefficients, the series (1) gives about 1000 decimal places for $L(f,1,2)$, and about 250 decimal places for $L(f,-4,2)$. However, the same 1000 Fourier coefficients will only give about 4 decimal places for $L(f,-191,2)$ or $L(f,197,2)$. Note that all the central values that appear in tables 1 and 2 were computed using the first 1000 Fourier coefficients of f, and thus their accuracy is actually less than what is displayed for the last few entries.

3. Two modular forms of weight 5/2

Consider the ternary lattice corresponding to R, namely

$$S^0 := \{ b \in \mathbb{Z} + 2R \ : \ \operatorname{Tr} b = 0 \} = \left\langle 2i,\ j,\ i+k \right\rangle.$$

In this section we only deal with quaternions in S^0, which we will write in that basis as triples of integers, so that (x,y,z) is the quaternion $x(2i) +$

$y(j) + z(i+k)$, and S^0 corresponds to $\mathbb{Z}^3$. With this convention, the norm restricted to S^0 is the ternary quadratic form

$$Q(x, y, z) := 4x^2 + 7y^2 + 8z^2 + 4xz,$$

whose corresponding bilinear form is

$$\big\langle (x, y, z), (x', y', z') \big\rangle := 8xx' + 14yy' + 16zz' + 4xz' + 4zx'.$$

As explained in the introduction, in order to obtain modular forms of weight 5/2 we need to compute a theta series of Q with spherical polynomials of degree 1. Since such polynomials are odd, we need to combine them with *odd* weight functions as defined in [MRVT].

3.1. **Imaginary twists.** Let ψ be the quadratic character of conductor 7. This is an odd character, and thus the weight function ω_7 associated to ψ is odd. To compute ω_7, we can use $b_0 = (1, 0, 2)$, of norm $44 \not\equiv 0 \pmod 7$. By computing $\langle (x, y, z), b_0 \rangle = 16x + 36z \equiv 2x + z \pmod 7$, we find that

$$\omega_7(x, y, z) := \left(\frac{2x + z}{7}\right).$$

We must now find a suitable spherical polynomial of degree 1; any homogeneous polynomial of degree 1 is indeed a spherical polynomial. Note that

$$Q(x + z, -y, -z) = Q(x, y, z),$$

so that $(x + z, -y, -z)$ is an automorphism of Q, and also

$$\omega_7(x + z, -y, -z) = \omega_7(x, y, z).$$

Thus, polynomials which are odd with respect to this involution, like y and z, will lead to null theta series; any other polynomial will give the same theta series as the unique even (i.e. invariant) polynomial $x + z/2$ (up to a constant). This is the natural candidate, although for the sake of simplicity we will use x instead.

d	$c_-(d)$	$L(f,-d,2)$	d	$c_-(d)$	$L(f,-d,2)$	d	$c_-(d)$	$L(f,-d,2)$
4	1	2.238791	71	0	0.000000	148	-12	1.432431
8	1	0.791532	79	-8	1.632461	151	14	1.891883
11	2	1.963697	88	4	0.347136	155	4	0.148499
15	2	1.233180	95	2	0.077371	163	-10	0.860640
23	-2	0.649489	107	-2	0.064727	179	-18	2.423088
39	-6	2.647336	116	12	2.064329	183	-6	0.260453
43	-6	2.286669	120	18	4.414450	184	2	0.028694
51	8	3.147228	123	4	0.210071	191	24	3.908201
67	2	0.130632	127	-2	0.050056			

TABLE 1. Coefficients of g_- and central values for $f = f_{\mathsf{7k4A}}$.

We are now ready to compute

$$\begin{aligned} g_- &:= \frac{1}{2} \sum_{(x,y,z)\in\mathbb{Z}^3} x\,\omega_7(x,y,z)\,q^{Q(x,y,z)} \\ &= \sum_{n=1}^{\infty} c_-(n)\,q^n \\ &= q^4 + q^8 + 2q^{11} + 2q^{15} - q^{16} - 2q^{23} + O(q^{32}), \end{aligned}$$

a weight $5/2$ modular form of level $4\cdot 7^2$ and character ψ_1, in Shimura correspondence with $f\otimes\psi$. Here $\psi_1(n) := \left(\frac{-1}{n}\right)\psi(n)$.

Table 1 shows the values of $c_-(d)$ and $L(f,-d,2)$, where $-200 < -d < 0$ is a fundamental discriminant such that $\left(\frac{-d}{7}\right) = -1$. The formula

$$L(f,-d,2) = k_-\,\frac{c_-(d)^2}{d^{3/2}}$$

is satisfied, where

$$k_- := 8\,L(f,-4,2) = 17.910324143488857621563653980249050613932 3...$$

Note that if $\left(\frac{-d}{7}\right) \neq -1$, i.e. $\left(\frac{d}{7}\right) \neq 1$, it is trivial that $c_-(d) = 0$, because the genus of Q only represents squares modulo 7 and $\omega_7 = 0$ for zeros modulo 7 of Q, and also that $L(f,-d,2)=0$, since the sign of the functional equation for $L(f,-d,s)$ is negative for such d.

3.2. **Real twists.** For this we need to use an odd weight function ω_l, for a suitably chosen prime l. First of all we need $l \equiv 3 \pmod 4$ so that ω_l is odd, but we should also require that $L(f,-l,2) \neq 0$. From table 1, the smallest such l is 11, for which $L(f,-11,2) \approx 2.238791$. In order to compute ω_{11}, we will use again $b_0 = (1,0,2)$, of norm $44 \equiv 0 \pmod{11}$. Now, $\langle (x,y,z), b_0\rangle \equiv 5x + 3z \equiv 3(-2x+z) \pmod{11}$, with $\left(\frac{3}{11}\right) = +1$, so that

$$\omega_{11}(x,y,z) := \begin{cases} 0 & \text{if } 11 \nmid Q(x,y,z), \\ \left(\frac{-2x+z}{11}\right) & \text{if } 2x \not\equiv z \pmod 7, \\ \left(\frac{x}{11}\right) & \text{otherwise.} \end{cases}$$

We claim that

$$\omega_{11}(x+z,-y,-z) = \omega_{11}(x,y,z).$$

Indeed, by the uniqueness of weight functions, the above equation is true up to a constant. It is thus enough to check the equality for a single nonzero value, such as $\omega_{11}(1+2,0,-2) = \omega_{11}(1,0,2) = +1$. Therefore, the same considerations as above apply, and lead us to choose x as the spherical polynomial.

We can then define and compute

$$\begin{aligned} g_+ &:= \frac{1}{4} \sum_{(x,y,z)\in\mathbb{Z}^3} x\,\omega_{11}(x,y,z)\, q^{Q(x,y,z)/11} \\ &= \sum_{n=1}^{\infty} c_+(n)\, q^n \\ &= q - 3q^4 + 7q^8 - 5q^9 - 5q^{16} + 14q^{21} + O(q^{25}), \end{aligned}$$

a weight 5/2 modular form of level $4 \cdot 7$ and trivial character, in Shimura correspondence with f.

d	$c_+(d)$	$L(f,d,2)$	d	$c_+(d)$	$L(f,d,2)$	d	$c_+(d)$	$L(f,d,2)$
1	1	0.599566	77	14	0.347846	141	84	2.526777
8	7	1.298369	85	-28	0.599825	149	0	0.000000
21	14	2.442273	88	42	1.281185	156	98	2.955305
28	-7	0.396576	92	0	0.000000	161	-42	1.035445
29	-28	3.009928	93	-28	0.524118	165	-84	1.996041
37	0	0.000000	105	42	1.965992	168	-42	0.971408
44	28	1.610550	109	28	0.413060	172	-28	0.208381
53	28	1.218258	113	14	0.097831	177	42	0.449134
56	-35	3.505267	120	-14	0.089397	184	-84	1.695001
57	-14	0.273074	133	-42	1.379076	193	42	0.394424
60	-42	2.275668	137	-28	0.293138	197	56	0.679989
65	14	0.224245	140	56	2.270132			

TABLE 2. Coefficients of g_+ and central values for $f = f_{\mathsf{7k4A}}$.

Table 2 shows the values of $c_+(d)$ and $L(f,d,2)$, where $0 < d < 200$ is a fundamental discriminant such that $\left(\frac{d}{7}\right) \neq -1$. The formula

$$L(f,d,2) = k_+ \frac{c_+(d)^2}{d^{3/2}} \cdot \begin{cases} 1 & \text{if } \left(\frac{d}{7}\right) = 1, \\ 2 & \text{if } \left(\frac{d}{7}\right) = 0, \\ 0 & \text{if } \left(\frac{d}{7}\right) = -1. \end{cases}$$

is satisfied, where

$$k_+ := L(f,1,2) = 0.599566157968617566581061167075228207656156...$$

Note that if $\left(\frac{d}{7}\right) = -1$, it is trivial that $c_+(d) = 0$, because the genus of Q only represents squares modulo 7, and also that $L(f,d,2) = 0$, since the sign of the functional equation for $L(f,d,s)$ is negative for such d.

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