COMPUTING CENTRAL VALUES OF TWISTED L-SERIES, THE CASE OF COMPOSITE LEVELS

ARIEL PACETTI AND GONZALO TORNARÍA

Abstract. We describe a general method to compute weight 3/2 modular forms "associated" to a given weight 2 modular form f of level N, and relate its Fourier coefficients to central values of quadratic twists (real and imaginary) of L(f,s). We will focus on examples for levels N = 27, N = 15 and N=75.

1. Introduction

Let $f \in S_2(N)$ be a newform of weight two and level N. If $f(z) = \sum_{m=1}^{\infty} a(m)\, q^m$ where $q = e^{2\pi i z}$, and D is a fundamental discriminant, we define the twisted L-series

$$L(f, D, s) = \sum_{m=1}^{\infty} \frac{a(m)}{m^s} \left(\frac{D}{m}\right)$$

We will assume that the twisted L-series are primitive (i.e. the corresponding twisted modular forms are newforms). There is no loss of generality in making this assumption: if this is not the case, then f would be a quadratic twist of a newform of smaller level, which we can choose instead.

The question of efficiently computing the family of central values $L(f, D, 1)$, for fundamental discriminants D, has been considered by several authors (see [Gr], [Bö-SP], [Pa-To1],[Pa-To2], [MRVT]). By Waldspurger's formula [Wa] these values are related to the Fourier coefficients of certain modular forms of weight 3/2.

Gross [Gr] gives a method to construct, for the case of prime level p and provided that $L(f, 1) \neq 0$, a weight 3/2 modular form of level $4p$, and gives an explicit version of Waldspurger's formula for the imaginary quadratic twists. In [Bö-SP] Böcherer and Schulze-Pillot extend Gross's method to the case of square free level, but their method works only for a fraction of imaginary quadratic twists (determined by

2000 *Mathematics Subject Classification.* Primary: 11F37; Secondary: 11F67.

Key words and phrases. Shimura Correspondence, L-series, Quadratic Twists.

The first author was supported by a CONICET grant.

quadratic residue conditions). Later in [Pa-To1] the case of level p^2 (p a prime) is considered, and this is used in [Pa-To2], provided $p \equiv 3 \pmod 4$, to compute central values for *real* quadratic twists.

In [MRVT] the non-vanishing condition is removed, and in the case of prime level two modular forms of weight 3/2 (one giving the imaginary quadratic twists and another one giving the real quadratic twists) are constructed.

The aim of this paper is to show how some of these ideas can be combined to handle the case of composite levels. In the case of odd squarefree level N, for instance, this method constructs 2^t modular forms, where t is the number of prime factors of N, whose coefficients give the central values of all the quadratic twists. We will focus on examples for levels $N = 27$, $N = 15$, and $N = 75$, which exhibit our methods for the non-square case. For the square case see [Pa-To1] and [Pa-To2].

2. The curve 27*A*

Let f be the modular form of level 27, corresponding to the elliptic curve $X_0(27)$, of minimal equation (see [Cr])

$$y^2 + y = x^3 - 7 \ .$$

The eigenvalue of f for the Atkin-Lehner involution W_{27} is -1, and the sign of the functional equation for $L(f, s)$ is $+1$.

Let $B = (-1, -3)$ be the quaternion algebra ramified at 3 and ∞, and consider the order $R = \langle 1, 3i, \frac{1+3j}{2}, \frac{i+k}{2} \rangle$, a Pizer order of reduced discriminant 27 (see [Pi] for the basic definitions of quaternion algebras, Brandt matrices and special orders). The class number of left R-ideals for such order is 2, and representatives for left R-ideals are $\{R, I\}$ where $I = \langle 4, 12i, \frac{7+6i+3j}{2}, \frac{6+13i+k}{2} \rangle$. The eigenvector for the Brandt matrices which corresponds to f is $(1, -1)$, with height 3.

The ternary quadratic forms associated to their right orders are

$$Q_1(x, y, z) = 4x^2 + 27y^2 + 28z^2 - 4xz \ ,$$

and

$$Q_2(x, y, z) = 7x^2 + 16y^2 + 31z^2 + 16yz + 2xz + 4xy \ ,$$

respectively.

D	$c(D)$	$L(f,D,1)$	D	$c(D)$	$L(f,D,1)$	D	$c(D)$	$L(f,D,1)$
-4	1	1.529954	-67	-1	0.373827	-139	3	2.335842
-7	-1	1.156537	-79	1	0.344267	-148	1	0.251523
-19	-1	0.701991	-88	-2	1.304749	-151	-1	0.249012
-31	0	0.000000	-91	1	0.320766	-163	-1	0.239670
-40	-2	1.935256	-103	1	0.301502	-184	2	0.902318
-43	2	1.866526	-115	-2	1.141352	-187	-2	0.895051
-52	1	0.424333	-127	-2	1.086092	-199	-3	1.952200
-55	2	1.650392	-136	2	1.049540			

TABLE 1. Coefficients of g and imaginary quadratic twists of $27A$

Note that, since the twist of f by the quadratic character of conductor 3 is f itself, we have

$$L(f,-3D,s) = L(f,D,s) \ ,$$

for $-3D$ a fundamental discriminant. We will thus assume that $3 \nmid D$.

2.1. **Imaginary quadratic twists.** Let $D<0$ be a fundamental discriminant. If $\left(\frac{D}{3}\right) = +1$, the sign of the functional equation for $L(f,D,s)$ is -1, so its central value vanishes trivially. Hence we can restrict to the case where $\left(\frac{D}{3}\right) = -1$. In this case we can follow Gross's method, using classical theta series

$$\Theta(Q_i) := \frac{1}{2}\sum_{(x,y,z)\in\mathbb{Z}^3} q^{Q_i(x,y,z)} \ ;$$

we obtain a weight 3/2 modular form of level $4\cdot 27$, namely

$$g = \Theta(Q_1) - \Theta(Q_2) = q^4 - q^7 - q^{19} + q^{28} - 2q^{40} + 2q^{43} + \cdots \ .$$

Table 1 shows the values of the Fourier coefficients $c(D)$ of g and of $L(f,D,1)$, where $-200 < D < 0$ is a fundamental discriminant such that $\left(\frac{D}{3}\right) = -1$. The Gross type formula

$$L(f,D,1) = k\,\frac{|c(D)|^2}{\sqrt{|D|}} \ , \quad D<0 \ ,$$

is satisfied, where $c(D)$ is the $|D|$-th Fourier coefficient of g, and

$$k = \frac{1}{3}\cdot\frac{(f,f)}{L(f,1)} = 2L(f,-4,1) \approx 3.0599080741143857498263883 45 \ .$$

2.2. **Real quadratic twists.** Let $D > 0$ be a fundamental discriminant. In this case, if $\left(\frac{D}{3}\right) = -1$ the sign of the functional equation for $L(f, D, s)$ will be -1, and its central value will vanish trivially. For $\left(\frac{D}{3}\right) = +1$, we will employ a method similar to the one used in [MRVT] for prime levels. We need to choose an auxiliary prime $l \equiv 3 \pmod 4$ such that $\left(\frac{-l}{3}\right) = -1$ and such that $L(f, -l, 1) \neq 0$, for example $l = 7$. Following [MRVT] we define a generalized theta series

$$\Theta_{-7}(Q_i) := \frac{1}{2} \sum_{(x,y,z)\in\mathbb{Z}^3} \omega_7^{(i)}(x, y, z)\, \omega_3^{(i)}(x, y, z)\, q^{Q_i(x,y,z)/7} \quad ,$$

where ω_7 and ω_3 are the two kinds of weight function introduced in §2.2 and §2.3 of [MRVT], respectively. The superscript in $\omega_3^{(i)}$ and $\omega_7^{(i)}$ indicates that we are writing the weight functions in the basis corresponding to the quadratic form Q_i.

The weight function of the first kind can be computed as

$$\omega_7^{(1)}(x, y, z) = \begin{cases} 0 & \text{if } 7 \nmid Q_1(x, y, z), \\ \left(\frac{x}{7}\right) & \text{if } 7 \nmid x, \\ \left(\frac{5z}{7}\right) & \text{otherwise;} \end{cases}$$

and

$$\omega_7^{(2)}(x, y, z) = \begin{cases} 0 & \text{if } 7 \nmid Q_2(x, y, z), \\ \left(\frac{3y+5z}{7}\right) & \text{if } 7 \nmid 3y + 5z, \\ \left(\frac{6x}{7}\right) & \text{otherwise.} \end{cases}$$

The weight function of the second kind can be computed as

$$\omega_3^{(1)}(x, y, z) = \left(\frac{x + z}{3}\right), \quad \text{and} \quad \omega_3^{(2)}(x, y, z) = \left(\frac{2x + y + 2z}{3}\right).$$

The generalized theta series will be

$$\Theta_{-7}(Q_1) = -2q^4 + 2q^{13} + 4q^{16} - 4q^{25} + 2q^{28} - 2q^{37} - 4q^{40} + \cdots \quad ,$$

and

$$\Theta_{-7}(Q_2) = q - q^4 - q^{13} + 2q^{16} - 3q^{25} - q^{28} + q^{37} + 2q^{40} + \cdots \quad .$$

Note that $\Theta_{-7}(Q_1) + 2\Theta_{-7}(Q_2) = 2q - 4q^4 + 8q^{16} - 10q^{25} + \cdots$, corresponding to the Eisenstein eigenvector for the Brandt matrices, has nonzero Fourier coefficients only at square indices. Since $\Theta_{-7}(Q_1)+2\Theta_{-7}(Q_2) \equiv \Theta_{-7}(Q_1)-\Theta_{-7}(Q_2) \pmod 3$, this explains the fact that the coefficients in Table 2, with the exception of $c_{-7}(1)$, are all divisible by 3.

D	$c_{-7}(D)$	$L(f,D,1)$	D	$c_{-7}(D)$	$L(f,D,1)$	D	$c_{-7}(D)$	$L(f,D,1)$
1	1	0.588880	76	-3	0.607942	136	6	1.817856
13	-3	1.469932	85	0	0.000000	145	6	1.760536
28	-3	1.001590	88	-6	2.259892	157	6	1.691917
37	3	0.871301	97	-3	0.538125	172	0	0.000000
40	6	3.351961	109	0	0.000000	181	-9	3.545457
61	3	0.678585	124	6	1.903786	184	-6	1.562860
73	-3	0.620308	133	-3	0.459561	193	3	0.381496

TABLE 2. Coefficients of g_{-7} and real quadratic twists of $27A$

Thus we obtain a modular form of weight 3/2, namely

$$g_{-7} = \Theta_{-7}(Q_1) - \Theta_{-7}(Q_2) = q + q^4 - 3q^{13} - 2q^{16} + q^{25} - 3q^{28} + 3q^{37} + 6q^{40} + \cdots \ ,$$

and the formula is now

$$L(f,D,1) = k_{-7}\,\frac{|c_{-7}(D)|^2}{\sqrt{|D|}} \ , \quad D > 0 \ ,$$

where $c_{-7}(D)$ is the D-th Fourier coefficient of g_{-7}, and

$$k_{-7} = \frac{1}{3}\cdot\frac{(f,f)}{L(f,-7,1)\sqrt{7}} = L(f,1) \approx 0.58887958342848331910456316668 \ .$$

Table 2 shows the values of the Fourier coefficients $c_{-7}(D)$ of g_{-7} and of $L(f,D,1)$, where $0 < D < 200$ is a fundamental discriminant such that $\left(\frac{D}{3}\right) = 1$.

3. The curve $15A$

Let f be the modular form of level 15, corresponding to the elliptic curve $X_0(15)$, of minimal equation

$$y^2 + xy + y = x^3 + x^2 - 10x - 10 \ .$$

The eigenvalues of f for the Atkin-Lehner involutions W_3 and W_5 are $+1$ and -1, and the sign of the functional equation for $L(f,s)$ is $+1$.

The method of Gross, as extended by Böcherer and Schulze-Pillot to the case of square-free levels, requires that the ramification of the quaternion algebra agrees with the Atkin-Lehner eigenvalues. In this case, it would be necessary to work with the quaternion algebra ramified at 5 and ∞. To exhibit the generality of our method, we will work with the quaternion algebra ramified at 3 and ∞ instead.

Let $B = (-1, -3)$ be such a quaternion algebra; an Eichler order of level 15 (index 5 in a maximal order) is given by $R = \left\langle 1, i, \frac{1+5j}{2}, \frac{1+i+3j+k}{2} \right\rangle$. The number of clases of left R-ideals is 2, and a set of representatives of the classes is given by $\{R, I\}$ where $I = \left\langle 2, 2i, \frac{3+2i+5j}{2}, \frac{3+i+3j+k}{2} \right\rangle$. The eigenvector for the Brandt matrices corresponding to f is $(1, -1)$, with height 4, and the ternary quadratic forms associated to R and I are

$$Q_1(x, y, z) = Q_2(x, y, z) = 4x^2 + 15y^2 + 16z^2 - 4xz \ .$$

3.1. **Imaginary quadratic twists.** Let $D < 0$ be a fundamental discriminant. We say that D is of type (s_1, s_2) if $\left(\frac{D}{3}\right) = s_1$ and $\left(\frac{D}{5}\right) = s_2$. We need the sign of the functional equation for $L(f, D, s)$ to be $+1$, so that its central value does not vanish trivially. For this to hold we need D to be of type $(-,+)$, $(+,-)$, $(+,0)$, $(0,-)$, or $(0,0)$ (see [At-Le]).

Note that the linear combination of classical theta series $\Theta(Q_1) - \Theta(Q_2)$ is trivially zero, since $Q_1 = Q_2$; this reflects the fact that the ramification does not match the Atkin-Lehner eigenvalues. Instead we set

$$\Theta_1(Q_i) := \frac{1}{4} \sum_{(x,y,z)\in\mathbb{Z}^3} \omega_3^{(i)}(x, y, z)\, \omega_5^{(i)}(x, y, z)\, q^{Q_i(x,y,z)} \ ,$$

where ω_3 and ω_5 are weight functions of the second kind as in [MRVT, §2.3]. We have $\Theta_1(Q_1) = -\Theta_1(Q_2)$, and hence we obtain a modular form of weight 3/2 and level $4 \cdot 15^2$, namely

$$g_1 = 2\,\Theta_1(Q_1) = q^4 + q^{16} + 2q^{19} + 2q^{31} + q^{64} + \cdots \ .$$

The corresponding formula is

$$L(f, D, 1) = k_1 \frac{|c_1(D)|^2}{\sqrt{|D|}} \ , \quad D < 0 \text{ of type } (-,+),$$

where $c_1(D)$ is the $|D|$-th Fourier coefficient of g_1, and

$$k_1 = \frac{1}{4} \cdot \frac{(f, f)}{L(f, 1)} = 2L(f, -4, 1) \approx 3.19248444426356702029793814 3 \ ,$$

c.f. Table 3 (top).

D	$c_1(D)$	$L(f,D,1)$	D	$c_1(D)$	$L(f,D,1)$	D	$c_1(D)$	$L(f,D,1)$
-4	1	1.596242	-91	-4	5.354613	-184	-4	3.765649
-19	2	2.929625	-136	-4	4.380053	-199	-2	0.905237
-31	2	2.293549	-139	-2	1.083132			
-79	-2	1.436730	-151	2	1.039203			

D	$c_{17}(D)$	$L(f,D,1)$	D	$c_{17}(D)$	$L(f,D,1)$	D	$c_{17}(D)$	$L(f,D,1)$
-3	2	0.921591	-83	4	0.350421	-152	8	1.035779
-8	-4	1.128714	-87	4	0.684541	-155	8	2.051412
-15	-2	0.824296	-95	0	0.000000	-167	4	0.247042
-20	4	1.427722	-107	4	0.308629	-168	8	1.970444
-23	4	0.665679	-120	-4	1.165730	-183	0	0.000000
-35	-4	1.079257	-123	-4	0.575713	-195	-4	0.914474
-47	-4	0.465672	-132	-8	2.222961			
-68	0	0.000000	-143	-8	1.067876			

TABLE 3. Coefficients of g_1 and g_{17}, and imaginary twists of $15A$

To obtain the other 4 types of negative D, we need to choose an auxiliary prime $l \equiv 1 \pmod 4$ such that $\left(\frac{l}{3}\right) = \left(\frac{l}{5}\right) = -1$, and such that $L(f,l,1) \neq 0$, e.g. $l = 17$. We then define the generalized theta series

$$\Theta_{17}(Q_i) := \frac{1}{4} \sum_{(x,y,z)\in\mathbb{Z}^3} \omega_{17}^{(i)}(x,y,z)\, q^{Q_i(x,y,z)/17} \quad ,$$

where ω_{17} is the weight function of the first kind defined in [MRVT, §2.2]. Now

$$g_{17} = 2\,\Theta_{17}(Q_1) = 2q^3 - 4q^8 - 2q^{15} + 4q^{20} + 4q^{23} + \cdots$$

is a weight 3/2 modular form of level $4 \cdot 15$. As expected by the multiplicity one theorem of Kohnen [Ko], this form turns out to be the same as the one constructed by Böcherer and Schulze-Pillot. The formula in this case is

$$L(f,D,1) = \star\, k_{17} \frac{|c_{17}(D)|^2}{\sqrt{|D|}} \quad , \quad D < 0 \text{ of type } (+,-),\ (+,0),\ (0,-), \text{ or } (0,0) \quad ,$$

and $\star = 1$, 2, 2, or 4 respectively; where $c_{17}(D)$ is the $|D|$-th Fourier coefficient of g_{17}, and

$$k_{17} = \frac{1}{4} \cdot \frac{(f,f)}{L(f,17,1)\sqrt{17}} \approx 0.199530277766472938768621134 0 \quad ,$$

D	$c_{-19}(D)$	$L(f,D,1)$	D	$c_{-19}(D)$	$L(f,D,1)$	D	$c_{-19}(D)$	$L(f,D,1)$
1	2	0.350151	76	-16	2.570563	141	-8	0.943616
21	-8	2.445093	109	16	2.146455	156	16	3.588416
24	8	2.287175	124	16	2.012446	181	0	0.000000
61	16	2.869261	129	-8	0.986530	184	-16	1.652061
69	-8	1.348902	136	0	0.000000			

D	$c_{-23}(D)$	$L(f,D,1)$	D	$c_{-23}(D)$	$L(f,D,1)$	D	$c_{-23}(D)$	$L(f,D,1)$
5	2	1.252737	77	-8	2.553816	152	0	0.000000
8	-4	1.980752	92	8	2.336367	173	-12	3.833492
17	4	1.358785	113	-4	0.527031	185	4	0.823795
53	4	0.769550	137	-4	0.478646	188	-8	1.634392
65	-4	1.389787	140	8	3.787922	197	12	3.592398

TABLE 4. Coefficients of g_{-19} and g_{-23}, and real twists of $15A$

c.f. Table 3 (bottom).

3.2. **Real quadratic twists.** Let $D > 0$ be a fundamental discriminant. In order for the sign of the functional equation of $L(f, D, s)$ to be $+1$, we need D to be of type $(+,+)$, $(0,+)$, $(-,-)$, or $(-,0)$.

For the first two types we need an auxiliary prime $l \equiv 3 \pmod 4$ such that $\left(\frac{-l}{3}\right) = -1$ and $\left(\frac{-l}{5}\right) = +1$, and such that $L(f, -l, 1) \neq 0$, e.g. $l = 19$. Again

$$\Theta_{-19}(Q_i) := \frac{1}{4} \sum_{(x,y,z)\in\mathbb{Z}^3} \omega_{19}^{(i)}(x,y,z)\, \omega_5^{(i)}(x,y,z)\, q^{Q_i(x,y,z)/19} \quad ,$$

with ω_{19} of the first kind and ω_5 of the second kind. The modular form

$$g_{-19} = 2\,\Theta_{-19}(Q_1) = 2q - 4q^4 + 2q^9 - 8q^{21} + 8q^{24} + \cdots$$

has level $4 \cdot 15 \cdot 5$, and the formula is

$$L(f, D, 1) = \star\, k_{-19}\, \frac{|c_{-19}(D)|^2}{\sqrt{|D|}} \quad , \quad D > 0 \text{ of type } (+,+) \text{ or } (0,+) \quad ,$$

$\star = 1$ or 2 respectively; $c_{-19}(D)$ is the D-th Fourier coefficient of g_{-19}, and

$$k_{-19} = \frac{1}{4} \cdot \frac{(f,f)}{L(f,-19,1)\sqrt{19}} = \frac{1}{4} L(f,1) \approx 0.087537690145787626448761302\,41 \quad .$$

Table 4 (top) shows the values of the coefficients $c_{-19}(D)$ and the central values

$L(f, D, 1)$ for $0 < D < 200$ a fundamental discriminant of type $(+,+)$ or $(0,+)$.

For the remaining two types we need an auxiliary prime $l \equiv 3 \pmod 4$ such that $\left(\frac{-l}{3}\right) = +1$ and $\left(\frac{-l}{5}\right) = -1$, and such that $L(f, -l, 1) \neq 0$, e.g. $l = 23$. As before we define

$$\Theta_{-23}(Q_i) := \frac{1}{4} \sum_{(x,y,z)\in\mathbb{Z}^3} \omega_{23}^{(i)}(x, y, z)\, \omega_3^{(i)}(x, y, z)\, q^{Q_i(x,y,z)/23} \ ,$$

with ω_{23} of the first kind and ω_3 of the second kind. The modular form

$$g_{-23} = 2\,\Theta_{-23}(Q_1) = 2q^5 - 4q^8 + 4q^{17} - 4q^{32} + 4q^{53} + \cdots$$

has level $4 \cdot 15 \cdot 3$, and the formula is

$$L(f, D, 1) = \star\, k_{-23} \frac{|c_{-23}(D)|^2}{\sqrt{|D|}} \ , \quad D > 0 \text{ of type } (-,-) \text{ or } (-,0) \ ,$$

$\star = 1$ or 2 respectively; $c_{-23}(D)$ is the D-th Fourier coefficient of g_{-23} and

$$k_{-23} = \frac{1}{4} \cdot \frac{(f, f)}{L(f, -23, 1)\sqrt{23}} \approx 0.35015076058315050579504520092 \ .$$

Table 4 (bottom) shows the values of the coefficients $c_{-19}(D)$ and the central values $L(f, D, 1)$ for $0 < D < 200$ a fundamental discriminant of type $(-,-)$ or $(-,0)$.

4. The curve 75A

Let f be the modular form of level 75 corresponding to the elliptic curve of minimal equation

$$y^2 + y = x^3 - x^2 - 8x - 7 \ .$$

The eigenvalue of f for the Atkin-Lehner involution W_3 is $+1$, for W_{25} is -1, and the sign of the functional equation for $L(f, s)$ is $+1$.

Let $B = (-1, -3)$ be the quaternion algebra ramified at 3 and ∞, and consider the order $R = \left\langle 1, i, \frac{1+5j}{2}, \frac{i+5k}{2} \right\rangle$, an Eichler order of level 75 (index 25 in a maximal order). The class number of left R-ideals is 6, and the eigenvector for the Brandt matrices which corresponds to f is $(1, -1, 1, -1, 0, 0)$, with height 6.

The ternary quadratic forms associated to the right orders of the choosen ideal class representatives are

$$Q_1(x, y, z) = Q_2(x, y, z) = 4x^2 + 75y^2 + 76z^2 - 4xz \ ,$$

$$Q_3(x, y, z) = Q_4(x, y, z) = 16x^2 + 19y^2 + 79z^2 + 4xy + 16xz + 2yz \ ,$$

and

$$Q_5(x,y,z) = Q_6(x,y,z) = 24x^2 + 31y^2 + 39z^2 + 24xy + 12xz + 6yz \ ,$$

respectively.

We will assume that $5 \nmid D$. Indeed, the twist of f by the quadratic character of conductor 5 is another modular form f' of level 75, thus we have

$$L(f, 5D, 1) = L(f', D, 1),$$

for $5D$ a fundamental discriminant. By applying the same procedure to the modular form f' we can compute the central values for these twists. So, we actually need 8 different modular forms of weight $3/2$ to compute all the twisted central values.

4.1. **Imaginary quadratic twists.** Let $D < 0$ be a fundamental discriminant. If the sign of the functional equation for $L(f, D, s)$ is $+1$, the type of D has to be either $(-,+)$ or $(-,-)$.

For the first case we look at the generalized theta series

$$\Theta_1(Q_i) := \frac{1}{4} \sum_{(x,y,z)\in\mathbb{Z}^3} \omega_3^{(i)}(x,y,z)\, \omega_5^{(i)}(x,y,z)\, q^{Q_i(x,y,z)} \ ;$$

we obtain the modular form

$$g_1 = 2\Theta_1(Q_1) - 2\Theta_1(Q_3) = q^4 - 2q^{16} - q^{19} - q^{31} - 2q^{64} + 3q^{76} + 4q^{79} - q^{91} + \cdots \ .$$

The formula

$$L(f, D, 1) = k_1 \frac{|c_1(D)|^2}{\sqrt{|D|}} \ , \quad D < 0 \text{ of type } (-,+) \ ,$$

is satisfied (c.f. Table 5, top), where $c_1(D)$ is the $|D|$-th Fourier coefficient of g_1 and

$$k_1 = \frac{1}{6} \cdot \frac{(f,f)}{L(f,1)} = 2\, L(f, -4, 1) \approx 4.66953274871871932795120676 1 \ .$$

In the second case we need to choose an auxiliary prime $l \equiv 1 \pmod 4$ such that $\left(\frac{l}{3}\right) = +1$, $\left(\frac{l}{5}\right) = -1$ and $L(f, l, 1) \neq 0$, for example $l = 13$, and define

$$\Theta_{13}(Q_i) := \frac{1}{4} \sum_{(x,y,z)\in\mathbb{Z}^3} \omega_{13}^{(i)}(x,y,z)\, \omega_3^{(i)}(x,y,z)\, \omega_5^{(i)}(x,y,z)\, q^{Q_i(x,y,z)/13} \ .$$

We obtain the modular form

$$g_{13} = 2\Theta_{13}(Q_1) - 2\Theta_{13}(Q_3) = 3q^7 + 3q^{28} + 3q^{43} + 3q^{52} - 3q^{67} - 6q^{88} + \cdots \ ,$$

D	$c_1(D)$	$L(f,D,1)$	D	$c_1(D)$	$L(f,D,1)$	D	$c_1(D)$	$L(f,D,1)$
-4	1	2.334766	-91	-1	0.489500	-184	2	1.376970
-19	-1	1.071264	-136	2	1.601637	-199	-5	8.275360
-31	-1	0.838673	-139	-2	1.584258			
-79	4	8.405816	-151	5	9.500030			
D	$c_{13}(D)$	$L(f,D,1)$	D	$c_{13}(D)$	$L(f,D,1)$	D	$c_{13}(D)$	$L(f,D,1)$
-7	3	5.294752	-88	-6	5.973286	-163	3	1.097238
-43	3	2.136291	-103	-6	5.521233	-187	0	0.000000
-52	3	1.942643	-127	-6	4.972248			
-67	-3	1.711423	-148	0	0.000000			

TABLE 5. Coefficients of g_1 and g_{13}, and imaginary twists of $75A$

and the formula

$$L(f,D,1) = k_{13}\,\frac{|c_{13}(D)|^2}{\sqrt{|D|}}\quad,\quad D<0 \text{ of type } (+,+)\ ,$$

is satisfied (c.f. Table 5, bottom), where $c_{13}(D)$ is the $|D|$-th Fourier coefficient of g_{13} and

$$k_{13} = \frac{1}{6}\cdot\frac{(f,f)}{L(f,13,1)\sqrt{13}} \approx 1.5565109162395731093170689 20\ .$$

4.2. **Real quadratic twists.** Let $D>0$ be a fundamental discriminant. The only possibilities so that the sign of the functional equation for $L(f,D,s)$ is $+1$ are the discriminants D of types $(+,+),(0,+),(+,-)$, and $(0,-)$.

For the first two cases we can use the generalized theta series

$$\Theta_{-19}(Q_i) := \frac{1}{2}\sum_{(x,y,z)\in\mathbb{Z}^3}\omega_{19}^{(i)}(x,y,z)\,\omega_5^{(i)}(x,y,z)\,q^{Q_i(x,y,z)/19}\ .$$

Thus we obtain a modular form of weight 3/2, namely

$$g_{-19} = q + q^4 + q^9 - q^{21} - 2q^{24} - q^{36} - 4q^{49} - q^{61} + \cdots\ ,$$

and the formula is

$$L(f,D,1) = \star\, k_{-19}\,\frac{|c_{-19}(D)|^2}{\sqrt{|D|}}\quad,\quad D>0 \text{ of type } (+,+) \text{ or } (0,+)\ ,$$

D	$c_{-19}(D)$	$L(f,D,1)$	D	$c_{-19}(D)$	$L(f,D,1)$	D	$c_{-19}(D)$	$L(f,D,1)$
1	1	1.402540	76	1	0.160882	141	2	0.944921
21	-1	0.612119	109	-1	0.134339	156	-1	0.224586
24	-2	2.290338	124	5	3.148795	181	3	0.938250
61	-1	0.179577	129	5	6.174338	184	-2	0.413586
69	2	1.350768	136	-6	4.329605			
12	3	2.429270	73	6	1.969859	168	6	2.596999
13	3	1.166984	88	-6	1.794135	172	3	0.320828
28	-3	0.795165	93	-3	0.872620	177	-6	2.530113
33	-6	5.859621	97	9	3.844972	193	9	2.725840
37	0	0.000000	133	-3	0.364847			
57	-3	1.114626	157	3	0.335805			

TABLE 6. Coefficients of g_{-19} and g_{-7}, and real twists of $75A$

$\star = 1$ or 2 respectively, $c_{-19}(D)$ the D-th Fourier coefficient of g_{-19}, and

$$k_{-19} = \frac{1}{6} \cdot \frac{(f,f)}{L(f,-19,1)\sqrt{19}} = L(f,1) \approx 1.4025399402162211198444940086 \ ,$$

c.f. Table 6 (top).

In the other two cases we can use the generalized theta series

$$\Theta_{-7}(Q_i) := \frac{1}{2} \sum_{(x,y,z)\in\mathbb{Z}^3} \omega_7^{(i)}(x,y,z)\, \omega_5^{(i)}(x,y,z)\, q^{Q_i(x,y,z)/7} \ .$$

We obtain a modular form of weight 3/2

$$g_{-7} = 3q^{12} + 3q^{13} - 3q^{28} - 6q^{33} + 6q^{48} - 9q^{52} - 3q^{57} + 6q^{73} + \cdots \ ,$$

satisfying the formula

$$L(f,D,1) = \star\, k_{-7}\, \frac{|c_{-7}(D)|^2}{\sqrt{|D|}} \ , \quad D > 0 \text{ of type } (+,-) \text{ or } (0,-) \ ,$$

$\star = 1$ or 2 respectively, $c_{-7}(D)$ the D-th Fourier coefficient of g_{-7}, and

$$k_{-7} = \frac{1}{6} \cdot \frac{(f,f)}{L(f,-7,1)\sqrt{7}} \approx 0.4675133134054070399481646950 \ ,$$

c.f. Table 6 (bottom).

5. Computation

Using the methods of the previous section we computed the coefficients up to 10^8 for all the theta series of weight 3/2 corresponding to elliptic curves $27A$ and $15A$. The computation of the theta series for the elliptic curve $75A$ are currently underway, and will be published online at [CNT]. All the computations were done on a cluster of 2.2GHz AMD Opteron processors funded by a NSF SCREMS grant and run by the Department of Mathematics of the University of Texas at Austin[1].

The computation for $27A$ is quite fast. Indeed, the form g_1 is a combination of classical theta series, and was computed in about 4 cpu-hours using the standard `qfrep` function of PARI/GP [GP]. The form g_7, on the other hand, requires using weight functions, and computing it took about 40 cpu-hours using a custom `qfrepmod` function written in C for this purpose, together with a collection of GP scripts to compute weight functions. The strategy is to use the fast `qfrepmod` function to compute theta series with congruences where the weight functions are constant, and combine them together in a GP script.

The computation for $15A$ is much longer. Indeed, the higher conductor of the weight functions requires too many congruence theta series except in the case $l = 1$. We actually divide the computation of the coefficients of the g_i by the congruence class of its index modulo 60. In particular, we avoid the need to reserve memory for coefficients that are trivially 0 (namely, only half the indices are actual discriminants, and from those half correspond to quadratic twists with sign $-$ in the functional equation). Moreover, each computation requires only a fraction of the space to keep all the coefficients in main memory while counting vectors. It also lends itself to a trivial way to parallelize the computation in 30 independent processes.

The computation used 30 cores in the above mentioned cluster, with a wall time of 26.5 days (this was the time for the two longest running processes, corresponding to discriminants congruent to 2 and 8 modulo 60). The accumulated running times

[1] `http://www.ma.utexas.edu/cluster/`

were as follows:

g_1	0.30 days
g_{17}	110.19 days
g_{-19}	106.88 days
g_{-23}	131.80 days
TOTAL	349.17 days

We believe the running times for all but g_1 are affected by the number of congruences, the combination of which is done by a GP script thus, we expect the times for the last three computations can be improved quite a lot with a careful rewriting in C of this code.

Note also that the modular form we are calling here g_{17} can also be computed as a difference of two classical theta series by working with the quaternion algebra ramified at 3 and ∞, and this will be much quicker in all cases. Thus, the totality of imaginary quadratic twists could be quickly computed.

6. Random Matrix Theory

The purpose of this section is to check some of the various conjectures of [CKRS] and [CKRS2]. We start by stating the conjectures; we checked each of them numerically with the computation discussed in the previous section.

An important comment should be made: in [CKRS] and [CKRS2], the conjectures are stated and checked, in case of non-prime level, only for a fraction of all quadratic twists, namely those that can be computed without weight functions using the methods of [Gr] and [Bö-SP]. In the case of the real quadratic twists they have been checked using the methods in [Pa-To1], [Pa-To2]. In both cases this has been based on a massive computation of *classical* theta series that was done in [CKRS2], using ternary quadratic forms data that was computed by the second author with aid from the first author [To], first published in december 2003, and in its final form since january 2004.

In this paper we have shown how to compute, in a few examples, enough weight 3/2 modular forms so as to be able to compute the central values for *all* the quadratic twists. Hence we state the conjectures for all the quadratic twists for which the sign of the functional equation is +, and give numerical evidence for the conjectures for all such twists. The task remains of doing a massive computation like the one done in [To] and [CKRS2] to check these conjectures for a very large and not partial set of quadratic twists for a large number of different elliptic curves.

In order to state the conjectures, fix an elliptic curve E defined over $\mathbb{Q}$. We let $S(X)$ be the set of fundamental discriminants, of absolute value up to X, such that the corresponding quadratic twist of E has positive sign in the functional equation.

We will refine the conjectures of [CKRS] sorting the discriminants by congruence classes in addition to sign: for M a positive integer and a an integer, we let

$$S(X;a,M) := \{d \in S(X) \;:\; d \equiv a \pmod{M}, ad > 0\}.$$

Among those, we consider the subset $S_p(X;a,M)$ of prime discriminants, i.e.

$$S_p(X;a,M) = \{d \in S(X;a,M) \;:\; \; d \text{ is prime}\}.$$

We are interested in the subsets

$$S^0(X;a,M) = \{d \in S(X;a,M) \;:\; L(E,d,1) = 0\},$$

and

$$S_p^0(X;a,M) = \{d \in S_p(X;a,M) \;:\; L(E,d,1) = 0\},$$

of discriminants with twisted central value vanishing (for non-trivial reasons, and to order at least 2, since the sign of the functional equation is $+$).

Conjecture 1. *There are constants* $c_E^p(a,M) \geq 0$ *such that*

$$\frac{\#S_p^0(X;a,M)}{\#S_p(X;a,M)} \sim c_E^p(a,M) \cdot X^{-1/4}\,(logX)^{3/8}$$

We remark that the constant $c_E^p(a,M)$ could be 0, as noted by [De]. In contrast, we believe that the constants $c_E(a,M)$ in the next conjecture should always be positive.

Conjecture 2. *There are constants* $c_E(a,M) \geq 0$ *such that*

$$\frac{\#S^0(X;a,M)}{\#S(X;a,M)} \sim c_E(a,M) \cdot X^{-1/4}\,(logX)^{11/8}$$

a	$\#S^0(X;a,12)$	$\#S(X;a,12)$	$c_E(X;a,12)$
1	295819	7599045	0.07087151
4	145496	3799561	0.06971437
-7	226182	7599088	0.05418776
-4	110886	3799541	0.05313127

TABLE 7. Numerics for $27A$, all discriminants, with $X = 10^8$.

a	$\#S_p^0(X;a,12)$	$\#S_p(X;a,12)$	$c_E^p(X;a,12)$
1	23700	1440021	0.55193748
-7	18233	1440496	0.42447923

TABLE 8. Numerics for $27A$, prime discriminants, with $X = 10^8$.

In table 7 we give the experimental numerics for

$$c_E(X;a,M) := \frac{\#S^0(X;a,M)}{\#S(X;a,M)} \cdot X^{1/4}\,(logX)^{-11/8}$$

for the elliptic curve $27A$, with $M = 12$ and $X = 10^8$. Only the values of a that lead to discriminants in $S(X)$ are displayed. In table 8 we show the corresponding numerics for prime discriminants, where

$$c_E^p(X;a,M) := \frac{\#S^0(X;a,M)}{\#S(X;a,M)} \cdot X^{1/4}\,(logX)^{-3/8}.$$

In tables 9 and 10 we investigate the dependence on a of the constants $c_E(a,M)$, for the elliptic curve $15A$ and $M = 60$. An interesting phenomenon can be observed in these tables: the constants $c_E(a,M)$ seem to only depend on the square class of a mod M.

Conjecture 3. *Let a and b be integers in the same square class modulo M, i.e. $ab > 0$ and there is an integer x relatively prime to M such that $a \equiv bx^2 \pmod{M}$. Then $c_E(a,M) = c_E(b,M)$.*

The case of conjectures 1 and 2 stated in [CKRS] correspond to the case $M = 1$, with $a = \pm 1$, and moreover restricted to partial subsets of discriminants. In figures 1 and 2 we show the numerics for the elliptic curve $27A$ in the case of prime discriminants (Conjecture 1), and in figures 3 and 4 we show the numerics in the case of all discriminants (Conjecture 2).

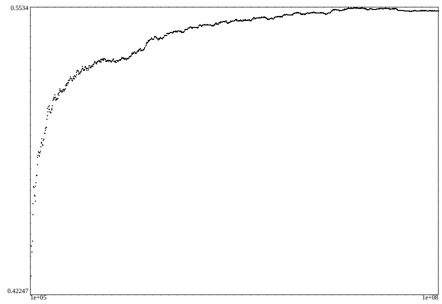

FIGURE 1. Value of $c_E^p(X; +1, 1)$ for $27A$.

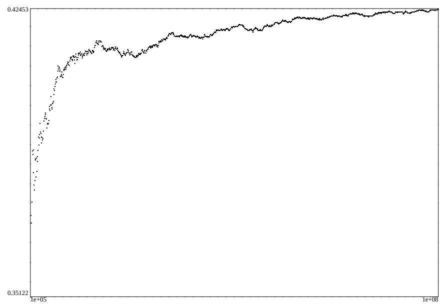

FIGURE 2. Value of $c_E^p(X; -1, 1)$ for $27A$.

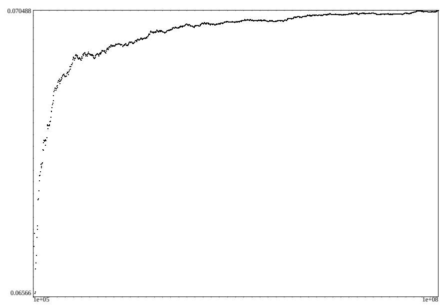

FIGURE 3. Value of $c_E(X; +1, 1)$ for $27A$.

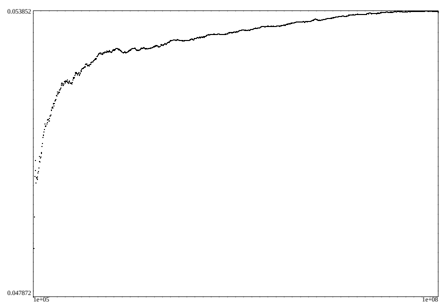

FIGURE 4. Value of $c_E(X; -1, 1)$ for $27A$.

a	$\#S^0(X;a,60)$	$\#S(X;a,60)$	$c_E(X;a,60)$
1	103871	1583103	0.11945101
49	103201	1583109	0.11868006
4	56689	791596	0.13037667
16	57272	791596	0.13171749
9	53190	1055442	0.09174878
21	53325	1055430	0.09198269
24	45765	527715	0.15788421
36	46085	527707	0.15899059
17	62882	1583163	0.07231117
53	63276	1583149	0.07276489
8	56117	791553	0.12906816
32	55560	791565	0.12778513
5	70561	1266445	0.10143389
20	46229	633300	0.13289532

TABLE 9. Numerics for real quadratic twists of $15A$, with $X = 10^8$.

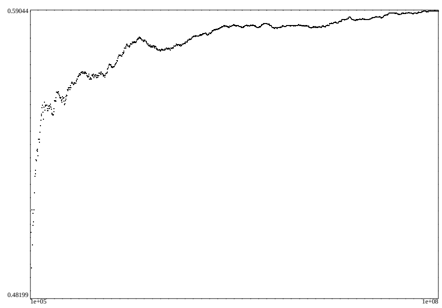

FIGURE 5. Value of $c_E^p(X;+1,1)$ for $15A$.

For the elliptic curve $15A$ we have the corresponding figure 5 for the case of prime discriminants, figures 6 and 7 for the case of all discriminants. By the work of Delaunay (see [De]), we know that the constant $c_E^p(-1,1)$ is 0, thus we only show positive prime discriminants for this curve. On the other hand, the graphs of $c_E(X;\pm1,1)$ for this curve seem to be too smooth, as if they had e.g. logarithmic growth. We do not have an explanation for this.

a	$\#S^0(X;a,60)$	$\#S(X;a,60)$	$c_E(X;a,60)$
-19	75626	1583138	0.08696751
-31	75333	1583128	0.08663111
-4	62536	791570	0.14382867
-16	62999	791558	0.14489573
-23	67381	1583166	0.07748465
-47	67794	1583158	0.07795997
-8	61142	791545	0.14062700
-32	60724	791565	0.13966207
-3	64191	1055419	0.11072710
-27	64178	1055408	0.11070583
-12	41844	527727	0.14435391
-48	41589	527728	0.14347394
-35	72803	1266486	0.10465345
-20	53586	633266	0.15405289
-15	50383	844328	0.10863694
-60	42661	422192	0.18396098

TABLE 10. Numerics for imaginary quadratic twists of $15A$, with $X = 10^8$.

We recall another conjecture from [CKRS]: let q be a prime, and consider the ratios

$$R_q^{\pm}(X) = \frac{\#\left\{d \in S^0(X;\pm1,1) \ : \ \left(\frac{d}{q}\right) = +1\right\}}{\#\left\{d \in S^0(X;\pm1,1) \ : \ \left(\frac{d}{q}\right) = -1\right\}}.$$

Let

$$R_q := \sqrt{\frac{q+1-a_q}{q+1+a_q}},$$

where $a_q = q + 1 - \#E(\mathbb{F}_q)$.

Conjecture 4. *Suppose E has good reduction modulo q. Then*

$$\lim_{X\to\infty} R_q^{\pm}(X) = R_q.$$

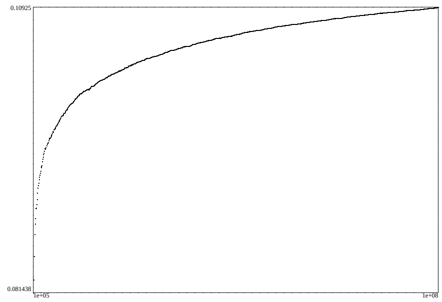

FIGURE 6. Value of $c_E(X;+1,1)$ for $15A$.

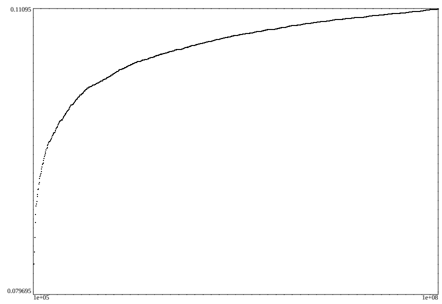

FIGURE 7. Value of $c_E(X;-1,1)$ for $15A$.

As noted in [CKRS], the conjectural value R_q of the limit is the square root of the ratio of $\#E(\mathbb{F}_q)$ to $\#E^\chi(\mathbb{F}_q)$, where χ is a quadratic character such that $\chi(q) = -1$.

In figures 8 and 9 we plot, for the elliptic curve $27A$, and for each prime number $q = 2, \dots, 3571$ the values $R_q^+(10^8) - R_q$ and $R_q^-(10^8) - R_q$, respectively. In figures 10 and 11 we do the same for the elliptic curve $15A$. It can be seen on the graphics that these values are close to 0 (the expected limit as X goes to infinity).

In figures 12 and 13 we plot the distribution of non-zero central values of the twisted L-series of the elliptic curve $27A$ by positive and negative fundamental discriminants, respectively. The same graphs for the elliptic curve $15A$ appear in figures 14 and 15.

The *Central Limit Conjecture* (see Conjecture 3.3 of [CKRS2]) states that the distribution of non-zero central values of the twisted L-series (scaled in a reasonable way) behaves like a standard Gaussian; concretely for any pair of real numbers $\alpha < \beta$ the percentage of discriminants $d \in S(X;\pm 1,1)$ with $\alpha < \frac{log(L(E,d,1))+\frac{1}{2}\log\log|d|}{\sqrt{\log\log|d|}} < \beta$ tends to $\frac{1}{\sqrt{2\pi}}\int_\alpha^\beta \exp(\frac{-t^2}{2})dt$ as X tends to infinity. In figure 16 we plot the value

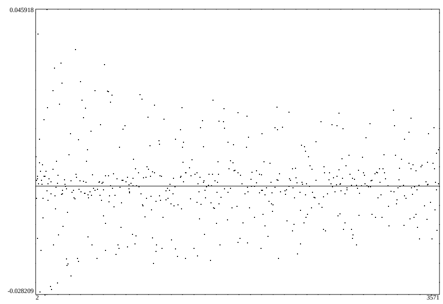

FIGURE 8. The values $R_q^+(10^8) - R_q$ for the elliptic curve $27A$ and $2 \leq q \leq 3571$ prime

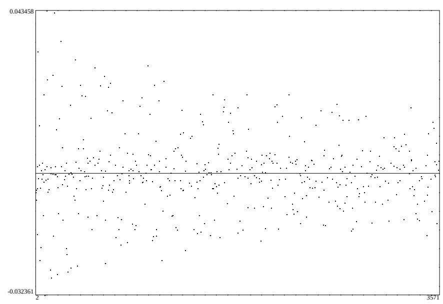

FIGURE 9. The values $R_q^-(10^8) - R_q$ for the elliptic curve $27A$ and $2 \leq q \leq 3571$ prime

distribution of the twisted L-series of the elliptic curves $27A$ and $15A$ by positive and negative fundamental discriminants compared to the standard Gaussian.

References

[At-Le] Atkin, A. O. L. and Lehner, J.: Hecke operators on $\Gamma_0(m)$ Math. Ann. **185** (1970), 134–160.

[Bö-SP] Böcherer, S., Schulze-Pillot, R.: On a theorem of Waldspurger and on Eisenstein series of Klingen type. Math. Ann. **288** (1990), 361–388.

[CNT] Computational Number Theory: `http://www.ma.utexas.edu/cnt/`.

[CKRS] Conrey, J. and Keating, J. and Rubinstein, M. and Snaith, N.: On the frequency of vanishing of quadratic twists of modular L-functions. Number theory for the millennium, I (Urbana, IL, 2000), 301–315.

[CKRS2] Conrey, J. and Keating, J. and Rubinstein, M. and Snaith, N.: Random Matrix Theory and the Fourier Coefficients of Half-Integral-Weight Forms. Exp. Math. 15 (2006), 67–82.

[Cr] Cremona, J. Elliptic Curve Data. `http://www.maths.nott.ac.uk/personal/jec/ftp/data/INDEX.html`

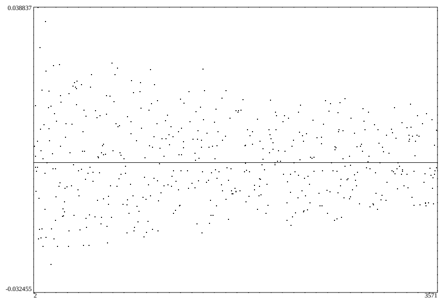

FIGURE 10. The values $R_q^+(10^8) - R_q$ for the elliptic curve $E15A$ and $2 \leq q \leq 3571$ prime

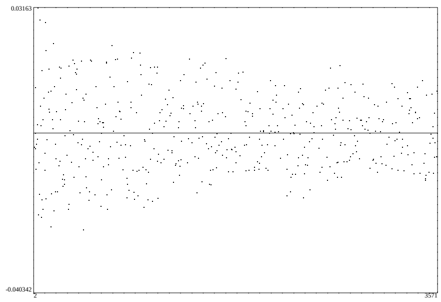

FIGURE 11. The values $R_q^-(10^8) - R_q$ for the elliptic curve $E15A$ and $2 \leq q \leq 3571$ prime

[De] Delaunay, C. Note on the frequency of vanishing of L-functions of elliptic curves in a family of quadratic twists. London Mathematical Society, Lecture Note Series 341: Ranks of elliptic curves and random matrix theory. Cambridge University Press. (2007) 195–200.

[GP] PARI/GP, version `2.2.11`, Bordeaux, 2005, `http://pari.math.u-bordeaux.fr/`

[Gr] Gross, B.: Heights and the special values of L-series. Canadian Math. Soc. Conf. Proceedings, volume 7 (1987), 115–187.

[Ko] Kohnen, W.: Newforms of half-integral weight. J. reine angew. Math. **333** (1982), 32–72.

[MRVT] Mao, Z., Rodriguez-Villegas, F., Tornaría, G.: Computation of central value of quadratic twists of modular L-functions. London Mathematical Society, Lecture Note Series 341: Ranks of elliptic curves and random matrix theory. Cambridge University Press. (2007) 273–288.

[Pa-To1] Pacetti, A., Tornaría, G.: Shimura correspondence for level p^2 and the central values of L-series. J. Number Theory **124** (2007) 396–414.

[Pa-To2] Pacetti, A., Tornaría, G.: Examples of Shimura correspondence for level p^2 and real quadratic twists. London Mathematical Society, Lecture Note Series 341: Ranks of elliptic curves and random matrix theory. Cambridge University Press. (2007) 289–314.

[Pi] Pizer, A.: An algorithm for computing modular forms on $\Gamma_0(N)$. J. Algebra **64** (1980), 340–390.

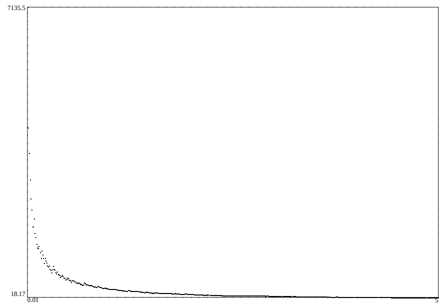

FIGURE 12. Value distribution of $L(27A, d, 1)$ for $0 < d < 10^8$

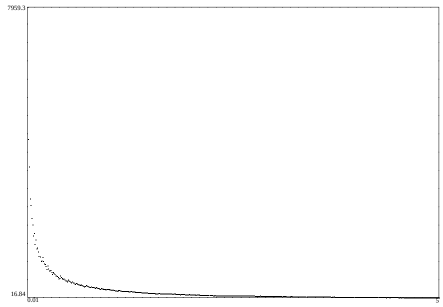

FIGURE 13. Value distribution of $L(27A, d, 1)$ for $0 > d > -10^8$

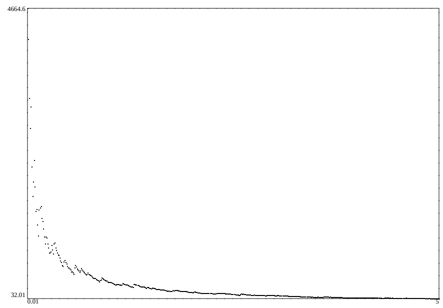

FIGURE 14. Value distribution of $L(15A, d, 1)$ for $0 < d < 10^8$

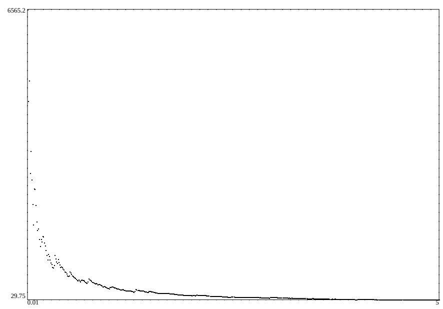

FIGURE 15. Value distribution of $L(15A, d, 1)$ for $0 > d > -10^8$

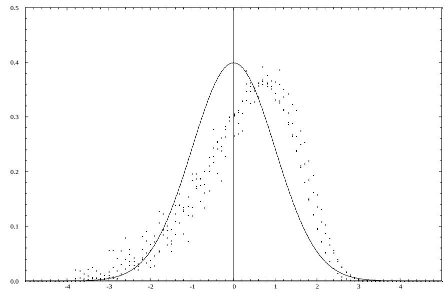

FIGURE 16. Value distribution of $(\log L(E,d,1) + \frac{1}{2}\log\log|d|)/\sqrt{\log\log|d|}$ for both $27A$ and $15A$, all discriminants, compared to the expected limit, the standard Gaussian.

[To] Tornaría, G.: Data about the central values of the L-series of (imaginary and real) quadratic twists of elliptic curves. Available from [CNT].

[Wa] Waldspurger, J-L.: Sur les coefficients de Fourier des formes modulaires de poids demi-entier. J. Math. pures Appl. **60** (1981), 375–484.

DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDAD DE BUENOS AIRES, PABELLÓN I, CIUDAD UNIVERSITARIA. C.P:1428, BUENOS AIRES, ARGENTINA,

E-mail address: `apacetti@dm.uba.ar`

CENTRO DE MATEMÁTICA, FACULTAD DE CIENCIAS, IGUÁ 4225 ESQ. MATAOJO, MONTEVIDEO, URUGUAY,

E-mail address: `tornaria@math.utexas.edu`