STARK-HEEGNER POINTS AND THE SHIMURA CORRESPONDENCE

HENRI DARMON AND GONZALO TORNARÍA

CONTENTS

1. INTRODUCTION

Let $N \geq 1$ be an odd square-free integer, and let $S_{2k}(N)$ denote the space of cusp forms of even weight $2k$ on the Hecke congruence group $\Gamma_0(N)$. Following the definitions in [Ko82], we denote by $S^+_{k+1/2}(4N)$ the space of modular forms which transform like $\theta(\tau)^{2k+1}$ under the action of $\Gamma_0(4N)$, and belong to the *plus space*: this means that they have a fourier

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expansion of the form $\sum_{D>0} c(D)q^D$, where $c(D) = 0$ unless

$$D^* := (-1)^k D \equiv 0, 1 \pmod 4.$$

The spaces $S_{2k}(N)$ and $S^+_{k+1/2}(4N)$ are equipped with an action of Hecke operators and with a notion of newforms. (For forms of integral weight, this is classical Atkin-Lehner theory, while for forms of half-integral weight these notions are made precise in [Ko82].) The Hecke operators acting on $S_{k+1/2}(4N)$ are indexed by squares of integers, and the operators T_{ℓ^2}, for ℓ a prime not dividing $2N$, preserve the subspace $S^{\rm new}_{k+1/2}(4N)$ of newforms.

A basic theorem of Shimura and Kohnen [Ko82, Theorem 2, Section 5], states that the spaces $S^{\rm new}_{2k}(N)$ and $S^{\rm new}_{k+1/2}(4N)$ are isomorphic as Hecke modules. More precisely, if $f = \sum a_n q^n$ is any normalised newform of weight $2k$ on $\Gamma_0(N)$, there is a newform $g \in S^{\rm new}_{k+1/2}(4N)$, which is unique up to scaling and satisfies

$$T_{\ell^2} g = a_\ell g, \quad \text{for all primes } \ell \nmid 2N.$$

The forms f and g are said to be in *Shimura correspondence*. We write

$$g(q) = \sum_{D>0} c(D) q^D$$

for the fourier expansion of g.

The fourier coefficients a_ℓ (for ℓ a prime) of the integral weight eigenform f can be recovered from those of g by the rule

$$a_\ell = \begin{cases} \frac{c(D\ell^2)}{c(D)} + \left(\frac{D^*}{\ell}\right) \ell^{k-1} & \text{if } \ell \nmid N; \\ \frac{c(D\ell^2)}{c(D)} & \text{if } \ell \mid N, \end{cases}$$

where D is any integer for which $c(D) \neq 0$. (Cf. formula (11) of [Ko85].)

The arithmetic significance of the coefficients $c(D)$ is revealed by the following fundamental formula of Kohnen and Waldspurger (cf. Corollary 1 of [Ko85]):

$$|c(D)|^2 = \begin{cases} \lambda_g D^{k-1/2} L(f, D^*, k) & \text{if } \left(\frac{D^*}{\ell}\right) = w_\ell \text{ for all } \ell \mid N, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where

(a) The complex number λ_g is a non-zero scalar which depends only on the choice of g;

(b) the function

$$L(f, D^*, s) = \sum_n a_n \chi_{D^*}(n) n^{-s}$$

is the twisted L-series attached to f and the quadratic Dirichlet character $\chi_{D^*} := \left(\frac{D^*}{\cdot}\right)$ of conductor D^*;

(c) the integers $w_\ell \in \{\pm 1\}$ are the eigenvalues of the Atkin-Lehner involutions W_ℓ acting on f.

For example, suppose that f is of weight 2 and has rational fourier coefficients, so that it corresponds to an elliptic curve E over $\mathbb{Q}$ of conductor N. The Birch and Swinnerton-Dyer conjecture then predicts that, if $c(D_1)$ and $c(D_2)$ are non-zero, we have

$$\frac{c(D_1)}{c(D_2)} = \pm \sqrt{\frac{\#Ш(E^{-D_1})}{\#Ш(E^{-D_2})}},$$

where E^D denotes the twist of the elliptic curve E by the quadratic Dirichlet character attached to D. In this way, the coefficients of g package arithmetic information concerning the twists of E. In particular, once a sign for one of the non-vanishing coefficients $c(D)$ has been fixed, the remaining coefficients pick out well-defined choices of square-roots for $\#Ш(E^{-D})$. The law that governs the variation of their signs is not well understood.

We remark that the sign $w(f, D^*) \in \{\pm 1\}$ that appears in the functional equation for $L(f, D^*, s)$ is equal to

$$w(f, D^*) = (-1)^k \chi_{D^*}(-N) w_N, \quad \text{where } w_N := \prod_{\ell \mid N} w_\ell.$$

In particular, $w(f, D^*) = 1$ whenever D satisfies $\chi_{D^*}(\ell) = w_\ell$ for all $\ell \mid N$.

When $w(f, D^*) = -1$, the central critical value $L(f, D^*, k)$ vanishes for parity reasons. It then becomes natural to study the central critical derivative $L'(f, D^*, k)$. One of the motivating questions behind the present paper is the following:

Question 1.1. *Can the data of $L'(f, D^*, k)$, (at least for certain D satisfying $w(f, D^*) = -1$) be packaged into the coefficients of a modular generating series?*

Our main result—Theorem 1.4 below—provides an element of answer to this question by relating some of these central critical derivatives to the fourier coefficients of p-*adic families* of modular forms of half-integral weight.

p-adic families. Denote by $\mathbb{C}_p$ the completion of an algebraic closure $\bar{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. Fix a compact open region $U \subset \mathbb{Z}_p$, and let $\mathcal{A}(U)$ denote the ring of $\mathbb{C}_p$-valued p-adic analytic functions on U. Given a formal q series

$$\underline{f} := \sum_{n \geq 0} \underline{a}_n q^n \tag{2}$$

whose coefficients $\underline{a}_n$ belong to $\mathcal{A}(U)$, we will denote by

$$f_k := \underline{f}(k) = \sum_{n \geq 0} \underline{a}_n(k) q^n$$

its specialisation to $k \in U$, viewed as a formal power series with coefficients in $\mathbb{C}_p$. A point $k \in U$ is said to be *classical* if

$$k \in \mathbb{Z}^{\geq 1} \quad \text{and } k \equiv 1 \pmod{p-1},$$

and the set of all classical points in U is denoted by U_{cl}.

For the purposes of this article, a p-adic family of modular forms on $\Gamma_0(N)$ is a formal q-series $\underline{f}$ as in (2) with the property that f_k is a normalised eigenform of weight $2k$ on $\Gamma_0(N)$, for all $k \in U_{cl}$. More precisely, the fourier coefficients of f_k are required to generate a finite extension K_{f_k} of $\mathbb{Q}$, and the image of f_k under any complex embedding $K_{f_k} \hookrightarrow \mathbb{C}$ is a classical modular form of the desired type. Note that K_{f_k} can be treated as an abstract field, but the datum of f_k also determines an embedding of K_{f_k} into $\mathbb{C}_p$.

Classical examples of p-adic families of modular forms include:

(a) Eisenstein series of varying weights,

$$E_k := \zeta^*(1-2k) + 2\sum_{n=1}^{\infty} \sigma^*_{2k-1}(n)q^n,$$

where $\zeta^*(s) = (1-p^{-s})\zeta(s)$ is the Riemann zeta function with its Euler factor at p removed, and $\sigma^*_{2k-1}(n) = \sum_{\substack{d|n \\ (p,d)=1}} d^{2k-1}$.

(b) The binary theta series associated to the powers of a fixed Hecke Grossencharacter Ψ of infinity type $(1,0)$ of an imaginary quadratic field K. These theta series are defined by letting, for all ideals $\mathfrak{a} \lhd \mathcal{O}_K$ of the ring of integers of K,

$$\Psi^*(\mathfrak{a}) = \begin{cases} \Psi(\mathfrak{a}) & \text{if } p \nmid \mathfrak{a}\bar{\mathfrak{a}}, \\ 0 & \text{otherwise,} \end{cases}$$

and setting

$$\theta_k := \sum_{\mathfrak{a} \lhd \mathcal{O}_K} \Psi^*(\mathfrak{a})^{2k-1} q^{\mathfrak{a}\bar{\mathfrak{a}}}.$$

A third class of examples arises from a theorem of Hida, which we will now state in a special case.

Assume for this statement that f is a newform of weight 2 on $\Gamma_0(N)$, and fix a prime p that divides N.

Proposition 1.2 (Hida). *There is a unique p-adic family $\underline{f}$ (defined on a suitable neighbourhood U of $k = 1$) satisfying $f_1 = f$.*

This proposition attaches to f an infinite collection of normalised eigenforms f_k, of varying weights, indexed by the $k \in U_{cl}$.

We mention in passing that Hida's theorem is considerably more general than the special case given in Proposition 1.2. For instance, f could be an eigenform of arbitrary weight, and the condition $p \mid N$ could have been relaxed to the assumption that f be *ordinary* at p. However, Proposition 1.2 is the only case that is germane to the concerns of this article.

For classical points $k > 1$, the modular form f_k obtained from Proposition 1.2 is *not* new at p. This is because f_k is ordinary at p, i.e., of slope 0, while newforms of level N and weight $2k$ have slope $k - 1$.

More precisely, writing $N = pM$, there is a unique normalised newform $f_k^\sharp$ in $S_{2k}(M)$ satisfying

$$T_\ell(f_k^\sharp) = \underline{a}_\ell(k) f_k^\sharp, \quad \text{for all primes } \ell \text{ with } (\ell, N) = 1.$$

Let

$$g_k = \sum_{D>0} c(D,k) q^D$$

denote the (unique, up to scaling) eigenform in $S^+_{k+1/2}(4M)$ that is associated to $f_k^\sharp$ by the Shimura-Kohnen correspondence, and let $g = g_1$ be the newform in $S^+_{3/2}(4N)$ associated to f.

It is crucial for our main result that the half-integral weight forms g_k have, a priori, "twice as many" non-vanishing fourier coefficients as the modular form g. The values of D for which $c(D,k)$ need not vanish can be divided into two categories.

I. The D for which $\chi_{-D}(\ell) = w_\ell$, for all ℓ dividing N.

II. The D for which

$$\begin{cases} \chi_{-D}(\ell) = w_\ell & \text{for all } \ell \mid M, \\ \chi_{-D}(p) = -w_p. \end{cases}$$

For these D, we have $c(D) = 0$; furthermore, $L(f, -D, 1) = 0$, because $w(f, -D) = -1$.

We will call discriminants $-D$ of the first type *Type I discriminants*, and those of the second type, *Type II discriminants*.

Since the functions $k \mapsto \underline{a}_n(k)$ on U_{cl} extend to analytic functions on U, it is natural to expect a similar principle for the functions $k \mapsto c(D, k)$. The fact that the individual forms g_k are only defined up to a nonzero scaling factor makes it necessary to introduce a normalisation. We do this by fixing a type I discriminant $-\Delta_0$ for which $c(\Delta_0) \neq 0$. A theorem of Hida and Stevens (cf. Theorem 5.5 and Lemma 6.1 of [St94]) then gives:

Proposition 1.3. *There is a p-adic neighbourhood of $k = 1$ in U on which $c(\Delta_0, k)$ is everywhere non-vanishing. After replacing U by such a neighbourhood, the normalised coefficient attached to $k \in U_{cl} \cap \mathbb{Z}^{>1}$ by*

$$\tilde{c}(D, k) = \frac{\left(1 - \left(\frac{-D}{p}\right) \underline{a}_p(k)^{-1} p^{k-1}\right) c(D, k)}{\left(1 - \left(\frac{-\Delta_0}{p}\right) \underline{a}_p(k)^{-1} p^{k-1}\right) c(\Delta_0, k)} = \frac{c(p^2 D, k)}{c(p^2 \Delta_0, k)}$$

extends to a p-adic analytic function on U, which satisfies

$$\tilde{c}(D, 1) = \frac{c(p^2 D)}{c(p^2 \Delta_0)} = \frac{c(D)}{c(\Delta_0)}.$$

In particular, if $-D$ is a type II discriminant, then $c(D) = 0$ and hence $\tilde{c}(D, 1) = 0$. It then becomes natural to consider the derivative of $\tilde{c}(D, k)$ with respect to k at $k = 1$.

Assume now, for simplicity, that the newform $f \in S_2(N)$ has *rational* fourier coefficients, so that it corresponds to an elliptic curve E of conductor N. Since p divides N exactly, the curve E has (split or non-split) multiplicative reduction at p. Let

$$\Phi_{\mathrm{Tate}} : \mathbb{C}_p^\times / q^{\mathbb{Z}} \longrightarrow E(\mathbb{C}_p)$$

be Tate's p-adic uniformisation of E, and let

$$\log_E : E(\mathbb{C}_p) \longrightarrow \mathbb{C}_p$$

be the p-adic formal group logarithm, which can be defined by

$$\log_E(P) = \log_q(\Phi_{\mathrm{Tate}}^{-1}(P)),$$

where $\log_q$ is the branch of the p-adic logarithm that vanishes on $q^{\mathbb{Z}}$. We extend this logarithm to $E(\mathbb{C}_p) \otimes \mathbb{Q}$ by $\mathbb{Q}$-linearity.

Let $E(\mathbb{Q}(\sqrt{-D}))^-$ denote the submodule of $E(\mathbb{Q}(\sqrt{-D}))$ on which the involution in $\mathrm{Gal}(\mathbb{Q}(\sqrt{-D})/\mathbb{Q})$ acts as -1. When $-D$ is a type II discriminant, we have

$$\chi_{-D}(p) = -w_p = a_p,$$

and hence the quadratic twist E^{-D} of E over $\mathbb{Q}(\sqrt{-D})$ has *split multiplicative reduction* at p. We also note that if P belongs to $E(\mathbb{Q}(\sqrt{-D}))^-$, then

$$\log_E(P) \text{ belongs to } \mathbb{Q}_p.$$

One of the main theorems of this article is:

Theorem 1.4. *Let $-D$ be a discriminant of type II. There exists an element $P_D \in E(\mathbb{Q}(\sqrt{-D}))^- \otimes \mathbb{Q}$ satisfying*

(1) $\frac{d}{dk}\tilde{c}(D,k)_{k=1} = \log_E(P_D)$;
(2) $P_D \neq 0$ *if and only if* $L'(E,-D,1) \neq 0$.

Although the points P_D only belong to $E(\mathbb{Q}(\sqrt{-D})) \otimes \mathbb{Q}$, the proof of Theorem 1.4 will show that the collection $\{P_D\}$ has *bounded denominators*:

there exists an integer t_E, depending on E but not on the type II discriminant $-D$, such that

$$t_E P_D \text{ belongs to } E(\mathbb{Q}(\sqrt{-D}))^{-}.$$

These global points, which are defined over a varying collection of quadratic fields, display an *a priori* unexpected coherence, which can be summarised by letting ϵ be a first order infinitesimal (satisfying $\epsilon^2 = 0$) and considering the formal q-series with coefficients in $\mathbb{Q}_p(\epsilon)$:

$$g_{1+\epsilon} := \left(\sum_{-D \text{ of type I}} c(D) q^D \right) + \epsilon \left(\sum_{-D \text{ of type II}} \log_E(P_D) q^D \right).$$

Theorem 1.4 amounts to the statement that $g_{1+\epsilon}$ is a "modular form of weight $(3/2 + \epsilon)$" associated to the Hida family $\underline{f}$ (or rather, to its specialisation to a first order neigbourhood of weight two) under the Shimura correspondence.

The proof of Theorem 1.4 rests on two ingredients.

(i) A formula of Kohnen expressing the products $c(D_1, k)c(D_2, k)$ of fourier coefficients of g_k in terms of certain *geodesic cycles integrals* associated to $f_k^\sharp$ and to binary quadratic forms of discriminant $D_1 D_2$.

(ii) The theory of *Stark-Heegner points*, which relates these period integrals to global points defined (conjecturally, in general) over ring class fields of real quadratic fields.

Sections 2 and 3 review these ingredients in turn. Section 4 describes the proof of Theorem 1.4. This proof suggests studying generating series whose coefficients are built out of Stark-Heegner points, in the spirit of the Gross-Kohnen-Zagier formula of [GKZ87] relating classical Heegner points to coefficients of modular forms of weight $3/2$. Several theorems of "Gross-Kohnen-Zagier type" are formulated and proved in Section 5. This section also formulates a more general Gross-Kohnen-Zagier conjecture for Stark-Heegner points which in some cases falls squarely outside the scope of the methods of this paper based on exploiting p-adic deformations of modular forms. Some of the cases of the Gross-Kohnen-Zagier conjecture for which a proof eludes us are discussed further in Section 6, and some numerical evidence for them is provided.

2. Kohnen's formula

In this section, and this section only, let f be a normalised newform of weight $2k$ on $\Gamma_0(M)$, and let $g = \sum_D c(D)q^D$ be the newform in $S_{k+1/2}(4M)$ which is attached to it under the Shimura-Kohnen correspondence.

Our purpose is to briefly recall a theorem of Kohnen which expresses the product $c(D_1)c(D_2)$ of two fourier coefficients of g in terms of *Shintani cycles* which we proceed to describe, following the treatment in [Ko85].

If $c(D_1)$ and $c(D_2)$ are non-zero, then D_1^* and D_2^* are discriminants of orders in a quadratic field, satisfying

$$\chi_{D_1^*}(\ell) = \chi_{D_2^*}(\ell) = w_\ell, \quad \text{for all } \ell \mid M, \tag{3}$$

by the definition of Kohnen's plus space. Assume for simplicity that D_1^* and D_2^* are both fundamental and prime to $2M$. Then the product $\Delta := D_1^* D_2^*$ is the discriminant of an order in a real quadratic field K, or of an order in the split quadratic algebra $\mathbb{Q} \times \mathbb{Q}$ if $D_1^* = D_2^*$. Note that all the $\ell \mid M$ are split in K. In particular, it is possible to choose an integer δ such that

$$\delta^2 \equiv \Delta \pmod{4M}.$$

A primitive binary quadratic form $Q(x,y) = Ax^2 + Bxy + Cy^2$ of discriminant Δ is said to be a *Heegner form* (relative to the level M) if

$$M \mid A, \qquad B \equiv \delta \pmod{M}.$$

Let C_Q be the image in $\Gamma_0(M)\backslash\mathcal{H}$ of the geodesic in the Poincaré upper half plane consisting of the complex numbers $z = x + iy$ satisfying

$$A\,|z|^2 + Bx + C = 0,$$

oriented from left to right if $A > 0$, from right to left if $A < 0$, and from $-C/B$ to $i\infty$ if $A = 0$. For example, if Δ is not a perfect square, let $r + s\sqrt{\Delta}$ be the fundamental unit of norm 1 in $\mathcal{O}_\Delta := \mathbb{Z}[\frac{\Delta+\sqrt{\Delta}}{2}]$, normalised so that $r, s > 0$. Then the path C_Q is equivalent to the path from τ to $\gamma_Q\tau$, where

$$\gamma_Q = \begin{pmatrix} r+sB & 2Cs \\ -2As & r-sB \end{pmatrix} \in \Gamma_0(M) \tag{4}$$

is a generator for the stabiliser subgroup of Q in $\Gamma_0(M)$, and $\tau \in \mathcal{H}$ is any base point.

To each indefinite quadratic form Q is associated the Shintani cycle

$$r(f,Q) = \int_{C_Q} f(z)Q(z,1)^{k-1}dz. \tag{5}$$

Note that $r(f,Q)$ depends only on the $\Gamma_0(M)$-equivalence class of Q.

Let

$$\mathcal{F}_\Delta = \{ \text{ Heegner forms } Ax^2 + Bxy + Cy^2 \quad \text{of discriminant } \Delta.\}.$$

This set of binary quadratic forms is preserved under the usual right action of $\Gamma_0(M)$. Define a function

$$\omega_{D_1^*,D_2^*} : \mathcal{F}_\Delta/\Gamma_0(M) \longrightarrow \pm 1$$

by the rule

$$\omega_{D_1^*,D_2^*}(Q) = \begin{cases} 0 & \text{if } \gcd(a,b,c,D_1) > 1; \\ \left(\frac{D_1^*}{Q(m,n)}\right) & \text{where } \gcd(Q(m,n),D_1)=1, \text{ otherwise.} \end{cases}$$

Genus theory shows that $\omega_{D_1^*,D_2^*}$ is well-defined, and that it is a quadratic character on the class group of primitive binary quadratic forms of discriminant Δ. This character cuts out the biquadratic field $\mathbb{Q}(\sqrt{D_1^*}, \sqrt{D_2^*})$, an unramified quadratic extension of $\mathbb{Q}(\sqrt{\Delta})$.

The Shintani cycle associated to the pair (D_1^*, D_2^*) is defined by the formula

$$r(f, D_1^*, D_2^*) = \sum_{Q \in (\mathcal{F}_\Delta/\Gamma_0(M))} \omega_{D_1^*,D_2^*}(Q) r(f, Q). \tag{6}$$

The following theorem of Kohhen (cf. Theorem 3 of [Ko85]) plays a key role in this paper:

Theorem 2.1. *For all* D_1 *and* D_2 *satisfying (3),*

$$\frac{c(D_1)\overline{c(D_2)}}{\langle g, g\rangle} = (-1)^{[k/2]} 2^k \times \frac{r(f, D_1^*, D_2^*)}{\langle f, f\rangle}.$$

3. Stark-Heegner points

For this section, we revert to the notations that were in use in the statement of Theorem 1.4 of the introduction. Thus, in particular, f is now a newform of weight 2 and level $N = pM$ associated to an elliptic curve E of conductor N.

Let Δ be a fundamental discriminant satisfying

$$\Delta > 0, \qquad \chi_\Delta(p) = -1, \qquad \chi_\Delta(\ell) = 1, \quad \text{for all } \ell \mid M. \tag{7}$$

The article [Dar01] (see also [BD06] and [Dar06]) introduces a conjectural p-adic variant of the Heegner point construction, associating to every equivalence class of binary quadratic forms of discriminant Δ a p-adic point in $E(\bar{\mathbb{Q}}_p)$. These local points are called *Stark-Heegner points* in [Dar01], and are predicted to be defined over ring class fields of $K = \mathbb{Q}(\sqrt{\Delta})$.

We will have no need for the original definition given in [Dar01] based on the modular symbols attached to f. Rather, we will recall here an alternate description exploited in [BD06] which relies on the p-adic family $\underline{f}_k$ and the classical newforms $f_k^\sharp$ attached to f by Hida's Proposition 1.2.

For each eigenform f_k with $k \in U_{cl}$, we begin by considering the complex period integrals

$$I_{\mathbb{C}}(f_k, P, r, s) := \int_r^s f_k(z) P(z) dz, \tag{8}$$

where $P(z) \in \mathbb{Q}[z]^{[2k-2]}$ is a polynomial of degree at most $2k-2$, and r, s are elements of $\mathbb{P}_1(\mathbb{Q})$ viewed as subsets of the extended Poincaré upper half plane $\mathcal{H}^*$. The integral (8) converges because f_k is a cusp form. Note that in order to define $I_{\mathbb{C}}(f_k, P, r, s)$ we had to choose a real embedding $K_{f_k} \hookrightarrow \mathbb{R}$, so that f_k could be viewed as a complex analytic function on $\mathcal{H}^*$.

Let

$$\begin{aligned} I_{\mathbb{C}}^{+}(f_k, P, r, s) &:= \text{Real}(I_{\mathbb{C}}(f_k, P, r, s)), \\ I_{\mathbb{C}}^{-}(f_k, P, r, s) &:= \text{Imag}(I_{\mathbb{C}}(f_k, P, r, s)). \end{aligned}$$

The following theorem of Shimura gives a rationality property for these complex numbers.

Proposition 3.1 (Shimura)**.** *There exist periods $\Omega_k^+, \Omega_k^- \in \mathbb{C}^\times$ depending only on f_k, for which*

$$I^{\pm}(f_k, P, r, s) := \frac{1}{\Omega_k^{\pm}} I_{\mathbb{C}}^{\pm}(f_k, P, r, s) \quad \textit{belongs to } K_{f_k},$$

for all $P \in \mathbb{Q}[z]^{[2k-2]}$, and $r, s \in \mathbb{P}_1(\mathbb{Q})$.

It will lighten the notation to fix a choice of sign $\epsilon = \{1, -1\}$ and set

$$I(f_k, P, r, s) := I^{\epsilon}(f_k, P, r, s). \tag{9}$$

For the proof of Theorem 1.4, it is only the value $\epsilon = -1$ that is relevant, so the reader may assume throughout Sections 3 and 4 that $\epsilon = -1$. However, the possibility to choose $\epsilon = 1$ allows a more general definition of Stark-Heegner points which will be crucial in the theorems, conjectures and numerical experiments described in Sections 5 and 6.

The identification of the complex number $I(f_k, P, r, s)$ with an element of K_{f_k} is made via the same complex embedding of K_{f_k} that was used to define $I_{\mathbb{C}}(f_k, P, r, s)$. Recall that the collection $\{f_k\}_{k \in U^{cl}}$ also determines p-adic embeddings of K_{f_k} into $\mathbb{C}_p$. Thanks to this data, we can—and will—view the integrals $I(f_k, P, r, s)$ as elements of $\mathbb{C}_p$. We can then extend their definition to $P \in \mathbb{C}_p[z]^{[2k-2]}$ by $\mathbb{C}_p$-linearity.

Let $\mathbb{Q}_{p^2} \subset \mathbb{C}_p$ be the quadratic unramified extension of $\mathbb{Q}_p$, let $\mathcal{O}_{p^2}$ be its ring of integers, and let

$$\mathcal{H}_p^0 := \left\{ \tau \in \mathcal{O}_{p^2} \text{ such that } \tau \not\equiv 0, 1, \ldots, p-1 \pmod{p} \right\}.$$

We now invoke the following theorem of Stevens which asserts that the quantities $I(f_k, P, r, s)$ can in some sense be p-adically interpolated.

Proposition 3.2 (Stevens). *After eventually replacing the region U by a smaller p-adic neighbourhood of 1, the complex periods Ω_k can be chosen in such a way that*

(1) *The function $k \mapsto I(f_k, (z-\tau)^{2k-2}, r, s)$ extends to an analytic function of $k \in U$, for all $\tau \in \mathcal{H}_p^0$, and all $r, s \in \mathbb{P}_1(\mathbb{Q})$.*
(2) *The function $(r, s) \mapsto I(f_1, 1, r, s)$ takes its values in $\mathbb{Q}$ and is not identically 0.*

Proof. This theorem is stated in Theorem 2.2.1 of [BD06]. In fact a more precise version is stated there, and since the statement will be used later, we recall it briefly here. Let $(\mathbb{Z}_p^2)'$ denote the set of *primitive vectors* in $\mathbb{Z}_p^2$, i.e., the vectors that are not divisible by p. Given a polynomial $P \in Q[z]^{[2k-2]}$, write $\tilde{P}(x, y)$ for its homogenisation of degree $2k - 2$. (Note in particular that $\tilde{P}(x, y)$ depends on k.) Theorem 2.2.1 of [BD06] asserts the existence, for each $r, s \in \mathbb{P}_1(\mathbb{Q})$, of a suitable p-*adic distribution* $\mu_{r,s}$ on $(\mathbb{Z}_p^2)'$ of total measure 0, satisfying

$$(10) \qquad I(f_k, P, r, s) = \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} \tilde{P}(x, y) d\mu_{r,s}(x, y), \qquad \text{for all } k \in U_{cl}.$$

□

For the most part, it is only the existence of the periods Ω_k and the distributions $\mu_{r,s}$ that will be used. Of course, the periods Ω_k are far from unique, but if $\{\Omega_k\}$ and $\{\Omega_k'\}$ are any two choices, then

$$\Omega_k' = \lambda(k)\Omega_k,$$

where λ extends to an analytic function on U satisfying $\lambda(1) \in \mathbb{Q}^\times$. All the statements that will be made concerning the periods $I(f_k, P, r, s)$ will be insensitive to a change in λ.

It is nonetheless useful to make a precise choice of periods $\{\Omega_k\}_{k \in U_{cl}}$ satisfying the conclusion of Proposition 3.2. This can be done in terms of the periods of (6) thanks to the following lemma.

Lemma 3.3. *Let Δ_0 be a fundamental discriminant satisfying*

$$\chi_{\Delta_0}(p) = w_p, \quad L(f, \Delta_0, 1) \neq 0.$$

Assume that Δ_0 is positive if the sign ϵ in (9) is 1, and that it is negative if $\epsilon = -1$. Then there exists a p-adic neighbourhood U of $k = 1$ such that

$$(11) \qquad \Omega_k := r(f_k, \Delta_0, \Delta_0) = (1 - \chi_{\Delta_0}(p)\underline{a}_p(k)^{-1}p^{k-1})^2 r(f_k^\sharp, \Delta_0, \Delta_0)$$

is non-zero for all $k \in U_{cl}$. *Furthermore, this choice of* Ω_k *satisfies the conclusion of Proposition 3.2.*

Proof. Proposition 3.2 implies that $r(f_k, \Delta_0, \Delta_0)$ has all the required properties, provided that it is non-zero. A formula of Birch and Manin expresses $r(f, \Delta_0, \Delta_0)$ as a non-zero multiple of $L(f, \Delta_0, 1)$. (Cf. for example the last equation on Page 241 of [Ko85].) The result follows. ☐

We next define the periods $I(f_k^\sharp, P, r, s) \in \mathbb{C}_p$ as in Proposition 3.1, but with f_k replaced by $f_k^\sharp$. (In particular, we use the same choice of complex periods Ω_k that was made define $I^\pm(f_k, P, r, s)$.)

Finally, we "regularize" these periods by setting, for all $k \in U_{cl}$,

$$J(f_k^\sharp, P, r, s) := (1 - \underline{a}_p(k)^{-2} p^{2k-2}) I(f_k^\sharp, P, r, s). \tag{12}$$

Proposition 3.4. *For all* $\tau \in \mathcal{H}_p^0$ *and all* $r, s \in \mathbb{P}_1(\mathbb{Q})$*, the function*

$$k \mapsto J(f_k^\sharp, (z - \tau)^{2k-2}, r, s)$$

extends to a p*-adic analytic function of* k *which vanishes at* $k = 1$.

Proof. This follows from Proposition 2.2.3 of [BD06]. This proposition also expresses $J(f_k^\sharp, P, r, s)$ in terms of the distributions $\mu_{r,s}$ alluded to in equation (10). More precisely,

$$J(f_k^\sharp, P, r, s) = \int_{(\mathbb{Z}_p^2)'} \tilde{P}(x, y) d\mu_{r,s}(x, y). \tag{13}$$

Proposition 3.4 then follows from the fact that $P = (z - \tau)^{2k-2}$ is a continuous function on the compact space $(\mathbb{Z}_p^2)'$ and is analytic as a function of k. The fact that the function $k \mapsto J(f_k^\sharp, (z - \tau)^{2k-2}, r, s)$ vanishes at $k = 1$ follows from the fact that $\mu_{r,s}$ has total measure 0. ☐

Recall the set $\mathcal{F}(\Delta)$ of Heegner forms of discriminant Δ attached to the level M. In order to define this set, we had to choose a square root δ of Δ

modulo M. We now choose a square root of Δ in $\mathbb{C}_p$, which we will simply denote by $\sqrt{\Delta}$. For $Q = [A, B, C] \in \mathcal{F}(\Delta)$, let

$$\tau_Q := \frac{-B + \sqrt{\Delta}}{2A}$$

be a root of $Q(z, 1) = 0$. Assumption (7) implies that this element belongs to $\mathcal{H}_p^0$, for all $Q \in \mathcal{F}(\Delta)$. Recall the matrix γ_Q defined in (4). We now choose an arbitrary base point $r \in \mathbb{P}_1(\mathbb{Q})$ and define

$$J(f, Q) = \frac{d}{dk}\left(J(f_k^\sharp, (z - \tau_Q)^{2k-2}, r, \gamma_Q r)\right)_{k=1}. \tag{14}$$

The following lemma collects some of the basic properties of the invariants $J(f, Q)$.

Lemma 3.5. (a) *The expression* $J(f, Q)$ *does not depend on the* $r \in \mathbb{P}_1(\mathbb{Q})$ *that was chosen to define it.*

(b) *The function* $Q \mapsto J(f, Q)$ *depends only on the image of* Q *in the class group* $\mathcal{F}_\Delta/\Gamma_0(M)$.

(c) *The element* $J(f, Q)$ *belongs to the quadratic unramified extension* $\mathbb{Q}_{p^2}$ *of* $\mathbb{Q}_p$. *If* $x \mapsto \bar{x}$ *denotes the non-trivial automorphism of this field, then*

$$J(f, Q) + \overline{J(f, Q)} = 2\frac{d}{dk}(\tilde{r}(f_k^\sharp, Q)),$$

where

$$\tilde{r}(f_k^\sharp, Q) = \frac{1}{\Omega_k} \times (1 - \underline{a}_p(k)^{-2} p^{2k-2}) r(f_k^\sharp, Q),$$

and $r(f_k^\sharp, Q)$ *is the invariant defined in (5).*

Proof. (a) A direct calculation shows that

$$\begin{aligned} J(f_k^\sharp, (z - \tau_Q)^{2k-2}, r, \gamma_Q r) \quad &- \quad J(f_k^\sharp, (z - \tau_Q)^{2k-2}, s, \gamma_Q s) \\ &= \quad (1 - \epsilon^{2k-2}) J(f_k^\sharp, (z - \tau_Q)^{2k-2}, r, s), \end{aligned}$$

where ϵ is a fundamental unit of the order of discriminant Δ, viewed as an element of $\mathbb{Q}_{p^2}^{\times}$. The expression on the right is a product of two p-adic analytic functions which vanish at $k = 1$, and hence vanishes to order at least 2 at $k - 1$. Part (a) follows. As for part (b), let $Q' = Q\alpha$, where $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. A direct calculation then shows that

$$J(f_k^\sharp, Q', \alpha^{-1}r, \gamma_{Q'}\alpha^{-1}r) = (c\tau_Q + d)^{2k-2} J(f_k^\sharp, Q, r, \gamma_Q r).$$

Part (b) now follows from the fact that $(c\tau + d)^{2k-2}$ is a p-adic analytic function of $k \in U$ which is equal to 1 at $k = 1$. To prove part (c), we first remark that, by (14),

$$\begin{aligned} J(f,Q) + \overline{J(f,Q)} &= \frac{d}{dk}\Big(J(f_k^\sharp, (z-\tau_Q)^{2k-2}, r, \gamma_Q r) \\ &\qquad + J(f_k^\sharp, (z-\overline{\tau_Q})^{2k-2}, r, \gamma_Q r)\Big)_{k=1}. \end{aligned}$$

By (13), the right-hand side in this equation can be rewritten as:

$$\begin{aligned} &\frac{d}{dk}\left(\int_{(\mathbb{Z}_p^2)'} (x-\tau_Q y)^{2k-2} + (x-\overline{\tau_Q} y)^{2k-2} d\mu_{r,\gamma_Q r}(x,y)\right)_{k=1} \\ &= 2\left(\int_{(\mathbb{Z}_p^2)'} (\log_p(x-\tau_Q y) + \log_p(x-\overline{\tau_Q} y)) d\mu_{r,\gamma_Q r}(x,y)\right) \\ &= 2\left(\int_{(\mathbb{Z}_p^2)'} (\log_p A + \log_p(x-\tau_Q y) + \log_p(x-\overline{\tau_Q} y)) d\mu_{r,\gamma_Q r}\right) \\ &= 2\left(\int_{(\mathbb{Z}_p^2)'} \log_p \tilde{Q}(x,y) d\mu_{r,\gamma_Q r}(x,y)\right), \end{aligned}$$

where we have used the fact that $\mu_{r,s}((\mathbb{Z}_p^2)') = 0$ in deriving the penultimate equality. We can rewrite this last expression as

$$\begin{aligned} J(f,Q)+\overline{J(f,Q)} &= 2\frac{d}{dk}\left(\int_{(\mathbb{Z}_p^2)'}\tilde{Q}(x,y)^{k-1}d\mu_{r,\gamma_Q r}(x,y)\right)_{k=1} \\ &= 2\frac{d}{dk}\left(J(f_k^\sharp,Q^{k-1},r,\gamma_Q r)\right)_{k=1} \\ &= 2\frac{d}{dk}\left((1-\underline{a}_p(k)^{-2}p^{2k-2})I(f_k^\sharp,Q^{k-1},r,\gamma_Q r)\right)_{k=1} \\ &= 2\frac{d}{dk}\left(\frac{1}{\Omega_k}\times(1-\underline{a}_p(k)^{-2}p^{2k-2})r(f_k^\sharp,Q)\right)_{k=1}, \end{aligned}$$

where the third equality follows from (12). The result now follows from the definition of $\tilde{r}(f_k^\sharp, Q)$. □

Thanks to part (b) of Lemma 3.5, we can attach to any factorisation of Δ as a product D_1D_2 of two fundamental discriminants, the invariants

$$\begin{aligned} \tilde{r}(f_k^\sharp,D_1,D_2) &= \sum_{Q\in\mathcal{F}_\Delta/\Gamma_0(M)}\omega_{D_1,D_2}(Q)\tilde{r}(f_k^\sharp,Q), \\ J(f,D_1,D_2) &= \sum_{Q\in\mathcal{F}_\Delta/\Gamma_0(M)}\omega_{D_1,D_2}(Q)J(f,Q), \end{aligned}$$

where ω_{D_1,D_2} is the genus character that was introduced in Section 2. We assume in this definition that the sign ϵ of (9) has been chosen to be 1 if D_1 and D_2 are positive, and -1 if they are both negative.

Note that, for any pair (D_1, D_2) of fundamental discriminants occurring in a factorisation of Δ, we necessarily have

$$D_1D_2>0, \quad \chi_{D_1}(p)=-\chi_{D_2}(p), \quad \chi_{D_1}(\ell)=\chi_{D_2}(\ell), \text{ for all } \ell \mid M.$$

In particular, the signs in the functional equations for the twisted L series $L(f, D_1, s)$ and $L(f, D_2, s)$ are *opposite*. By interchanging D_1 and D_2 if necessary, assume that

$$w(f,D_1)=1, \qquad w(f,D_2)=-1.$$

It is not hard to see that $J(f, D_1, D_2)$ is in an eigenspace for the non-trivial element in $\mathrm{Gal}(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$. More precisely, we have

$$\overline{J(f, D_1, D_2)} = \begin{cases} J(f, D_1, D_2) & \text{if } \chi_{D_2}(p) = -w_p; \\ -J(f, D_1, D_2) & \text{if } \chi_{D_2}(p) = w_p. \end{cases} \tag{15}$$

We can now recall the following conjecture on "Stark-Heegner points" that is formulated in [BD06].

Conjecture 3.6. *Let* H *be the ring class field associated to the order* $\mathbb{Z}[\frac{\Delta+\sqrt{\Delta}}{2}]$ *of* $K = \mathbb{Q}(\sqrt{\Delta}))$.

(a) *For each* $Q \in \mathcal{F}_\Delta/\Gamma_0(M)$*, there exists* $P_Q \in E(H) \otimes \mathbb{Q}$ *such that* $J(f, Q) = \log_E(P_Q)$.

(b) *There is a global point* $P(f, D_1, D_2) \in E(\mathbb{Q}(\sqrt{D_2}))^- \otimes \mathbb{Q}$ *such that*

$$J(f, D_1, D_2) = \log_E(P(f, D_1, D_2)).$$

Furthermore, $P(f, D_1, D_2)$ *is of infinite order if and only if*

$$L'(E/\mathbb{Q}(\sqrt{\Delta}), \omega_{D_1,D_2}, 1) = L(f, D_1, 1)\, L'(f, D_2, 1)$$

is non-zero.

While Conjecture 3.6 appears difficult, we have been able to prove the following special case which will play a key role in the proof of Theorem 1.4

Theorem 3.7. *Assume that*

(a) *The level* N *is divisible by at least two primes.*

(b) *We have* $\chi_{D_2}(p) = -w_p$*, (so that in particular* $J(f, D_1, D_2)$ *belongs to* $\mathbb{Q}_p$*).*

Then

(1) *We have*

$$J(f, D_1, D_2) = \log_E(P(f, D_1, D_2)),$$

for some element $P(f, D_1, D_2) \in E(\mathbb{Q}(\sqrt{D_2}))^- \otimes \mathbb{Q}$.

(2) *There exists an integer* t_E*, depending on* E *but not on* (D_1, D_2)*, such that* $t_E P(f, D_1, D_2)$ *belongs to* $E(\mathbb{Q}(\sqrt{D_2}))^-$.

(3) *The invariant* $P(f, D_1, D_2)$ *corresponds to a point of infinite order if and only if*

$$L'(E/\mathbb{Q}(\sqrt{\Delta}), \omega_{D_1,D_2}, 1) = L(f, D_1, 1)\, L'(f, D_2, 1)$$

is non-zero.

Theorem 3.7 is a special case of Theorem 1 of [BD06]. To recall the proof here would take us too far afield. (The key idea is to use the main theorem of [BD07] to express the periods $J(f, D_1, D_2)$ in terms of actual *Heegner points* arising from parametrisations of E by Shimura curves associated to quaternion algebras that are ramified at p.) We will content ourselves with drawing attention to the important relationship between $J(f, D_1, D_2)$ and the periods $\tilde{r}(f_k^\sharp, D_1, D_2)$ that arise in Kohnen's formula.

Lemma 3.8. *Under the assumptions of Theorem 3.7,*

$$J(f, D_1, D_2) = \frac{d}{dk}\left(\tilde{r}(f_k^\sharp, D_1, D_2)\right)_{k=1}.$$

Proof. Since $J(f, D_1, D_2)$ belongs to $\mathbb{Q}_p$,

$$2J(f, D_1, D_2) = J(f, D_1, D_2) + \overline{J(f, D_1, D_2)} = 2\frac{d}{dk}\left(\tilde{r}(f_k^\sharp, D_1, D_2)\right)_{k=1},$$

where the last equality follows from part (c) of Lemma 3.5. ☐

4. Proof of Theorem 1.4

Let $-D_2$ be a discriminant of type II. Choose an auxiliary discriminant $-D_1$ of type I satisfying

(a) $\gcd(D_1, D_2) = 1$;

(b) $c(D_1) \neq 0$.

This implies that the product $\Delta = D_1 D_2$ satisfies assumption (7). Note also that in this case

$$w(f, -D_1) = 1, \quad w(f, -D_2) = -1, \qquad \chi_{-D_2}(p) = -w_p.$$

Hence all the conditions in the statement of Theorem 3.7 are satisfied.

By definition of the normalised coefficients, we have

$$\tilde{c}(D_1, k)\tilde{c}(D_2, k) = \frac{(1 - \underline{a}_p(k)^{-2} p^{2k-2}) c(D_1, k) c(D_2, k)}{(1 - w_p \underline{a}_p(k)^{-1} p^{k-1})^2 c(\Delta_0, k)^2}.$$

Hence by Theorem 2.1,

$$\begin{aligned} \tilde{c}(D_1, k)\tilde{c}(D_2, k) &= \frac{(1 - \underline{a}_p(k)^{-2} p^{2k-2}) r(f_k^\sharp, -D_1, -D_2)}{(1 - w_p \underline{a}_p(k)^{-1} p^{k-1})^2 r(f_k^\sharp, -\Delta_0, -\Delta_0)} \\ &= \frac{(1 - \underline{a}_p(k)^{-2} p^{2k-2}) r(f_k^\sharp, -D_1, -D_2)}{\Omega_k}, \end{aligned}$$

where the last equality follows from the choice of Ω_k that was made in (11), in light of the fact that $\chi_{-\Delta_0}(p) = w_p$. By the definition of the period $\tilde{r}(f_k^\sharp, -D_1, -D_2)$ given in Lemma 3.5, it follows that

$$\tilde{c}(D_1, k)\tilde{c}(D_2, k) = \tilde{r}(f_k^\sharp, -D_1, -D_2).$$

Differentiating both sides with respect to k, evaluating at $k = 1$ and applying Lemma 3.8, yields

$$\tilde{c}(D_1)\frac{d}{dk}\tilde{c}(D_2, k)_{k=1} = J(f, -D_1, -D_2). \tag{16}$$

The first part of Theorem 1.4 is now a consequence of Theorem 3.7 combined with the rationality of $\tilde{c}(D_1)$. The second part follows from the fact that $\tilde{c}(D_1) \neq 0$ if and only if $L(f, -D_1, 1) \neq 0$.

5. GROSS-KOHNEN-ZAGIER FORMULAE FOR STARK-HEEGNER POINTS

The conjectures of [Dar01] predict that Stark-Heegner points should have many of the properties of their classical counterparts. It is therefore natural to look for analogues of the theorem of Gross, Kohnen and Zagier relating Stark-Heegner points to the fourier coefficients of modular forms of weight $3/2$. In fact, the method of proof of Theorem 1.4 yields some results in this direction.

For example, we have:

Theorem 5.1. *Let $-D_2$ be a fixed discriminant of type II associated to f. Then*

(a) *The periods $b(D_1) := J(f, -D_1, -D_2)$, as $-D_1$ varies over the type I discriminants attached to f, are (proportional to) the fourier coefficients $c(D_1)$ of a Shimura-Kohnen lift $g \in S^{+}_{3/2}(4N)$ attached to f.*

(b) *Assume further that N is the product of at least two primes. Then the function $D_1 \mapsto b(D_1)$ is non-zero if and only if*

$$L'(f, -D_2, 1) \neq 0.$$

Proof. This follows directly from (16), which shows that the ratio between $b(D_1)$ and $c(D_1)$ is equal to the expression $\frac{d}{dk}\tilde{c}(D_2, k)_{k=1}$, which does not depend on D_1. When N is divisible by at least two primes, Theorem 3.7 relates this expression to a global point on $E(\mathbb{Q}(\sqrt{-D_2}))^{-} \otimes \mathbb{Q}$ which is non-zero precisely when $L'(f, -D_2, 1) \neq 0$. $\square$

Observe that Theorem 3.7 is not needed to prove part (a) of Theorem 5.1. In particular, when f is a form of prime conductor, the invariants $J(f, -D_1, -D_2)$ (as D_1 varies, and D_2 is fixed) can still be related to the fourier coefficients of a modular form of weight $3/2$, even though the proof of Theorem 3.7 breaks down for such modular forms and we are unable to relate $J(f, -D_1, -D_2)$ to a global point on $E(\mathbb{Q}(\sqrt{-D_2}))^-$. This remark leads to the following corollary which gives further evidence for the general conjectures on Stark-Heegner points formulated in [Dar01].

Corollary 5.2. *Let $-D_2$ be a fixed discriminant of type II. Assume that there exists a type I discriminant $-D_1$ for which*

$$J(f, -D_1, -D_2) = \log_E(P(f, -D_1, -D_2)) \neq 0,$$

with $P(f, -D_1, -D_2) \in E(\mathbb{Q}(\sqrt{-D_2}))^- \otimes \mathbb{Q}$. Then for all type I discriminants $-D_1$, the expressions $J(f, -D_1, -D_2)$ are equal to the formal group logarithm of global points in $E(\mathbb{Q}(\sqrt{-D_2}))^- \otimes \mathbb{Q}$.

In order to generalise this discussion, let D_2 be any fixed discriminant (either positive, or negative) satisfying

$$w(f, D_2) = -1,$$

but not necessarily of type II. Then the invariants $J(D_1, D_2)$ are defined on all fundamental discriminants D_1 satisfying

(17) $D_1 D_2 > 0, \quad \chi_{D_1}(p) = -\chi_{D_2}(p), \quad \chi_{D_1}(\ell) = \chi_{D_2}(\ell)$ for all $\ell \mid M$.

The coefficients $b(D_1) := J(f, D_1, D_2)$, as D_1 varies over fundamental discriminants satisfying (17), are really only defined up to sign, since they depend on the choice of a p-adic square root of $D_1 D_2$. But for discriminants D_1 that are congruent to each other modulo p, it is possible to make a consistent choice of square root and remove the sign ambiguity in the definition of $b(D_1)$ for D_1 in a fixed residue class modulo p. Section 6.2 discusses this issue in more detail, and explains how in certain cases (for example, when $p \equiv 3$ modulo 4) the coefficient $b(D_1)$ can even be defined unambiguously for all D_1.

Theorem 5.1 suggests the following conjecture.

Conjecture 5.3. *The coefficients $b(D)$ are proportional to the Dth fourier coefficients of a modular form of weight $3/2$ on $\Gamma_1(4N^2)$ associated to f by a (suitably generalised) Shimura-Kohnen correspondence. Furthermore, these coefficients vanish identically if and only if $L'(f, D_2, 1) = 0$.*

Conjecture 5.3 can be divided into two cases:

Case 1: The case where $\chi_{D_2}(p) = -w_p$.

In that case the invariants $J(f, D_1, D_2)$ belong to $\mathbb{Q}_p$, and the key Lemma 3.8 still holds. One therefore has a hope of proving Conjecture 5.3 by suitably generalising Kohnen's formula. Some progress in this direction has been made by work of Mao, Rodriguez-Villegas and Tornaría [MRVT07] and of Mao [Ma07]. See also [PT07c] for more examples of the generalized Shimura-Kohnen correspondence in the case of composite levels.

Case 2: The case where $\chi_{D_2}(p) = w_p$.

In that case one has

$$\overline{J(f, D_1, D_2)} = -J(f, D_1, D_2),$$

and the proof of Lemma 3.8 breaks down completely. In fact, the periods $\tilde{r}(f_k^\sharp, D_1, D_2)$ vanish identically in this case, and there is therefore little hope of controlling the Stark-Heegner points $J(f, D_1, D_2)$ by exploiting the p-adic variation of modular forms. In this setting, Conjecture 5.3 is more mysterious, and we can give little theoretical evidence for it. We have however gathered some numerical evidence in its support in the next section.

6. Numerical evidence

In this section we present some numerical evidence for Conjecture 5.3, in the case—which is the simplest and most natural for calculations—where

(1) The elliptic curve E has prime conductor p and odd analytic rank, so that $w_p = 1$;
(2) the auxiliary discriminant D_2 is equal to 1.

Under these hypotheses, we find ourselves in case 2 in the discussion of Conjecture 5.3. This setting is therefore the most interesting from a theoretical point of view because *both* hypotheses in Theorem 3.7 fail, and Lemma 3.8 does not hold.

For fundamental discriminants $\Delta > 0$ satisfying $\chi_\Delta(p) = -1$ (i.e. for real quadratic fields K in which p is inert), we can use the computer package `shp` of [DP-SHP] to check and (conjecturally) compute the corresponding global point $P(\Delta) \in E(\mathbb{Q})$ (the "trace" of the Stark-Heegner points for K).

On the other hand, a construction given in [MRVT07] associates to f a modular form g_+ of weight $3/2$ on $\Gamma_1(4p^2)$ whose coefficients $c(\Delta)$ are indexed by *real* quadratic discriminants $\Delta > 0$ satisfying $\chi_\Delta(p) = -1$, and are related to a square root of the central values $L(f, \Delta, 1)$. This computation is done by a `PARI/GP` [PARI] package which computes modular forms of half integral weight as linear combinations of generalised theta series associated to positive definite ternary quadratic forms. The linear combinations are determined from Brandt matrices using the package `qalgmodforms` from [CNT].

6.1. **The curve** $37a$.

The smallest prime conductor for which there is an elliptic curve with sign -1 in its functional equation is $p = 37$. Indeed, the quotient of the modular curve $X_0(37)$ by the Atkin-Lehner involution is an

Δ	$h(\Delta)$	$P(\Delta)$	m_Δ	Δ	$h(\Delta)$	$P(\Delta)$	m_Δ
5	1	$(0,0)$	1	61	1	∞	0
8	1	$(0,0)$	1	69	1	∞	0
13	1	$(0,-1)$	-1	76	1	$(0,-1)$	-1
17	1	$(0,-1)$	-1	88	1	$(0,-1)$	-1
24	1	$(0,-1)$	-1	89	1	$(0,-1)$	-1
29	1	$(1,0)$	2	92	1	$(1,-1)$	-2
56	1	$(0,-1)$	-1	93	1	$(1,0)$	2
57	1	$(0,-1)$	-1	97	1	∞	0
60	2	$(0,0)$	1	105	2	$(0,-1)$	-1

TABLE 1. Traces of Stark-Heegner points in $37a$

elliptic curve, denoted by $37a$ in the tables of Cremona, and given by the equation

$$E : y^2 + y = x^3 - x.$$

Note that this curve is unique in its $\mathbb{Q}$-isogeny class.

The computation of the Stark-Heegner points for this curve has been discussed in [DP06], which focuses on a few small discriminants with relatively large class number. This time we computed the *traces* of the Stark-Heegner points over a much larger range—for discriminants $\Delta \leq 10000$ with $\chi_\Delta(37) = -1$. The computations were carried to 20 significant 37-adic digits, which was enough to recognize all of the traces but one (see below).

To compare these traces with coefficients of modular forms of weight $3/2$, we will write $P(\Delta) = m_\Delta P_0$, where $m_\Delta \in \mathbb{Z}$ and $P_0 = (0,0)$ is a fixed generator of $E(\mathbb{Q})$. The traces $P(\Delta)$ and the values m_Δ for fundamental discriminants $\Delta \leq 105$ are shown in Table 1.

Note that the values of m_Δ in the table being rather small, the points $P(\Delta)$ are usually of small height, and thus rational reconstruction from the 37-adic approximation is very easy. For instance

$$P(461) \equiv (3606438279313387, 3005365232761155) \pmod{37^{10}},$$

is easily recognized as the global point $P(461) = \left(\frac{1}{4}, -\frac{5}{8}\right) = 5P_0$.

As the discriminant Δ gets larger, so does m_Δ. In our computation for $\Delta \leq 10000$ we found one discriminant for which the working precision of 20 significant 37-adic digits is not enough to recognize $P(\Delta)$:

$$P(8357) = \left(\frac{51678803961}{12925188721}, -\frac{12133184284073305}{1469451780501769}\right) = -22P_0.$$

It turns out this is the largest value of m_Δ in our range. Of course we *expect* $|m_{8357}| = 22$ from reading a coefficient of a modular form of weight $3/2$ (see Table 2), and we can still check *a posteriori* that this is consistent with the value of $P(8357)$ computed modulo 37^{20}.

The computation of the modular form g_+ of weight $3/2$ corresponding to $37a$ can be done following [MRVT07]. The fourier expansion of g_+ begins

$$g_+ = q^5 - q^8 - q^{13} - q^{17} + q^{20} + q^{24} - 2q^{29} + O(q^{30}).$$

Note that due to some choices in the construction, the sign of the coefficients is not well defined. The sign of the coefficients here may differ from the ones given in [MRVT07].

The coefficients of $g_+ = \sum c(\Delta)q^\Delta$ for the fundamental discriminants $\Delta \leq 105$, $\Delta = 461$ and $\Delta = 8357$ are shown in Table 2.

6.2. **The mysterious signs.** The Stark-Heegner points as we have described them have an inherent ambiguity of sign. This is because their calculation depends essentially on a choice of $\sqrt{\Delta} \in \mathbb{C}_p$. In the case of classical Heegner points in the original Gross-Kohnen-Zagier formula,

Δ	5	8	13	17	24	29	56	57	60	61	69
$c(\Delta)$	1	-1	-1	-1	1	-2	-1	-1	-1	0	0

Δ	76	88	89	92	93	97	105	...	461	...	8357
$c(\Delta)$	-1	1	-1	-2	2	0	-1	...	5	...	22

TABLE 2. Coefficients of g_+ corresponding to $37a$

square roots in $\mathbb{C}$ of imaginary quadratic discriminants are canonically chosen to be in the upper half plane. At this stage, Conjecture 5.3 only makes sense up to a sign; this is tantamount to a Gross-Zagier type formula for the heights of the (traces of) Stark-Heegner points, and does not include a statement about the "mysterious signs".

We remark that g_+ itself is not unique (not even up to a single constant, unlike for the case of classical Heegner points). However, having the level bounded at $4p^2$ restricts the choices quite a lot. There is still room to change signs of the different coefficients *so long* as the change is periodic modulo p. In particular, the sign of $c(\Delta_1)/c(\Delta_2)$ is well defined, provided $\Delta_1 \equiv \Delta_2 \pmod{p}$.

In harmony with this observation, notice that if $\Delta_1 \equiv \Delta_2 \not\equiv 0 \pmod{p}$, there is a natural way to choose $\sqrt{\Delta_2} \in \mathbb{C}_p$ once $\sqrt{\Delta_1} \in \mathbb{C}_p$ has been choosen: namely, make the unique choice such that $\sqrt{\Delta_2}$ is congruent to $\sqrt{\Delta_1}$ modulo p. Thus, the ambiguity in sign can be resolved for Stark-Heegner points attached to like discriminants modulo p.

For instance, the traces $P(\Delta)$ and the values m_Δ for fundamental discriminants $\Delta \equiv 2 \pmod{37}$ with $\Delta \leq 2000$ are shown in Table 3. The coefficients of g_+, which are displayed in Table 4, agree with the m_Δ including the sign.

Δ	$h(\Delta)$	$P(\Delta)$	m_Δ	Δ	$h(\Delta)$	$P(\Delta)$	m_Δ
76	1	$(0,-1)$	-1	1001	2	$(0,0)$	1
113	1	∞	0	1112	1	∞	0
409	1	∞	0	1149	1	$(1,0)$	2
520	4	∞	0	1297	11	$(0,-1)$	-1
557	1	$(\frac{21}{25},-\frac{56}{125})$	-8	1704	2	$(0,-1)$	-1
668	1	$(-\frac{5}{9},\frac{8}{27})$	7	1741	1	$(0,-1)$	-1
705	2	$(0,-1)$	-1	1852	1	∞	0
853	1	$(1,0)$	2	1889	1	$(-1,-1)$	3

TABLE 3. Traces of Stark-Heegner points in $37a$ for $\Delta \equiv 2 \pmod{37}$

Δ	76	113	409	520	557	668	705	853
$c(\Delta)$	-1	0	0	0	-8	7	-1	2

Δ	1001	1112	1149	1297	1704	1741	1852	1889
$c(\Delta)$	1	0	2	-1	-1	-1	0	3

TABLE 4. Coefficients of g_+ for $\Delta \equiv 2 \pmod{37}$

There is also a special case in which the signs can be completely well defined. When $p \equiv 3 \pmod 4$ we can employ the fact that the quadratic character χ_{-p} of conductor p is *odd*. Once $\sqrt{\Delta_0}$ has been chosen, it determines a canonical choice for $\sqrt{\Delta} \in \mathbb{C}_p$ by requiring

$$\chi_{-p}(\sqrt{\Delta}/\sqrt{\Delta_0}) = 1.$$

On the side of half-integral weight modular forms, the form g_+ is also determined uniquely up to a single constant by requiring it to have character $\chi_{4p} = \chi_{-4}\chi_{-p}$ of conductor $4p$. This weight $3/2$ modular form g_+ can be characterised as being in Shimura correspondence with the quadratic twist $f \otimes \chi_{-p}$, of level p^2. The Shimura correspondence for level p^2 has been worked out explicitly in [PT07a], and examples of its application to central values of real quadratic twists, and in particular to computing the fourier coefficients of g_+, appear in [PT07b].

6.3. **The curve $43a$.** The smallest conductor $p \equiv 3 \pmod 4$ for which there is an elliptic curve with sign -1 in its functional equation is $p = 43$. The quotient of the modular curve $X_0(43)$ by the Atkin-Lehner involution is again an elliptic curve, denoted by $43a$ in the tables of Cremona, and is given by the equation

$$E : y^2 + y = x^3 + x^2.$$

This curve is unique in its $\mathbb{Q}$-isogeny class.

We computed the traces of the Stark-Heegner points for discriminants $\Delta \leq 10000$ with $\chi_\Delta(43) = -1$, using a precision of 20 significant 43-adic digits, which was enough to recognize all of the traces as global points in $E(\mathbb{Q})$ except for $P(7613) = 21P_0$.

We will write $P(\Delta) = m_\Delta P_0$ with $m_\Delta \in \mathbb{Z}$ and $P_0 = (0,0)$ is a fixed generator of $E(\mathbb{Q})$. These data, for discriminants $\Delta \leq 104$, are show in Table 5.

The computation of the modular form g_+ of weight $3/2$ corresponding to $43a$ can be done following either [PT07b] or [MRVT07]. The fourier expansion of g_+ begins

$$g_+ = q^5 - q^8 + q^{12} + q^{20} - q^{28} - q^{29} + O(q^{30}).$$

In this case the sign of its coefficients is canonically defined, after setting $c(g^+, 5) = 1$. We verified that the resulting coefficients agree with the values of m_Δ computed above, for all $\Delta \leq 10000$.

Δ	$h(\Delta)$	$P(\Delta)$	m_Δ	Δ	$h(\Delta)$	$P(\Delta)$	m_Δ
5	1	$(0,0)$	1	69	1	$(0,-1)$	-1
8	1	$(0,-1)$	-1	73	1	$(0,0)$	1
12	1	$(0,0)$	1	76	1	∞	0
28	1	$(0,-1)$	-1	77	1	$(1,1)$	-3
29	1	$(0,-1)$	-1	85	2	$(0,0)$	1
33	1	$(0,-1)$	-1	88	1	$(0,-1)$	-1
37	1	$(-1,-1)$	2	89	1	$(0,0)$	1
61	1	$(0,0)$	1	93	1	$(1,-2)$	3
65	2	$(0,-1)$	-1	104	2	$(0,0)$	1

TABLE 5. Traces of Stark-Heegner points in $43a$

Δ	5	8	12	28	29	33	37	61	65
$c(\Delta)$	1	-1	1	-1	-1	-1	2	1	-1

Δ	69	73	76	77	85	88	89	93	104
$c(\Delta)$	-1	1	0	-3	1	-1	1	3	1

TABLE 6. Coefficients of g_+ corresponding to $43a$

The coefficients of $g_+ = \sum c(\Delta)q^\Delta$ for the fundamental discriminants $\Delta \leq 104$ are shown in Table 6. A few more coefficients (for all $\Delta < 200$) can be found in [MRVT07].

6.4. **Other curves.** In addition to the elliptic curves of $37a$ and $43a$ already discussed, we also did computations for the following 12 curves:

$$
\begin{aligned}
53a &: \ y^2 + xy + y = x^3 - x \\
61a &: \ y^2 + xy = x^3 - 2x + 1 \\
53a &: \ y^2 + xy + y = x^3 - x \\
61a &: \ y^2 + xy = x^3 - 2x + 1 \\
79a &: \ y^2 + xy + y = x^3 + x^2 - 2x \\
83a &: \ y^2 + xy + y = x^3 + x^2 + x \\
89a &: \ y^2 + xy + y = x^3 + x^2 - x \\
101a &: \ y^2 + y = x^3 + x^2 - x - 1 \\
131a &: \ y^2 + y = x^3 - x^2 + x \\
163a &: \ y^2 + y = x^3 - 2x + 1 \\
197a &: \ y^2 + y = x^3 - 5x + 4 \\
229a &: \ y^2 + xy = x^3 - 2x - 1 \\
269a &: \ y^2 + y = x^3 - 2x - 1 \\
277a &: \ y^2 + xy + y = x^3 - 1
\end{aligned}
$$

These are all the isogeny classes of elliptic curves of prime conductor $p < 300$ and rank 1. Note that for p up to 131, these are isomorphic to the quotient of the modular curve $X_0(p)$ by the Atkin-Lehner involution. All of these curves are unique in their $\mathbb{Q}$-isogeny class.

For each of these curves we computed the traces of the Stark-Heegner points for discriminants $\Delta \leq 10000$ with $\chi_\Delta(p) = -1$. All the computations were carried to 20 significant p-adic digits, which was enough to recognize almost all of the traces. In Table 7 we indicate the number of discriminants for each curve, and the number of discriminants for which the precision was insufficient to recognize the point.

For the 5 curves with conductor $p \equiv 3 \pmod 4$, the corresponding modular forms of weight $3/2$ level $4p^2$ and character χ_{4p} (and many more) can be obtained from the data in [To04]. We verified that their coefficients agree with the values of m_Δ, for $\Delta \leq 10000$.

E	$\#\Delta$	x	E	$\#\Delta$	x
$37a$	1483	1	$101a$	1514	0
$43a$	1491	1	$131a$	1515	0
$53a$	1490	3	$163a$	1508	2
$61a$	1504	0	$197a$	1524	1
$79a$	1504	0	$229a$	1525	10
$83a$	1513	10	$269a$	1519	4
$89a$	1509	0	$277a$	1524	106

TABLE 7. Number of discriminants $\Delta \leq 10000$ we used for each curve. The column labeled x indicates the number of such discriminants for which the precision of 20 significant p-adic digits was not enough to recognize the trace of the Stark-Heegner point.

For the other 7 curves with conductor $p \equiv 1 \pmod 4$, we computed the corresponding modular forms of weight $3/2$ for $\Gamma_1(4p^2)$ following the method explained in [MRVT07], and verified that their coefficients agree with the values of m_Δ up to a sign function defined modulo p, for all $\Delta \leq 10000$.

The complete set of data that we computed, comprising the traces of the Stark-Heegner points and the values m_Δ for discriminants $\Delta \leq 10000$ with $\chi_\Delta(p) = -1$, for each of the curves mentioned above, is available online at [DT-SHP].

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