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Instituto de Matemática Pura e Aplicada

PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA
(Doutorado)

# About Transitivity of Surface Endomorphisms Admitting Critical Points

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Rio de Janeiro - RJ
2017

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Tese apresentada para obtenção do título de Doutor em Matemática, área de Sistemas Dinâmicos junto ao Programa de Pós-graduação em Matemática do Instituto de Nacional de Matemática Pura e Aplicada.

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Rio de Janeiro
2017

da Silva, Wagner Ranter Gouveia.
About Transitivity of Surface Endomorphisms Admitting Critical Points / Wagner Ranter Gouveia da Silva. - Rio de Janeiro: [s.n.], 2017.
55 f. : il. ; 30cm.

Orientador: Enrique Ramiro Pujals
Co-Orientador: Rafael Potrie
Tese (doutorado) - Instituto Nacional de Matemática Pura e Aplicada

1. Sistemas Dinâmicos. 2. Topology. 3. Endomorphisms. I. Pujals, Enrique. II. Potrie, Rafael. III. Instituto Nacional de Matemática Pura e Aplicada. IV. Título.

CDU - ???.??

# Acknowledgements

I would especially like to thank my advisors, E. Pujals and R. Potrie for their dedication, commitment and patience. They, besides excellent mathematicians, are incredible as people, generous, caring, respectful and very attentive. I am infinitely grateful by they accept to participate in this project, which for me is a dream. As much as I try to express their contributions in both my mathematical formation and my personal formation, it would be insufficient. Then, as retribution, I will try of the best way possible to replicate everything they taught me.

At the National Institute of Pure and Applied Mathematics (Instituto Nacional de Matemática Pura e Aplicada - IMPA), I appreciate the opportunity to be one of the students its. It is an incredible institute that offers all its students the best conditions to work mathematics, With an organized, clean, structured ambience and with extremely qualified people. In addition, I will express my gratitude to the University of the Republic (Universidad de la Republica-Udelar) in Montevideo, Uruguay, where I have been for four months. In these four months, I have kown many people and wish to thank them by the math discussions with Verônica, Santiago, León, Nacho, Alejo, Rambo, Luis Pedro and the professor, J. Iglesias, A. Portela, A. Passegi and M. Sambarino.

At the IMPA, I had the pleasure of living with excellent mathematicians and formidable people as Cayo, Leandro, Eduardo, Mauricío, Davi, Fernando e Janaína, Gisele, Javier, Martin Andersson, Robertinho, Robertão, Pedrão, Alex Zamudio, Ítalo, Jamerson, Hudson, Alcides, Ermerson, Juan David, Rafael, Edileno, Plínio, Daniel Marroquin, Wállace, Sandoel and other. I thank these people for great math discussions, but mainly for conversations that contributed to my improvement as a person and a citizen. There is another person who would like to thank, C. Lizana was important for a project of this project, for his partner, patience and dedication.

Finally, I wish to thank my family for the support, for believes and sharing this dream. In special, my wife, my partner and my guardian angel, Ana

Carla (Aninha).

The thesis could not have been done without the financial help by the following institutions: CAPES and CNPq.

Wagner Ranter Gouveia da Silva.

# Abstract

We present a study on the transitivity of surface endomorphisms admitting critical points. In this work, we obtain some answers about necessary conditions for the existence of robustly transitive endomorphisms. We show that the only surfaces that admits a robustly transitive endomorphism are the torus and the Klein bottle. Furthermore, we show that a robustly transitive endomorphism exhibits dominated splitting and must be homotopic to a linear endomorphism with at least one eigenvalue with modulus bigger than one.

We also give sufficient conditions over the critical set in order to get transitivity. We show that a non-wandering endomorphism on the torus with topological degree at least two, hyperbolic linear part and for which the critical points are in some sense "generic" is transitive. This is an improvement of a result by Andersson [And16] since it allows critical points and relaxes the volume preserving hypothesis.

Key-words: Transitivity, linear Algebra, topology, dynamical systems, dominated splitting.

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# Chapter 1

# Introduction

In order to understand the $C^1$-maps on surfaces is important to study topological properties. We say that a system is transitive if there exists a point with dense forward orbit. In this work, we study sufficient conditions for continuous maps on the torus to be transitive. Moreover, we study robust transitivity. That is, every $C^1$-close system to a transitive one is also transitive.

The problem of sufficient conditions for a map to be transitive was address by Andersson, in [And16]. He proved that volume preserving non-invertible covering maps of the torus with hyperbolic linear part must be transitive. This also implies robust transitivity in the conservative context. In this work, we improve Andersson's result by allowing critical points and relaxing the volume preserving hypothesis, assuming that the non-wandering set is the whole torus. We call an endomorphism verifying the previous property by non-wandering endomorphism. The critical points that we allow are in some sense "generic"(see definition in Chapter 2). This result is the following:

**Theorem A.** *Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a non-wandering endomorphism with topological degree at least two and generic critical set. If the linear part of $f$ is hyperbolic, then $f$ is transitive.*

The other problem that we address is to find necessary conditions for robust transitivity. This kind of problem is well understood in the context of diffeomorphisms. It was proved in [DPU99] and [BDP03] for compact manifolds of any dimension that robust transitivity implies a weak form of hyperbolicity, so-called dominated splitting. In dimension two, R. Mañé proved in [Mañ82] that robust transitivity implies hyperbolicity and moreover, that the only surface that admits such systems is the torus.

We contribute to this problem obtaining positive results in the context of $C^1$-maps on surface which admits critical points. In a joint work with

C. Lizana, we obtain that dominated splitting (see Definition 3.1.4), a weak form of hyperbolicity, is a necessary condition for robust transitivity of endomorphisms admitting critical points. More precisely, we proved the following result:

**Theorem B.** *If* $f \in \mathrm{End}^1(\mathrm{M})$ *is a robustly transitive endomorphism and the set of its critical points is nonempty, then* $M$ *admits a dominated splitting for* $f$*.*

In order to prove the Theorem B, first we will prove the following result:

**Theorem B'.** *If* $f \in \mathrm{End}^1(\mathrm{M})$ *is a robustly transitive endomorphism and the set of its critical points has nonempty interior, then* $M$ *admits a dominated splitting for* $f$*.*

Theorem B is almost a consequence of Theorem B'. "Almost" is because, we still have to prove that the dominated splitting extends to the limit.

The result is a weaker version of Mañé's result previously mentioned for endomorphisms with critical points. The techniques used to prove it are different. In this proof, we do not use any of the classical results such as the Closing Lemma, or the Connecting Lemma, like it is used for diffeomorphisms. As a consequence of this result, one has the following topological obstruction:

**Theorem C.** *If* $M$ *admits a robustly transitive endomorphism, then* $M$ *is either the torus* $\mathbb{T}^2$ *or the Klein bottle* $\mathbb{K}^2$*.*

We also prove that there is no robustly transitive map with critical points and homotopic to the identity. In a more general way, we show the following result:

**Theorem D.** *If* $f \in \mathrm{End}^1(\mathrm{M})$ *is a robustly transitive endomorphism, then* $f$ *is homotopic to a linear map having at least one eigenvalue with modulus larger than one.*

The proof of this theorem is based on the proof Brin, Burago and Ivanov's result (see [BBI09]) which shows that on three manifolds the action of a partially hyperbolic diffeomorphism on the first homology group is also partially hyperbolic. In both cases, it is necessary to have arcs where the length of their iterates grows exponentially. For diffeomorphisms, this is obtained through the Ergodic Closing Lemma which is unknown for endomorphisms admitting critical points. In our case, we use similar topological arguments as in [PS07].

Finally, we build new examples of robustly transitive map admitting critical points. The first examples of maps of this kind appeared in [BR13] and in [ILP16], both of them on $\mathbb{T}^2$. We construct a new example on $\mathbb{T}^2$ extending the class of examples in [BR13] and [ILP16].

This thesis is organized as follows, in the second chapter we present and prove the Main Theorem 1 that implies Theorem A. In the third chapter, we present the results obtained in a joint work with C. Lizana. We show two technical results, called Main Theorem 2 and Main Theorem 3, which implies Theorems B, C and D. Finally, in the last chapter, we build a new example of robustly transitive endomorphism on $\mathbb{T}^2$, this is also a joint work with C. Lizana.

# Chapter 2

# Transitive endomorphisms with critical points

The interplay between the dynamics on the homology group and properties of dynamical systems have attracted recently a lot of attention. One of the most well known problems in this topic is the Entropy conjecture of Shub (see [Shu74]). In a sense, one tries to obtain some dynamical properties (which are of asymptotic nature) by the a priori knowledge of how a certain map wraps the manifold in itself.

In this chapter we are interested in properties of the action induced by a map on the torus on the homology group of $\mathbb{T}^2$ that allow to promote a mild recurrence property (being non-wandering) to a stronger one (i.e., transitivity). This improves a recent result by Andersson (see [And16]) by allowing the presence of critical points.

Let us fix some notations. Let $\mathbb{T}^2$ be two-dimensional torus and let $\mathcal{M}(2,\mathbb{Z})$ be the set of all square matrices with integer entries. A toral endomorphism or, simply, an endomorphism is a continuous map $f : \mathbb{T}^2 \to \mathbb{T}^2$. It is well known that given two endomorphisms $f, g : \mathbb{T}^2 \to \mathbb{T}^2$ then $f$ and $g$ are homotopic if and only if $f_* = g_* : H_1(\mathbb{T}^2) \to H_1(\mathbb{T}^2)$. From this fact, we have that given a continuous map $f : \mathbb{T}^2 \to \mathbb{T}^2$ there is a unique square matrix $L \in \mathcal{M}(2,\mathbb{Z})$ such that the linear endomorphism induced by $L$, denoted by $L : \mathbb{T}^2 \to \mathbb{T}^2$ as well, is homotopic to $f$. The matrix $L$, we call linear part of $f$. When $L$ is a hyperbolic matrix[1], we call it hyperbolic linear part.

Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be an endomorphism with linear part $L \in \mathcal{M}(2,\mathbb{Z})$. We define the *topological degree of* $f$ by the determinant of $L$.

The following question naturally arises:

**Question 1:** Under which conditions an endomorphism with hyperbolic

[1]The matrix has no eigenvalues of modulus one.

linear part is transitive?

Recently, Andersson (in [And16]) showed that volume preserving non-invertible covering maps of the torus with hyperbolic linear part is transitive. Thus, some natural questions about the critical set and the volume preserving condition given by Andersson can be posed:

**Question 2:** Can the result be extended to allow critical points?

**Question 3:** Can the volume preserving condition be relaxed?

In this direction, we are interested to give some answer about questions 2 and 3. We show that it is possible to obtain an analogous result changing the volume preserving property given by a milder topological property even in the case where there are critical points. Notice that one can create sinks for maps of $\mathbb{T}^2$ in any homotopy class, so at least some sort of a priori recurrence is necessary to obtain such result.

In order to state the main result of this work, let us introduce some notations before.

A point $p \in \mathbb{T}^2$ is a *non-wandering point for* $f$ if for every neighborhood $B_p$ of $p$ in $\mathbb{T}^2$ there exists an integer $n \geq 1$ such that $f^n(B_p) \cap B_p$ is nonempty. The set $\Omega(f)$ of all non-wandering points is called non-wandering set. Clearly $\Omega(f)$ is closed and $f$-forward invariant. We call an endomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ by non-wandering endomorphism if $\Omega(f) = \mathbb{T}^2$. A point $p$ belonging to $\mathbb{T}^2$ is said to be a *critical point* for $f$ if for every neighborhood $B_p$ of $p$ in $\mathbb{T}^2$, we have that $f : B_x \to f(B_x)$ is not a homeomorphism. We will denote by $S_f$ the set of all the critical points. Clearly $S_f$ is a closed set in $\mathbb{T}^2$. A critical point $p$ is called *generic critical point* if for any neighborhood $B$ of $p$ in $\mathbb{T}^2$, $f(B) \backslash \{f(p)\}$ is a connected set. When all critical points are generics, $S_f$ we will be called generic critical set. It is easy to see that the fold and cusp critical points are *generic critical points*, this justifies the name since by H. Whitney (see [Whi55]) the maps whose critical points are folds and cusps are generic in the $C^\infty$-topology.

In this chapter, we will prove the following result:

**Theorem A.** *Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a non-wandering endomorphism with topological degree at least two and generic critical set. If the linear part of $f$ is hyperbolic, then $f$ is transitive.*

It is not known whether the hypothesis of generic critical set is necessary. It is utilized as a technical hypothesis.

The Theorem-A can be rephrased as follows:

**Main Theorem 1.** *Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a non-wandering endomorphism with topological degree at least two and generic critical set. If $f$ is not transitive, then its linear part has a real eigenvalue of modulus one.*

Before starting of the proof, we give some immediate consequences of the main theorem:

**Corollary 2.1.** *Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a volume preserving endomorphism with topological degree at least two and generic critical set. If $f$ is not transitive, then its linear part has a real eigenvalue of modulus one.*

The proof follows from of the fact that volume preserving implies that the non-wandering set is the whole torus. Furthermore, in the case that the critical set is empty. That is, when the endomorphism is a covering maps. We also have the following consequence:

**Corollary 2.2.** *Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a non-wandering endomorphism with topological degree at least two and without critical points (i.e., $S_f = \emptyset$). If $f$ is not transitive, then its linear part has a real eigenvalue of modulus one.*

The chapter is organized as follows. In section 2.1, we give a sketch of the proof of the main theorem. In sections 2.2 and 2.3, we prove some results that will be used in the proof of the main theorem which is presented in section 2.4.

## 2.1 Sketch of the proof of the main theorem

We prove in the section 2.2 that if a non-wandering endomorphism is not transitive, then we can divide the torus in two complementary open sets which are $f$-invariant. After, in the section 2.3, we use the generic critical points to prove that those open sets are essential (see Definition 2.3.1) and their fundamental groups have just one generator. Then, in section 2.4, we prove that the action of $f$ on the fundamental group of the torus has integer eigenvalues and that at least one has modulus one.

## 2.2 Existence of invariant sets

An open subset $U \subset \mathbb{T}^2$ is called *regular* if $U = \operatorname{int}(\overline{\mathrm{U}})$ where $\overline{U}$ is the closure of $U$ in $\mathbb{T}^2$ that sometimes we will also be denoted like $\mathbf{cl}(U)$.

Given a subset $A \subset \mathbb{T}^2$ we write $A^{\perp} := \mathbb{T}^2 \backslash \overline{A}$. Note that for any open set $U \subseteq \mathbb{T}^2$, we have $U^{\perp} = \operatorname{int}(\overline{\mathrm{U}^{\perp}})$, i. e., $U^{\perp}$ is regular.

We say that a subset $A \subseteq \mathbb{T}^2$ is *f-backward invariant* if $f^{-1}(A) \subseteq A$ and *f-forward invariant* if $f(A) \subseteq A$. We say that $A \subseteq \mathbb{T}^2$ is *f-invariant* when it is $f$-backward and $f$-forward invariant set.

An endomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ is transitive if for every open set $U$ in $\mathbb{T}^2$ we have that $\cup_{n\geq 0} f^{-n}(U)$ is dense in $\mathbb{T}^2$. This definition is equivalent to previous definition given in the Introduction.

The lemma below gives a topological obstruction for a non-wandering endomorphism to be transitive.

**Lemma 2.3.** *Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a non-wandering endomorphism. Then, the following are equivalent:*

- $(a)$ *$f$ is not transitive;*
- $(b)$ *there exist $U, V \subseteq \mathbb{T}^2$ disjoint $f$-backward invariant regular open sets. Furthermore, $\overline{U}$ and $\overline{V}$ are $f$-forward invariant.*

*Proof.* $(b) \Rightarrow (a)$: It is clear. Since $f^{-n}(U) \cap V = \emptyset$ for every $n \geq 0$.

$(a) \Rightarrow (b)$: Since $f$ is not transitive, there exist $U_0'$ and $V_0'$ open sets such that

$$f^{-n}(U_0') \cap V_0' = \emptyset \quad \text{for every} \quad n \geq 0.$$

*Claim* 1: $U' = \cup_{n\geq 0} f^{-n}(U_0')$ and $V' = \cup_{n\geq 0} f^{-n}(V_0')$ are disjoint $f$-backward invariant open sets.

Indeed, it is clear that $U'$ and $V'$ are $f$-backward invariant open sets. Then, we must show only that $U'$ and $V'$ are disjoint sets. For this, suppose by contradiction that $U' \cap V' \neq \emptyset$. That is, suppose that there exist $n, m \geq 0$ such that

$$f^{-n}(U_0') \cap f^{-m}(V_0') \neq \emptyset.$$

Let $x \in f^{-n}(U_0') \cap f^{-m}(V_0')$. Then $f^n(x) \in U_0'$ and $f^m(x) \in V_0'$.

Then, we have the following possibilities:

- $n \geq m$ : $\quad f^{n-m}(f^m(x)) \in U_0' \Rightarrow f^{-n+m}(U_0') \cap V_0' \neq \emptyset.$

- $n < m$ : By continuity of $f$, we can take a neighborhood $B \subseteq U_0'$ of $f^n(x)$ such that $f^{m-n}(B) \subseteq V_0'$. Since $\Omega(f) = \mathbb{T}^2$, we can take $B$ and $k \geq m - n$ such that $f^k(B) \cap B \neq \emptyset$. Hence, $f^{-(k-m+n)}(U_0') \cap V_0' \neq \emptyset$.

In both cases, we have a contradiction.

The following statement will be used to choose the sets $U$ and $V$.

*Claim* 2: $f^{-1}(U')$ is dense in $U'$. The same holds for $V'$.

Indeed, given any open subset $B$ of $\mathbb{T}^2$ contained in $U'$, since $f$ is a non-wandering endomorphism there exists $n \geq 1$ such that $f^n(B) \cap B \neq \emptyset$. Then, $f^{-n}(B) \cap B \neq \emptyset$, in particular, $f^{-n}(U') \cap B \neq \emptyset$. Therefore $f^{-1}(U') \cap B \neq \emptyset$ , since $f^{-m}(U') \subseteq f^{-1}(U')$ for all $m \geq 1$. In particular, $\overline{U'} = \overline{f^{-1}(U')}$. This proves the claim 2.

Finally, we define

$$U = \operatorname{int}(\overline{U'}) \quad \text{and} \quad V = \operatorname{int}(\overline{V'}). \tag{2.2.1}$$

*Claim* 3: $U$ and $V$ satisfy:

$(i)$ $U$ and $V$ are regular;

$(ii)$ $f^{-1}(U) \subseteq U$ and $f^{-1}(\overline{U}) \supseteq \overline{U}$, the same holds for $V$.

Item $(i)$ follows from the fact that $\overline{U} = \overline{U'}$.

To prove item $(ii)$, it is sufficient to show that

$$\operatorname{int}(f^{-1}(\overline{U'})) = U.$$

Because $f^{-1}(U) \subseteq \operatorname{int}(f^{-1}(\overline{U'}))$, since $f^{-1}(U) = f^{-1}(\operatorname{int}(\overline{U'})) \subseteq f^{-1}(\overline{U'})$. Hence, we have $f^{-1}(U) \subseteq U$ and $\overline{U} = \overline{f^{-1}(U)} \subseteq f^{-1}(\overline{U})$, by Claim 2.

Now, we will prove that

$$\operatorname{int}(f^{-1}(\overline{U'})) = U. \tag{$*$}$$

Note that $U = \operatorname{int}(\overline{U'}) \subseteq \operatorname{int}(f^{-1}(\overline{U'}))$, since $\overline{U'} = \overline{f^{-1}(U')} \subseteq f^{-1}(\overline{U'})$. Hence, we have to show only that

$$\operatorname{int}(f^{-1}(\overline{U'})) \subseteq U. \tag{$**$}$$

To prove this, let $B$ be an open set contained in $f^{-1}(\overline{U'})$. Suppose that $B$ is not contained in $\overline{U'}$. Then, we may take an open subset $B'$ of $\mathbb{T}^2$ contained in $B$ such that $B' \cap \overline{U'} = \emptyset$. Since $\Omega(f) = \mathbb{T}^2$ and $f^n(B') \subseteq f^n(\overline{U'}) \subseteq \overline{U'}$ for every $n \geq 1$, we have a contradiction because $f^n(B') \cap B' \neq \emptyset$ for $n \geq 1$. Therefore, $B$ is contained in $\overline{U'}$. Thus, we conclude $(**)$, and so, $(*)$. This proves the Claim 3. $\square$

Henceforth, we assume that $f$ is a non-wandering endomorphism with topological degree at least two and $U, V$ are the sets given by proof of the item $(b)$ of the lemma above.

**Remark 2.4.** *Note that as $f^{-1}(U) \subseteq U$ and $f^{-1}(\overline{U}) \supset \overline{U}$, one gets $f(\overline{U}) = \overline{U}$ and, consequently, $f(\partial\overline{U}) = \partial\overline{U}$. Moreover, since* $\text{int}(\overline{U}) = U$, $\partial U = \partial\overline{U}$, *one has*
$\partial U_i \subseteq \partial U$ *for every* $U_i$ *connected component of* $U$. *Thus, given* $U_i$ *a connected component of* $U$, *we have* $f(\partial U_i) \subseteq \partial U$.

The following proposition shows that the points belonging to $U$ whose images are in the boundary of $U$ are critical points.

**Proposition 2.5.** *Let $p \in U$. If $f(p) \in \partial U$ then $p \in S_f$.*

*Proof.* Suppose that there exist a neighborhood $B$ of $p$ contained in $U$ such that $f : B \to f(B)$ is a homeomorphism and $f(B)$ is an open set contained in $\overline{U}$. In particular, $f(B) \subseteq \text{int}(\overline{U}) = U$. □

The following lemma shows that the image of a component of $U$ which intersect two other components of $U$ intersects the boundary of $U$ in a unique point.

**Lemma 2.6.** *Given $U_0$, $U_1$ and $U_2$ connected components of $U$ such that $U_1$ and $U_2$ are disjoint and let $U_{01}$ and $U_{02}$ be connected components of $f^{-1}(U_1)$, $f^{-1}(U_2)$ contained in $U_0$, respectively. If $C := \partial U_{01} \cap \partial U_{02}$ is a non-empty set contained in $U_0$, then $f(C)$ is a point.*

*Proof.* Consider $C' := f(C)$, without loss of generality, suppose that $C$ is a nontrivial connected set. Then, as $f(\partial U_i) \subseteq \partial U$, we have $C' \subseteq \partial U_1 \cap \partial U_2$ is a connected set.

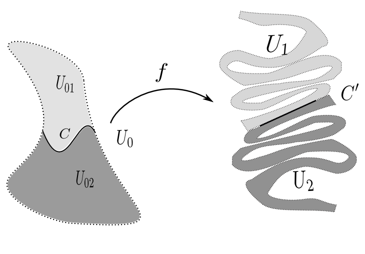


Figure 2.1: Components $U_{01}$ and $U_{02}$ in $U_0$.

Given $y \in C'$, denote by $B_\epsilon(y)$ a ball in $\mathbb{T}^2$ centered in $y$ and radius $\epsilon$.

*Claim* 1: For every $\epsilon \ll 1$, we have that $B_\epsilon(y) \cap \overline{U}_1$ or $B_\epsilon(y) \cap \overline{U}_2$ has infinitely many connected components.

Indeed, suppose that for every $\epsilon > 0$, $B_\epsilon(y) \cap \overline{U}_1$ and $B_\epsilon(y) \cap \overline{U}_2$ has finitely many connected components. Denote by $W^+$ the connected component of $B_\epsilon(y) \cap \overline{U}_1$ and by $W^-$ the connected component of $B_\epsilon(y) \cap \overline{U}_2$ which intersect $C'$. Note that, up to subsets of $C'$, we may suppose that $C' \subseteq B_\epsilon(y)$ and that $C' = W^+ \cap W^-$.

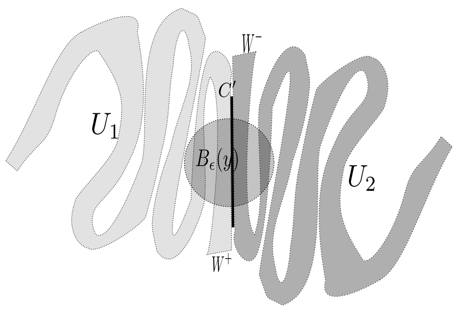


Figure 2.2: connected components.

Hence, we can choose $\epsilon_0 > 0$ such that $W^+ \cup W^-$ contain an open set and $B_\epsilon(y)$ is contained in $W^+ \cup W^-$ for every $0 < \epsilon < \epsilon_0$. In particular, $B_\epsilon(y)$ is contained in $\overline{U}_1 \cup \overline{U}_2$. Contradicting the fact that $U = \mathrm{int}(\overline{U})$ and $U_1, U_2$ are connected components of $U$. This proves of claim 1.

To finish the proof of the lemma, we may suppose, without loss generality, that
$B_{\epsilon_0}(y) \cap U_1$ has infinitely many connected components. Then, we know, by continuity of $f$, that for $0 < \epsilon < \frac{\epsilon_0}{2}$ there is $\delta > 0$ such that

$$d(x, y) < \delta \Rightarrow d(f(x), f(y)) < \epsilon, \forall x, y \in \mathbb{T}^2.$$

Now, we consider $x \in C$ such that $y = f(x)$ and a curve $\gamma$ in $B_\delta(x)$ that intersect $C$ at $x$ and $\gamma(0) \in U_{01}, \gamma(1) \in U_{02}$. Then $f(\gamma)$ is a curve such that $f(\gamma) \cap B_{\epsilon_0}(y)$ has infinitely many components. In particular, there exist $t, s \in [0, 1]$ such that

$$d(f(\gamma(t)), f(\gamma(s))) \geq \epsilon_0 > \epsilon,$$

which is a contradiction, because $f$ is uniformly continuous, the desired result follows.

□

The lemma below is important because it shows the existence of critical points that are not generic for $f$.

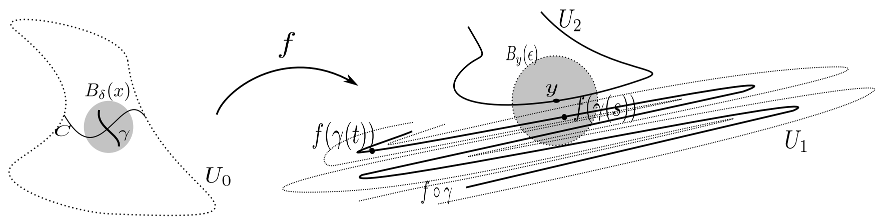


Figure 2.3: Connected components.

**Corollary 2.7.** *Let $U_{01}$ and $U_{02}$ be as in Lemma* 2.6*. If $p$ belongs to $C = \partial U_{01} \cap \partial U_{02}$ then $p$ is not a generic critical point.*

*Proof.* By item $(b)$ of the Lemma 2.3 and by Remark 2.4, $U$ and $V$ are disjoint $f$-backward invariant open sets satisfying:

- $\mathbb{T}^2 = \overline{U} \cup V$;
- $f(\partial U) = \partial U$ and $f(\partial V) = \partial V$.

Then, $f^{-1}(f(p))$ has empty interior. Otherwise, $f(\mathrm{int}(f^{-1}(f(p))) = f(p) \in \partial U$ that is $f$-forward invariant, contradicting the fact that $f$ is a non-wandering endomorphism. Now, we can choose a neighborhood $B$ of $p$ contained in $U_0$ such that $B \backslash \{f^{-1}(f(p))\}$ has at least two connected components which are contained in $U_{01}$ and $U_{02}$. By Lemma 2.6, it follows that the boundary component of $U_{0i}$ contained in $U_0$ has as image a point, where $U_{0i}$ is a component connected of $f^{-1}(U_i)$ contained in $U_0$. Then, as $\overline{U}_0 = \{\overline{U}_{0i} : U_{0i} \subset U_0\}$, we have that

$$f(B)\backslash\{f(p)\} = f(B\backslash\{f^{-1}(f(p))\}) \subset \{f(B \cap \overline{U}_{0i}) : U_{0i} \subset U_0\}.$$

In particular, $f(B \cap U_{01}) \subseteq U_1$ and $f(B \cap U_{02}) \subseteq U_2$.

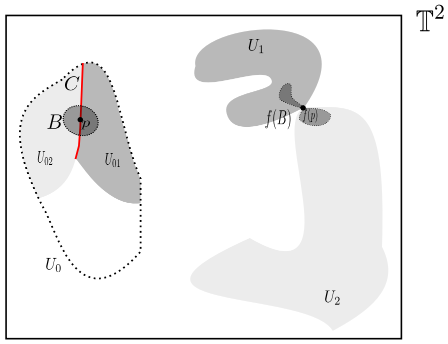


Figure 2.4: $p$ is not generic critical point.

Therefore, one has that $p$ is not a generic critical point.

$\square$

In the following lemma we will show that $f$ satisfies: for each $U_i$ connected component of $U$ there exists a unique connected component $U_j$ of $U$ such that $\overline{U}_j = f(\overline{U}_i)$. Hence, we will say that $f$ *preserves the connected components of* $U$.

Now, we suppose, in addition to the hypothesis of $f$ be non-wandering endomorphism of degree at least two, that the critical points of $f$ are generics.

**Lemma 2.8.** *$f$ preserves the connected components of $U$. Moreover, every connected component $U_i$ of $U$ is periodic (i.e., $\exists n_i \geq 1$ such that $f^{n_i}(\overline{U}_i) = \overline{U}_i$ and $f^{-n_i}(U_i) \subseteq U_i$).*

*Proof.* Suppose that $f(U_i)$ intersect at least two connected components of $U$. Then, by Corollary 2.7, it follows that there exists a non-generic critical point, contradicting that $S_f$ is a generic critical set. Thus, we have that for each connected component $U_i$ of $U$, $f(U_i)$ must intersect a unique connected component $U_{j_i}$ of $U$. In particular, since $f(\partial U_i) \subseteq \partial U$, one has $f(\overline{U}_i) \subseteq \overline{U}_{j_i}$. More precisely, one has that for each connected component $U_i$ of $U$ there exists a unique $U_{j_i}$ such that $f(\overline{U_i}) \subseteq \overline{U}_{j_i}$.

We want to prove that every connected component $U_i$ of $U$ is periodic but before that, we prove that for each $U_i$ there exists a unique $U_j$ such that $f^{-1}(U_i) \subseteq U_j$.

Indeed, suppose that $f^{-1}(U_i)$ intersects at least two connected components $U_j$ and $U_k$ of $U$. Then, by we saw above, we have that $f(\overline{U}_j) \subseteq \overline{U}_i$ and $f(\overline{U}_k) \subseteq \overline{U}_i$. Since $\Omega(f) = \mathbb{T}^2$, there exist $n_i, n_k \geq 1$ such that $f^{n_j}(\overline{U}_j) \subseteq \overline{U}_j$ and $f^{n_j}(\overline{U}_j) \subseteq \overline{U}_j$ that imply $f^{n_j-1}(U_i) \subseteq U_j$ and $f^{n_k-1}(U_i) \subseteq U_k$. Hence, one has $n_j = n_k$ and $U_j = U_k$.

Therefore, for each connected component $U_i$ of $U$ there exist unique $U_{j_i}$ and $U_{k_i}$ such that $f^{-1}(U_i) \subseteq U_{j_i}$ and $f(\overline{U}_i) \subseteq \overline{U}_{k_i}$ implying that $f$ preserves the connected components of $U$, $f^{n_i}(\overline{U}_i) = \overline{U}_i$, and $f^{-n_i}(U_i) \subseteq U_i$. $\square$

**Corollary 2.9.** *There is a finite number of connected components of $U$.*

*Proof.* By definition of $U$ (see equation 2.2.1), we can take a connected component $U_0$ of $U$ such that $\overline{U} = \cup_{n\geq 0} f^{-n}(U_0)$. Hence and by Lemma 2.8, for each connected component $U_j$ of $U$ there exists $n_j \geq 1$ and $n_0 \geq 1$ such that $f^{n_j}(\overline{U}_j) = \overline{U}_0$ and $f^{n_0}(\overline{U}_0) = \overline{U}_0$. Therefore, $U$ has finitely many connected components.

$\square$

## 2.3 Essential sets

Now, our goal is to show that $f$ for a non-wandering endomorphism with topological degree at least two and generic critical set that is not itself transitive, every connected component of $U$ has fundamental group with just one generator in the fundamental group of the torus. Before to formalize this idea, let us fix some notations. Let $L$ be the linear part of $f$ which is an invertible matrix in $\mathcal{M}(2,\mathbb{Z})$ and has determinant of modulus at least two. Let $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ be the universal covering of the torus and let $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ be a lift of $f$. It is known that $\tilde{f}(\tilde{x} + v) = L(v) + \tilde{f}(\tilde{x})$ for every $\tilde{x} \in \mathbb{R}^2$ and $v \in \mathbb{Z}$.

**Definition 2.3.1.** *We say that a connected open set $A$ in $\mathbb{T}^2$ is essential if for every connected component $\widetilde{A}$ of $\pi^{-1}(A)$ in $\mathbb{R}^2$,*

$$\pi := \pi|_{\widetilde{A}} : \widetilde{A} \to A$$

*is not a homeomorphism. Otherwise, we say that $A$ is inessential.*

The following proposition shows properties of the essential sets.

**Proposition 2.10.** *Let $W \subseteq \mathbb{T}^2$ be a connected open set. Then the following are equivalent:*

- $(i)$ *$W$ is essential;*
- $(ii)$ *$W$ contains a loop homotopically non-trivial in $\mathbb{T}^2$;*
- $(iii)$ *there is a non-trivial deck transformation $T_w : \mathbb{R}^2 \to \mathbb{R}^2$ such that every connected component of $\pi^{-1}(W)$ is $T_w$-invariant.*

*Moreover, if $W$ is path connected in $\mathbb{T}^2$, then $i_* : \pi_1(W, x) \to \pi_1(\mathbb{T}^2, x)$ is a non-trivial map, where $x \in W$ and $i : W \hookrightarrow \mathbb{T}^2$ is the inclusion.*

Heuristically, an essential set is a set that every connected component of its lift has infinite volume.

The following lemma is fundamental in the proof of Main Theorem. That lemma is interesting, because it shows that every closure of a connected component of $U$ contains a closed curve homotopically non-trivial in $\mathbb{T}^2$.

**Lemma 2.11.** *Let $U_j$ be any connected component of $U$. Then $\overline{U}_j$ contains a closed curve homotopically non-trivial in $\mathbb{T}^2$.*

*Proof.* By Corollary 2.9, we can suppose $\overline{U}_j = f^j(\overline{U}_0)$ and $U_{n_0} = U_0$. If $U_0$ is essential there is nothing to prove. Now, suppose that $U_0$ is inessential. Let $\widetilde{U}_0 \subset \mathbb{R}^2$ be a connected component of $\pi^{-1}(U_0)$, then, $\pi : \widetilde{U}_0 \to U_0$ is injective. Consider $w \in \mathbb{Z}^2 \backslash L(\mathbb{Z}^2)$, such $w$ exists because $|\det(L)| \geq 2$. We denote by $W'$ the interior of the set $\pi(\tilde{f}^{-1}(w + \tilde{f}(\widetilde{U}_0)))$ that is not empty, because $f(U_0)$ has interior non-empty. Then

$$f(W') = f \circ \pi(\tilde{f}^{-1}(w + \tilde{f}(\widetilde{U}_0))) = \pi(w + \tilde{f}(\widetilde{U}_0)) = f(U_0).$$

But as $W'$ is a open set and $f(W') \subset \overline{U}_1$, and so $W' \subset U$. Then,

$$f^n(W') \cap W' \neq \emptyset \Longleftrightarrow n = kn_0, \quad \text{for some} \quad k \geq 1.$$

In particular, $W'$ must intersect to $U_0$. Hence $W'$ is contained in $U_0$.

Since $W'$ is contained in $U_0$, we have that $\tilde{f}^{-1}(w + \tilde{f}(\widetilde{U}_0))$ is contained in $\widetilde{U}_0$. Thus, $\tilde{f}(\widetilde{U}_0)$ contains $w + \tilde{f}(\widetilde{U}_0)$ and $\tilde{f}(\widetilde{U}_0)$. Hence, there exist $\tilde{x}$ and $\tilde{y}$ in $\widetilde{U}_0$ such that $\tilde{f}(\tilde{y}) = w + \tilde{f}(\tilde{x})$, and so taking a curve $\tilde{\gamma}$ in $\widetilde{U}_0$ joining $\tilde{x}$ to $\tilde{y}$, one has $\tilde{f}(\tilde{\gamma})$ is a curve joining $\tilde{f}(\tilde{x})$ to $\tilde{f}(\tilde{y})$. In particular, $\gamma := \pi \circ \tilde{\gamma}$ is a curve such that $\gamma_f := f \circ \gamma$ is a closed curve whose homology class is $w$. Therefore, $f^{j-1} \circ \gamma_f$ is a closed curve in $\overline{U}_j$ whose homology class is $L^{j-1}(w)$. ☐

Next lemma is important because it shows that the closure of the connected components of $U$ and $V$ obtained in Lemma 2.3 are essential sets.

**Lemma 2.12.** *If $U_j$ is a connected component of $U$ such that $f^n(\overline{U}_j) = \overline{U}_j$ for some $n \geq 1$. Then, $U_j$ is essential.*

*Proof.* Suppose, without loss of generality, $j = 0$. Let $\widetilde{U}_0$ be a connected component of $\pi^{-1}(U_0)$ in $\mathbb{R}^2$. Suppose that $U_0$ is an inessential set. Since the degree of $f$ is at least two and $f(\partial U_0) \subseteq \partial U$, one has that $\tilde{f}(\widetilde{U}_0)$ contains at least two connected components of $\pi^{-1}(U_0)$ and $\tilde{f}(\partial\widetilde{U}_0) \subseteq \partial\pi^{-1}(U_0)$. Then, there exist at least two connected components of $\pi^{-1}(U_0)$, suppose, without loss of generality, that $\widetilde{U_0}$ and $\widetilde{U_0} + v$ for some $v \in \mathbb{Z}^2$ are contained in $\tilde{f}(\widetilde{U}_0)$ and that the component components $\widetilde{U}_{00}$ and $\widetilde{U}_{0v}$ of $\tilde{f}^{-1}(\widetilde{U}_0)$ and $\tilde{f}^{-1}(\widetilde{U}_w)$, respectively, contained in $\widetilde{U}_0$ so that $\widetilde{C} = \partial\widetilde{U}_{00} \cap \partial\widetilde{U}_{0w}$ is a nonempty set in $\widetilde{U}_0$.

Then, from the proof of Lemma 2.6, $f(\pi(\widetilde{C}))$ is a point and ,by the proof of the Corollary 2.7, there exists $p \in \pi(\widetilde{C})$ so that $p$ is not a generic critical point.

☐

The lemma below shows what happens when two essential sets are linearly independent.

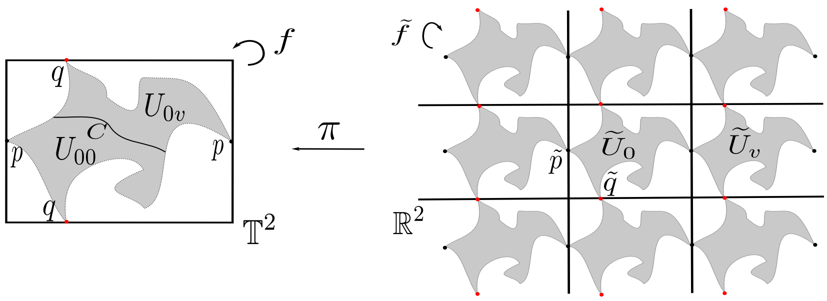


Figure 2.5: The components $\widetilde{U}_0$ and $\widetilde{U}_v$.

**Lemma 2.13.** *Suppose that $\gamma$ and $\sigma$ are loops in $\mathbb{T}^2$ such that $[\gamma]$ and $[\sigma]$ are linearly independent in $\mathbb{Z}^2$. Then $\gamma$ and $\sigma$ intersect.*

*Proof.* See Lemma 4.2 in [And16]. □

The lemma below shows the existence of integer eigenvalues of $L$.

**Lemma 2.14.** *The eigenvalues of $L$ are integers.*

*Proof.* By Lemma 2.12, the connected components $U_j$ and $V_i$ of $U$ and $V$ are essentials. We consider two loops $\gamma$ and $\sigma$ in $U_j$ and $V_i$ such that $[\gamma]$ and $[\sigma]$ are different to zero in $\mathbb{Z}^2$. As $U_j \cap V_i = \emptyset$, it follows, by Lemma 2.13, that $[\gamma]$ and $[\sigma]$ are linearly dependent in $\mathbb{Z}^2$. analogously, as $\overline{U}_{j+1} \cap V_i = \emptyset$ and $f \circ \gamma$ is loop in $\overline{U}_{j+1}$, we have that $[f \circ \gamma] = L[\gamma]$ and $[\sigma]$ are linearly dependent in $\mathbb{Z}^2$, in particular $L[\gamma]$ and $[\gamma]$ are linearly dependent in $\mathbb{Z}^2$. Therefore, there exits $k \in \mathbb{Z}\backslash\{0\}$ such that $L[\gamma] = k[\gamma]$. This proves the lemma. □

The lemma below is fundamental. It shows that all connected components of $U$ and $V$ are essential.

## 2.4 The proof of Main Theorem

Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a non-wandering endomorphism with generic critical set and degree at least two which is not transitive. Then we know from Lemma 2.3 that there exist $U$ and $V$ in $\mathbb{T}^2$ $f$-backward invariant regular open sets such that $\overline{U}$ and $\overline{V}$ are $f$-forward invariant sets. Since all critical points are generics, from Lemma 2.12 and Corollary 2.9 follow that all connected component of $U$ and $V$ are essential and that $\overline{U}_0$ is periodic. Let $\overline{U}_0, f(\overline{U}_0), \cdots, f^{n-1}(\overline{U}_0)$ be all connected components of $\overline{U}$ with $\overline{U}_0 = f^n(\overline{U}_0)$. Then, consider two connected components $\widetilde{U}_0$ and $\widetilde{V}_0$ of

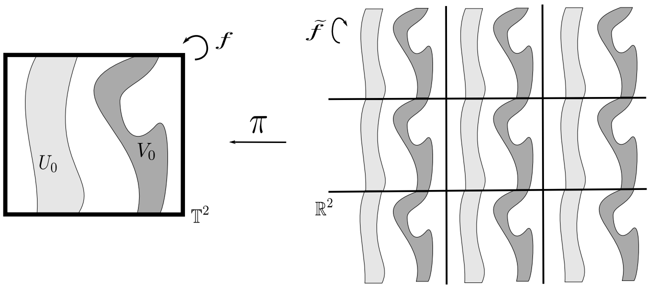


Figure 2.6: The sets $U_0$ and $V_0$.

$\pi^{-1}(U_0)$ and $\pi^{-1}(V_0)$, respectively, and choose $\tilde{f}:\mathbb{R}^2\to\mathbb{R}^2$ a lift of $f$ such that $\tilde{f}^n(\widetilde{U}_0)\subseteq \mathbf{cl}(\widetilde{U}_0)$.

Let us now prove that $L$ has a real eigenvalue of modulus one. First, note that as $\overline{U}_0$ and $V_0$ are disjoints, Lemma 2.14 implies that $L$ has integer eigenvalues $l$ and $k$. Let $w$ and $u$ be the eigenvectors of $L$ associated to $l$ and $k$, respectively, are in $\mathbb{Z}^2$. Suppose, without loss generality, that $w$ and $u=e_2$. That is, as $\widetilde{U}_0$ is $T_u$-invariant, we have that $\widetilde{U}_0$ is a "vertical" component of $\pi^{-1}(U_0)$.

To finish, suppose that $|k|\geq 2$. Then, consider in $\mathbb{R}^2$ a curve $\widetilde{\gamma}$ which $\widetilde{\gamma}(0)\in\widetilde{U}_0$ and $\widetilde{\gamma}(1)=\widetilde{\gamma}(0)+e_1$. Thus, $\widetilde{f}^n\circ\widetilde{\gamma}$ is a curve with $\widetilde{f}^n\circ\widetilde{\gamma}(0)\in\widetilde{U}_0$ and $\widetilde{f}^n\circ\widetilde{\gamma}(1)=\widetilde{f}^n\circ\widetilde{\gamma}(0)+L^n(e_1)$. However, there exist $a$ and $b$ in $\mathbb{Z}$ with $b$ different from zero such that $e_1=ae_2+bw$. Hence, we have $L(e_1)=ake_2+blw$ and, in particular,

$$\tilde{f}(\widetilde{U}_0+e_1)\subseteq \mathbf{cl}(\widetilde{U}_0+L(e_1)).$$

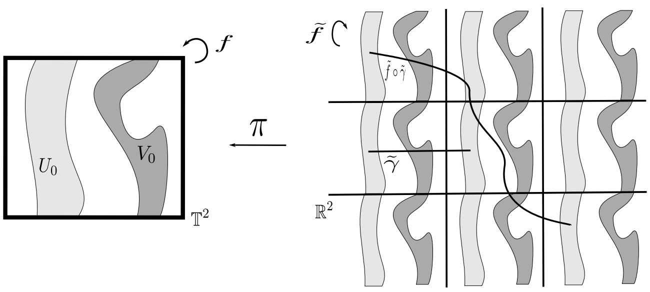


Figure 2.7: The curves $\widetilde{\gamma}$ and $\widetilde{f}\circ\widetilde{\gamma}$.

Then, if $|l|\geq 2$, we have that the first coordinate of $L(e_1)$ has modulus at least two. Hence there is $c\in\mathbb{Z}$ such that $\widetilde{U}_r+ce_1$ is between $\widetilde{U}_0$ and $\widetilde{U}_0+L(e_1)$, and so, we have a contradiction because $U_r's$ are disjoint and $f^n$-

backward invariant. Hence there is not a set $W$ in between $\widetilde{U}_0$ and $\widetilde{U}_0 + e_1$ such that $\tilde{f}^n(W) = \widetilde{U}_0 + ce_1$.

Therefore, $|l| = 1$. And so, $L$ is not an Anosov endomorphism, contradiction. This proves the Main Theorem.

## 2.5 Examples

Consider a map $f$ on $\mathbb{S}^1$ itself of form:

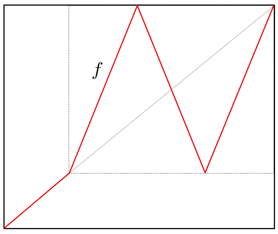


Figure 2.8: The graph of $f$.

such that $f$ is not transitive map, but is volume preserving. Let $g : \mathbb{S}^1 \to \mathbb{S}^1$ be any volume preserving degree 2 map and let $H : \mathbb{T}^2 \to \mathbb{T}^2$ be any volume preserving endomorphism without critical points homotopic to $(x, y) \mapsto (2x, y)$. Then $F : \mathbb{T}^2 \to \mathbb{T}^2$ given by

$$F(x, y) = H(f(x), g(y))$$

is a volume preserving endomorphism with generic critical points and homotopic to $(x, y) \mapsto (2x, 2y)$. Therefore, by the Main Theorem, $F$ is transitive. More general, given endomorphisms $f$ and $g$ on $\mathbb{S}^1$ itself which $f$ has critical points and $g$ is an expanding such that $f \times g : \mathbb{T}^2 \to \mathbb{T}^2$, $f \times g(x, y) = (f(x), g(y))$, is a volume preserving endomorphism, then for every $H : \mathbb{T}^2 \to \mathbb{T}^2$ a volume preserving covering map such that $F = H \circ (f \times g)$ is homotopic to Anosov endomorphism degree at least two, we have $F$ is transitive.

# Chapter 3

# Topological obstruction for robustly transitive endomorphism on surfaces

This chapter is a joint work with C. Lizana. One of the main goal in dynamics is to study robust phenomena. That is, phenomena that are persistent under perturbations. Robust transitivity ($C^1$-robustly transitive systems are systems that are transitive and all systems nearby in the $C^1$- topology are transitive as well, that is, they have a dense orbit) is an important topological phenomenon in dynamical systems.

It was first studied for diffeomorphisms on surfaces by Mañé (see [Mañ82]). He showed that robustly transitive diffeomorphisms are Anosov diffeomorphisms and the surface must be $\mathbb{T}^2$. In higher dimension, Bonatti, Diaz, Pujals and Ures in (see [DPU99] and [BDP03]) showed that robustly transitive diffeomorphisms exhibit a kind of hyperbolicity, called volume hyperbolic. However, in higher dimension there is no topological obstructions for the manifolds as it is obtained by Mañé on surface.

Typically studied in the context of diffeomorphisms, the interest in robust transitivity for endomorphisms ($C^1$-maps on a manifold to itself) is growing. However, in the endomorphisms setting, this kind of behavior is no longer true. For endomorphisms, hyperbolicity is not a necessary condition in order to have robust transitivity.

The first work that address the problem about necessary and sufficient conditions for robustly transitive endomorphisms was [LP13]. In [LP13] is proved that volume expanding is a $C^1$-necessary condition for endomorphisms without critical points (local diffeomorphisms) not exhibiting dominated splitting (in a robust way). However, it is not a sufficient condition.

So far the study of robust transitivity has been done in the local diffeomorphisms setting, meanwhile the case of endomorphisms admitting critical points has received less far attention. In this context, we present necessary conditions for the existence of robustly transitive endomorphisms on surface with critical points. The following result is similar to Mañé's results (see [Mañ82]). However, we can not exhibit hyperbolicity as for diffeomorphisms, but we obtain a weaker property called dominated splitting (see definition 3.1.2):

**Theorem B.** *If $f \in \mathrm{End}^1(\mathrm{M})$ is a robustly transitive endomorphism and the set of its critical points is nonempty, then $M$ admits a dominated splitting for $f$.*

For the proof of this result, we will use the following theorem:

**Theorem B'.** *If $f \in \mathrm{End}^1(\mathrm{M})$ is a robustly transitive endomorphism and the set of its critical points has nonempty interior, then $M$ admits a dominated splitting for $f$.*

For proving Theorem B, observe that there exists a sequence of endomorphisms exhibiting dominated splitting converging to $f$. Then, we show that the angle between the subbundles of the splitting for the sequence are uniformly bounded away from zero. Hence, we prove that $f$ exhibits a dominated splitting.

The proof of the Theorem B' is different from the proof in the diffeomorphisms case. For diffeomorphisms, it is used the existence of sinks or sources as obstruction for transitivity.It is defined a splitting on the vector bundle over periodic points, then it is proved that the periodic points are hyperbolic. Hence, using classical results such as Closing Lemma and Connecting Lemma, we are able to extend the splitting to the whole manifold.

To prove the Theorem B', we will consider a dense subset of the surface where it is possible to define a dominated splitting. Then, we use the fact that the existence of critical points whose kernel has full dimension (see Proposition 3.7) is an obstruction for robust transitivity to prove the domination. In this proof, we are not using classical results such as Closing Lemma, neither Connecting Lemma. Moreover, we avoid the periodic points as candidates to define a dominated splitting. This is very different from the diffeomorphisms case, since the periodic points has an important role in the development of the proof.

A consequence of this result is that there are not robustly transitive endomorphisms on the sphere $\mathbb{S}^2$. Otherwise, we would have a robustly transitive endomorphism on $\mathbb{S}^2$ and, as $\mathbb{S}^2$ is simply-connected and by Mañé's result [Mañ82], it admits critical points. Up to a perturbation, we can suppose the critical set has nonempty interior, and by theorem above follows that $\mathbb{S}^2$ admitting a dominated splitting implies the existence of a continuous vector field without singularity. More general, we prove the following result:

**Theorem C.** *If $M$ admits a robustly transitive endomorphism, then $M$ is either the torus $\mathbb{T}^2$ or the Klein bottle $\mathbb{K}^2$.*

Furthermore, we also obtain another homological necessary condition for robust transitivity. We prove the following result:

**Theorem D.** *If $f \in \mathrm{End}^1(\mathrm{M})$ is a robustly transitive endomorphism. Then $f$ is homotopic to a linear map having at least one eigenvalue with modulus larger than one.*

The proof of this result is based in the proof of Brin, Burago and Ivanov's result (see [BBI09]) which shows that the action of a partially hyperbolic diffeomorphism on the first homology group of a three-manifold is partially hyperbolic.

## 3.1 Preliminaries

### 3.1.1 Linear algebra

Let $V$ and $W$ be inner product spaces. If $L : V \to W$ is a linear map we denote

$$\|L\| = \sup_{\|v\|=1} \|Lv\| \quad \text{and} \quad m(L) = \inf_{\|v\|=1} \|Lv\,|$$

which are called norm and conorm of $L$, respectively.

**Remark 3.1.** *Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map which is not a rotation composed with a homothety. It is known from standard linear algebra that if $v$ and $w$ are unit vector such that $\|L(v)\| = \|L\|$ and $\|L(w)\| = m(L)$, then $v$ and $w$ are orthogonal vectors.*

*Indeed, consider $\xi : \mathbb{S}^1 \to \mathbb{R}$ defined by $\xi(v) = \langle L(v), L(v)\rangle$ or $\langle L^*L(v), (v)\rangle$, where $L^*$ denotes the adjoint operator of $L$. Using that $v$ and $w$ are the maximum and minimum of $\xi$, we have that they are critical points of $\xi$ and*

$$d\xi_v(u) = \langle L^*L(v), u\rangle = 0, \quad \forall\, u \in T_v\mathbb{S}^1$$

*and*

$$d\xi_w(u) = \langle L^*L(w), u\rangle = 0, \quad \forall\, u \in T_w\mathbb{S}^1.$$

*Hence, it follows that $v$ and $w$ are the eigenvectors of $L^*L$. But, $v^\perp$ is an eigenvector of $L^*L$, because*

$$\langle L^*L(v^\perp), v\rangle = \langle v^\perp, L^*L(v)\rangle = \langle v^\perp, \|L\|v\rangle = 0.$$

*Therefore, $v^\perp = w$.*

**Definition 3.1.1.** *Let $V$ and $W$ be two one-dimensional subspaces of an inner product space $Z$ such that $Z = V \oplus W$. Let $L : V \to V^\perp$ be the unique linear map such that*

$$W = \operatorname{graph}(L) = \{v + L(v) : v \in V\}.$$

*Then, we can define the angle between $V$ and $W$ by*

$$\sphericalangle(V, W) = \|L\|.$$

**Definition 3.1.2** (Cones)**.** *Given two one-dimensional subspaces $V$ and $W$ of a bi-dimensional vector space $Z$. We define a cone of length $\eta > 0$ containing the subspace $V$ by:*

$$\mathscr{C}_{V,W}(\eta) = \{v + w \in V \oplus W : \|w\| \leq \eta\|v\|\}$$

*and when $W = V^\perp$, we denote by:*

$$\mathscr{C}_V(\eta) = \{v + w \in V \oplus V^\perp : \|w\| \leq \eta\|v\|\},$$

*or equivalently*

$$\mathscr{C}_V(\eta) = \{w \in Z : \sphericalangle(V, \mathbb{R}\langle w\rangle) \leq \eta\}.$$

*Moreover, we define the dual cone $\mathscr{C}^*_{V,W}(\eta)$ by the closure of $Z \backslash \mathscr{C}_{V,W}(\eta)$. When $W = V^\perp$, we denote $\mathscr{C}^*_{V,V^\perp}(\eta)$ by $\mathscr{C}^*_V(\eta)$.*

### 3.1.2 Dominated splitting

Let us consider $f : M \to M$ a surjective endomorphism from a surface $M$ into itself.

#### Dominated splittings for endomorphisms by cones

Let $\Lambda_f$ be a compact, $f$-invariant subset of $M$ (i.e., $f(\Lambda_f) = \Lambda_f$). Consider a (not necessarily invariant) one-dimensional subbundle $\tilde{E}$ defined over $\Lambda_f$.

**Definition 3.1.3** (Dominated splitting for endomorphisms)**.** *We say that $\Lambda_f$ admits a dominated splitting for $f$ if: there exist $\eta > 0$ and $m \geq 1$ such that the cone field on $\Lambda_f$,*

$$\mathscr{C} : x \in \Lambda_f \mapsto \mathscr{C}_{\tilde{E}}(x, \eta) := \mathscr{C}_{\tilde{E}(x)}(\eta) \subseteq T_x M, \tag{3.1.1}$$

*satisfies for every $x \in \Lambda_f$,*

(a) *Transversality: $\mathscr{C}_{\tilde{E}}(x, \eta) \cap \ker(Df_x) = \{0\}$;*

(b) *Invariance: $Df_x^m\big(\mathscr{C}_{\tilde{E}}(x, \eta)\big) \subseteq \operatorname{int}(\mathscr{C}_{\tilde{E}}(f^m(x), \eta))$.*

Next, we give another definition of dominated splitting for endomorphisms which can be seen as a natural definition of dominated splitting for non-invertible linear cocycle over shift map.

### Inverse limit dominated splitting

Let $M$ be a closed surface and $M^{\mathbb{Z}}$ denote the compact product space $M^{\mathbb{Z}} = \{(x_j)_j : x_j \in M, \forall j \in \mathbb{Z}\}$. A homeomorphism $\sigma : M^{\mathbb{Z}} \to M^{\mathbb{Z}}$, defined by $\sigma((x_j)_j) = (x_{j+1})_j$, is called *shift map.* For $f$, we define the *inverse limit* of $f$ by the set

$$M_f = \{\bar{x} \in M^{\mathbb{Z}} : f(x_j) = x_{j+1}, \forall j \in \mathbb{Z}\}.$$

Then $M_f$ is a closed subset of $M^{\mathbb{Z}}$. Since $M_f$ is $\sigma$-invariant (i.e., $\sigma(M_f) = M_f$), we may define a homeomorphism

$$\sigma_f := \sigma\,|_{M_f} : M_f \to M_f.$$

The restriction $\sigma_f$ is called the shift map determined by $f$.

Let $j \in \mathbb{Z}$ and denote as $\pi_j : M_f \to M$ the projection defined by $(x_j)_j \mapsto x_j$. Then $\pi_j \circ \sigma_f = f \circ \pi_j$ holds. That is, the diagram

$$\begin{tikzcd} M_f \arrow[r, "\sigma_f"] \arrow[d, "\pi_j"'] & M_f \arrow[d, "\pi_j"] \\ M \arrow[r, "f"'] & M \end{tikzcd}$$

commutes, and $\pi_0 \circ \sigma_f^j = f^j \circ \pi_0$ for every $j \geq 1$.

Denote by $TM_f$ the vector bundle $\pi_0^*(TM)$, called the *pullback* of $TM$ by $\pi_0$. That is,

$$TM_f = \{(\bar{x}; v) \in M_f \times TM \mid v \in T_{\pi_0(\bar{x})}M\}.$$

It is the unique maximal subset of $M_f \times TM$ which makes the following diagram commute

$$\begin{tikzcd} TM_f \arrow[r, phantom, "\simeq"] \arrow[d, "\bar{\pi}"'] & TM \arrow[d, "\pi"] \\ M_f \arrow[r, "\pi_0"'] & M \end{tikzcd}$$

where $\bar{\pi} : TM_f \to M_f$ and $\pi : TM \to M$ are the natural projections. The fiber $T_{\bar{x}}M_f$ of $TM_f$ over $\bar{x} \in M_f$ is isomorphic to the vector space $T_{\pi_0(\bar{x})}M$. We denote such isomorphism by $TM_f \simeq TM$.

Let $\Lambda$ be a subset of $M_f$. We define by $T_\Lambda M_f$ the vector bundle $TM_f \mid_\Lambda$. That is,

$$T_\Lambda M_f = \{(\bar{x}, v) \in TM_f : \bar{x} \in \Lambda\} = \bigsqcup_{\bar{x}\in\Lambda} T_{\bar{x}}M_f.$$

If $\Lambda$ is a $\sigma_f$-invariant set, we define the *linear cocycle of $f$ over $\Lambda$* by

$$Df : T_\Lambda M_f \to T_\Lambda M_f,\ Df(\bar{x}, v) = (\sigma_f(\bar{x}), Df_{\pi_0(\bar{x})}v).$$

That is, $Df((x_j)_j, v) = ((x_{j+1})_j, Df_{x_0}v)$.

We define the norm of $Df_{\bar{x}}$ by

$$\|Df_{\bar{x}}\| = \sup\{\|Df_{\pi_0(\bar{x})}(v)\| : \|v\| = 1, v \in T_{\bar{x}}M_f\} = \|Df_{\pi_0(\bar{x})}\|.$$

Given $\Lambda \subseteq M_f$, a *splitting over* $\Lambda$, $T_\Lambda M_f = E \oplus F$, is a linear decomposition such that for every $\bar{x} \in \Lambda$, one has

- $\dim(E(\bar{x})) = \dim(F(\bar{x})) = 1$;
- $T_{\bar{x}}M_f = E(\bar{x}) \oplus F(\bar{x})$.

Note that as the fiber $T_{\bar{x}}M_f$ of $TM_f$ over $\bar{x} \in M_f$ is the vector space $T_{\pi_0(\bar{x})}M$, we have that $E(\bar{x})$ and $F(\bar{x})$ are one-dimensional subspaces in $T_{\pi_0(\bar{x})}M$.

**Definition 3.1.4** (Dominated splitting for linear cocycle)**.** *Let $\Lambda \subseteq M_f$ $\sigma_f$-invariant. We say that $\Lambda$ admits a dominated splitting for the linear cocycle $Df$ if: there exists a splitting over $\Lambda$,*

$$T_\Lambda M_f = E \oplus F,$$

*satisfying:*

($a'$) *Invariance: The subbundles $E$ and $F$ are invariant for the linear cocycle $Df$ (or $Df$-invariant). That is, for every $\bar{x} \in \Lambda$,*

$$Df(E(\bar{x})) \subseteq E(\sigma_f(\bar{x})) \quad \text{and} \quad Df(F(\bar{x})) = F(\sigma_f(\bar{x})).$$

(*b*′) *Uniform angle: The angle between* $E$ *and* $F$ *is uniformly away from zero. That is, there exists* $\alpha > 0$ *such that*

$$\measuredangle(E(\bar{x}), F(\bar{x})) \geq \alpha, \forall \bar{x} \in \Lambda.$$

(*c*′) *Domination: There exists* $m \geq 1$ *such that for every* $\bar{x} \in \Lambda$,

$$\|Df^m|_{E(\bar{x})}\| \leq \frac{1}{2}\|Df^m|_{F(\bar{x})}\|, \tag{3.1.2}$$

*where* $Df^m|_{E(\bar{x})}$ *and* $Df^m|_{F(\bar{x})}$ *denote the linear maps:*

$$Df^m_{\bar{x}}|_{E(\bar{x})}\colon E(\bar{x}) \to E(\sigma_f^m(\bar{x})) \quad \text{and} \quad Df^m_{\bar{x}}|_{F(\bar{x})}\colon F(\bar{x}) \to F(\sigma_f^m(\bar{x})),$$

*and* $\|Df^m|_{E(\bar{x})}\|$ *and* $\|Df^m|_{F(\bar{x})}\|$ *denote their norms, respectively.*

The proposition below gives a natural candidate for the dominated subbundle in the splitting over the critical points.

**Proposition 3.2.** *Let* $T_\Lambda M_f = E \oplus F$ *be a dominated splitting for the cocycle* $Df$. *If* $x_0 = \pi_0(\bar{x}) \in S_f$, *then* $\ker(Df_{x_0}) = E(\bar{x})$. *In particular,* $\dim(\ker(Df_x)) \leq 1$ *for every* $x \in M$.

*Proof.* Let $v \in E(\bar{x})$ and $w \in F(\bar{x})$ be vectors such that $v+w \in \ker(Df_{x_0})\setminus\{0\}$. Then, $Df_{x_0}(v) = Df_{x_0}(-w)$. Since $E$, $F$ are $Df$-invariant, we get that $v$, $w \in \ker(Df_{x_0})$. By domination $w = 0$, and so $\ker(Df_{x_0}) = E(\bar{x})$. □

The following proposition shows the uniqueness of the dominated splitting $E \oplus F$.

**Proposition 3.3** (Uniqueness)**.** *For* $f \in \text{End}^1(M)$, *if* $\Lambda \subseteq M_f$ *is a* $\sigma_f$*-invariant set admitting two dominated splittings* $E \oplus F$ *and* $G \oplus H$. *Then,* $E(\bar{x}) = G(\bar{x})$ *and* $F(\bar{x}) = H(\bar{x})$ *for every* $\bar{x} \in \Lambda$.

*Proof.* First, suppose $x_j = \pi_0(\sigma_f^j(\bar{x})) \notin S_f$ for every $j \in \mathbb{Z}$. Then, given a vector $v \in E(\bar{x})$ one decomposes $v = v_G + w_H$ in an unique way where $v_G \in G(\bar{x})$ and $w_H \in H(\bar{x})$. Similarly, one can decompose $v_G = v'_E + w'_F$ and $w_H = v''_E + w''_F$ with $v'_E, v''_E \in E(\bar{x})$ and $w'_F, w''_F \in F(\bar{x})$. Then, by domination, there exists $n_0 \geq 1$ such that for $n \geq n_0$, we have

$$\begin{aligned}
\|Df^n_{x_0}(v)\| &\geq \|Df^n_{x_0}(w_H)\| - \|Df^n_{x_0}(v_G)\| \geq \|Df^n_{x_0}(v_G)\| \\
&\geq \|Df^n_{x_0}(w'_F)\| - \|Df^n_{x_0}(v'_E)\| \geq \frac{1}{2}\|Df^n_{x_0}(w'_F)\|
\end{aligned}$$

or

$$\begin{aligned}\|Df_{x_0}^n(v)\| &\geq \|Df_{x_0}^n(w_H)\| - \|Df_{x_0}^n(v_G)\| \geq \frac{1}{2}\|Df_{x_0}^n(w_H)\| \\ &\geq \frac{1}{2}\big(\|Df_{x_0}^n(w_F'')\| - \|Df_{x_0}^n(v_E'')\|\big) \geq \frac{1}{4}\|Df_{x_0}^n(w_F'')\|.\end{aligned}$$

Since $v \in E(\bar{x})$ and $F(\sigma_f^j(\bar{x})) \cap \ker(Df_{x_j}) = \{0\}$ for every $j \in \mathbb{Z}$, then both $w_F'$ and $w_F''$ must be zero. Therefore, one deduces that $v_G \in E(\bar{x}) \cap G(\bar{x})$ and $w_H \in E(\bar{x}) \cap H(\bar{x})$. Symmetrically one deduces that if $v \in G(\bar{x})$ is decomposed as $v = v_E + w_F$ with $v_E \in E(\bar{x})$ and $w_F \in F(\bar{x})$ one has $v_E \in G(\bar{x}) \cap E(\bar{x})$ and $w_F \in G(\bar{x}) \cap F(\bar{x})$.

Assume by contradiction that $E(\bar{x})$ is not contained in $G(\bar{x})$. One can choose $v \in E(\bar{x})$ such that $w_H \neq 0$ and is contained in $E(\bar{x}) \cap G(\bar{x})$. Since $\dim(E(\bar{x})) = \dim(G(\bar{x}))$ and as they do not coincide, one gets a non-zero vector $w \in F(\bar{x}) \cap G(\bar{x})$ by the same argument. Using the fact that $H$ dominates $G$ one deduces that $\|Df_{x_0}^n(v)\|$ grows faster than $\|Df_{x_0}^n(w)\|$ contradicting the fact that $F$ dominates $E$.

Note that, in this case, we can consider the inverse cocycle $Df^{-1}$ over the orbit $(\sigma_f^j(\bar{x}))_j$, and use the same argument to get that $F(\bar{x}) = H(\bar{x})$. Therefore, $E(\bar{x}) = G(\bar{x})$ and $F(\bar{x}) = H(\bar{x})$ for every $\bar{x} \in \Lambda$ such that $\pi_0(\sigma_f^j(\bar{x})) \notin S_f$ for every $j \in \mathbb{Z}$.

Now, assume $x_{j_0} = \pi_0(\sigma_f^{j_0}(\bar{x})) \in S_f$ for some $j_0 \in \mathbb{Z}$. Then, we have, by Proposition 3.2, that $E(\sigma_f^j(\bar{x})) = G(\sigma_f^j(\bar{x})) = \ker(Df_{x_j})$ for every $x_j \in S_f$. In particular, for $j_0$. For $x_j \notin S_f$, consider $n_j = \min\{n \geq j : x_n \in S_f\}$, we get by the invariance of $E$ and $G$ that

$$E(\sigma_f^j(\bar{x})) = \mathbb{R}\langle v\rangle = G(\sigma_f^j(\bar{x})),\ Df_{x_j}^{n_j - j}(v) \in \ker(Df_{x_{n_j}}).$$

Therefore, $E(\sigma_f^j(\bar{x})) = G(\sigma_f^j(\bar{x}))$ for every $j \in \mathbb{Z}$.

In order of proving that $F(\sigma_f^j(\bar{x})) = H(\sigma_f^j(\bar{x}))$, we will consider two cases:

*Case* 1: $x_j \notin S_f$ for every $j < j_0$.

We may suppose, without loss of generality, that $j_0 = 0$ and consider the inverse cocycle $Df^{-1}$ over the orbit $(\sigma_f^j(\bar{x}))_{j<0}$. Since $F$ and $H$ are dominated by $E$ and $G$ for $Df^{-1}$, respectively. We can repeat the same arguments in the first part of the proof and get $F(\sigma_f^j(\bar{x})) = H(\sigma_f^j(\bar{x}))$ for $j < 0$. Finally, as $F$ and $H$ are $Df$-invariants, one has: $F(\sigma_f^j(\bar{x})) = H(\sigma_f^j(\bar{x}))$ for every $j \in \mathbb{Z}$.

*Case* 2: $x_{j_n} \in S_f$ for $n \geq 1$ for some sequence $(j_n)_{n\geq 0} \in \mathbb{Z}$, $j_n \searrow -\infty$.

Assume, without loss of generality, that $x_j \notin S_f$ for $j_n < j < j_{n+1}$, $n \geq 0$. Since $E(\sigma_f^{j_n}(\bar{x})) = G(\sigma_f^{j_n}(\bar{x})) = \ker(Df_{x_{j_n}})$ for $n \geq 0$, we have

$$F(\sigma_f^{j_n+1}(\bar{x})) = Df(F(\sigma_f^{j_n}(\bar{x})) = Df(H(\sigma_f^{j_n}(\bar{x})) = H(\sigma_f^{j_n+1}(\bar{x})).$$

Therefore, by invariance of $F$ and $H$, we get

$$F(\sigma_f^{j}(\bar{x})) = Df^{j-j_n}(F(\sigma_f^{j_n}(\bar{x}))) = Df^{j-j_n}(H(\sigma_f^{j_n}(\bar{x}))) = H(\sigma_f^{j}(\bar{x})).$$

This proves the proposition. □

**Remark 3.4.** *It follows from the proof of the uniqueness of the subbundles $E$ and $F$ that the subbundle $E$ only depend of forward iterates. That is,*

$$E(\bar{x}) = E(\bar{y}) \quad \textit{whenever} \quad \pi_0(\bar{x}) = \pi_0(\bar{y}).$$

The next proposition, gives the continuity of the dominated splitting. Moreover, it shows that a dominated splitting can be extended to the closure.

**Proposition 3.5** (Continuity and extension to the closure)**.** *The map*

$$\bar{x} \in \Lambda_f \longmapsto E(\bar{x}) \oplus F(\bar{x})$$

*is continuous. Moreover, it can be extended to the closure $\bar{\Lambda}$ of $\Lambda$ continuously.*

*Proof.* Let $(\bar{x}_n)_{n\geq 1} \subseteq \Lambda_f$ be a sequence such that $\bar{x}_n \to \bar{x}$ when $n \to \infty$. Suppose, unless of a subsequence, that $E(\bar{x}_n)$ and $F(\bar{x}_n)$ converge to subspaces $\tilde{E}(\bar{x})$ and $\tilde{F}(\bar{x})$ (e.g., taking unit vectors of $E(\bar{x}_n)$ and $F(\bar{x}_n)$). By item $(c')$ of Definition 3.1.4, the angle between $\tilde{E}(\bar{x})$ and $\tilde{F}(\bar{x})$ is at least $\alpha$. In particular, $\tilde{E}(\bar{x}) \cap \tilde{F}(\bar{x}) = \{0\}$. Furthermore, by continuity of $Df$, the subspaces $\tilde{E}(\bar{x})$ and $\tilde{F}(\bar{x})$ are $Df$-invariant, and they satisfy the domination property (3.1.2).

By Proposition 3.3, we have that $\tilde{E}(\bar{x})$ and $\tilde{F}(\bar{x})$ do not depend of the subsequence. Then, $\tilde{E}(\bar{x})$ and $\tilde{F}(\bar{x})$ are well defined, and we denote it by

$$E(\bar{x}) = \lim E(\bar{x}_n) \quad \text{and} \quad F(\bar{x}) = \lim F(\bar{x}_n).$$

Therefore, we can extend the subbundles $E$ and $F$ continuously to the closure of $\Lambda_f$. □

**Equivalence of the definitions**

In what follows, we show that the two definitions of dominated splitting, by cones and inverse limit, given above are equivalents.

Indeed, suppose, without loss of generality, that $\bar{\Lambda} \subseteq M_f$ admits a dominated splitting. Let $T_{\bar{\Lambda}}M_f = E \oplus F$ be the dominated splitting over $\bar{\Lambda}$. Then, we define $\Lambda_f$ by the projection of $\bar{\Lambda}$ by $\pi_0$, that is, $\Lambda_f = \pi_0(\bar{\Lambda})$. Since $\bar{\Lambda}$ is a compact subset of $M_f$ and $\sigma_f$-invariant, we have that $\Lambda_f$ is a compact subset of $M$ and $f$-invariant.

Since, by Remark 3.4, $E : \bar{x} \in \bar{\Lambda} \mapsto E(\bar{x})$ only depend of forward iterates, we can define a subbundle on $\Lambda_f$, which we also denote by $E$, as follows

$$E : x \in \Lambda_f \longmapsto E(\bar{x}) \subseteq T_{\bar{x}}M_f \simeq T_xM,$$

where $x = \pi_0(\bar{x})$. Now, we define the dual cone in the natural way by

$$\mathscr{C}^* : x \in \Lambda_f \longmapsto \mathscr{C}^*_E(x, \eta)$$

where $\dfrac{\alpha}{2} < \eta < \alpha$ with $\alpha > 0$ given by the property $(b')$ in Definition 3.1.4.

Next, we show that $\mathscr{C}^*$ satisfies the items $(a)$ and $(b)$ of Definition 3.1.3.

First, by definition of the dual cone $\mathscr{C}^*$, note that:

$$\mathscr{C}^*_E(x, \eta) \cap E(x) = \{0\}.$$

In particular, as $E(x) = \ker(Df_x)$ for $x \in S_f$, $\mathscr{C}^*$ satisfies the item $(a)$. Furthermore, we have, for any $\bar{x} \in \bar{\Lambda}$ such that $\pi_0(\bar{x}) = x$, that

$$F(\bar{x}) \in \mathscr{C}^*_E(x, \eta).$$

Fix any $F(\bar{x})$ and denote it by $F(x)$.

Given $v \in \mathscr{C}^*_E(x, \eta)$, we consider $\beta$ and $\theta$ the angles $\sphericalangle(E(x), F(x))$ and $\sphericalangle(E(x), \mathbb{R}\langle v \rangle)$, respectively. Using elementary trigonometry theory, we get that

$$\frac{\|v_E\|}{\|v_F\|} = |\sin(\beta)||\theta^{-1} - \beta^{-1}| \tag{3.1.3}$$

where $v_E \in E(x)$ and $v_F \in F(x)$ are such that $v = v_E + v_F$. In other words,

$$\frac{\|v_E\|}{\|v_F\|} = |\sin(\beta)| \left| \sphericalangle(E(x)^{\perp}, \mathbb{R}\langle v \rangle) - \sphericalangle(E(x)^{\perp}, F(x)) \right|.$$

Then, for $Df_x^{km}(v) = Df_x^{km}(v_E) + Df_x^{km}(v_F)$, we have

$$\frac{\|Df_x^{km}(v_E)\|}{\|Df_x^{km}(v_F)\|} = |\sin(\beta_{km})| \left| \sphericalangle(E(f^{km}(x))^{\perp}, \mathbb{R}\langle Df_x^{km}(v) \rangle) - \sphericalangle(E(f^{km}(x))^{\perp}, F(f^{km}(x))) \right|.$$

On the other hand,

$$\frac{\|Df_x^{km}(v_E)\|}{\|Df_x^{km}(v_F)\|} = \frac{\|Df_x^{km}\ |_E\ \|}{\|Df_x^{km}\ |_F\ \|}\frac{\|v_E\|}{\|v_F\|} \leq \frac{1}{2^k}\frac{\|v_E\|}{\|v_F\|},$$

so it is sufficient to prove the case $\sphericalangle(E(x)^{\perp}, \mathbb{R}\langle v\rangle) = \eta^{-1}$ and $\beta_{km} = \alpha$. Thus, we get

$$|\eta^{-1} - \sphericalangle(E(x)^{\perp}, F(x))| \geq 2^k \frac{|\sin(\alpha)|}{|\sin(\beta)|}|\sphericalangle(E(f^{km}(x))^{\perp}, \mathbb{R}\langle Df_x^{km}(v)\rangle) - \alpha^{-1}| \tag{3.1.4}$$

Since $\sphericalangle(E(x)^{\perp}, F(x)) \geq 1/\alpha$ and $\beta \geq \alpha$, we have

$$\begin{aligned} (3.1.4) &\implies |\eta^{-1} - \alpha^{-1}| \geq 2^k|\sphericalangle(E(f^{km}(x))^{\perp}, \mathbb{R}\langle Df_x^{km}(v)\rangle) - \alpha^{-1}| \\ &\iff \quad |\sphericalangle(E(f^{km}(x))^{\perp}, \mathbb{R}\langle Df_x^{km}(v)\rangle)| \leq \frac{1}{2^k}(\eta^{-1} + \alpha^{-1}) \leq \frac{1}{2^k}\eta^{-1}. \end{aligned}$$

Therefore, $Df_x^{km}(\mathscr{C}_E^*(x, \eta)) \subseteq \mathrm{int}(\mathscr{C}_E^*(f^{km}(x), \eta))$. Thus, we conclude that Definition 3.1.4 implies Definition 3.1.3.

Reciprocally, let $\Lambda_f$ be a $f$-invariant compact subset of $M$, and we consider a cone field $\mathscr{C} : x \in \Lambda_f \mapsto \mathscr{C}(x, \eta)$ satisfying $(a)$ and $(b)$ in the Definition 3.1.3. By item $(b)$, we get that the dual cone $\mathscr{C}^* : x \in \Lambda_f \mapsto \mathscr{C}^*(x, \eta)$ satisfies:

$$Df_x^{-1}(\mathscr{C}^*(f(x), \eta)) \subseteq \mathrm{int}(\mathscr{C}^*(x, \eta)), \quad \forall\, x \notin S_f,$$

since $\mathscr{C}^*(x, \eta)$ is the closure of $T_xM \backslash \mathscr{C}(x, \eta)$.

Define

$$\Lambda = \{\bar{x} \in M_f : x_j = \pi_0(\sigma_f^j(\bar{x})) \in \Lambda_f, \forall j \in \mathbb{Z}\}.$$

Then, for each $\bar{x} \in \Lambda$, we define $n^+(\bar{x}) = \min\{n \geq 0 : x_n = \pi_0(\sigma_f^n(\bar{x})) \in S_f\}$ and the following sets

$$E(\bar{x}) = \begin{cases} \ker(Df_{x_0}^{n^+}); \\ \bigcap_{j\geq 0} Df_{x_0}^{-j}(\mathscr{C}^*(x_j, \eta)), \text{ otherwise;} \end{cases}$$

and

$$F(\bar{x}) = \bigcap_{j\geq 0} Df_{x_{-j}}^j(\mathscr{C}(x_{-j}, \eta)).$$

It follows by item $(a)$ of Definition 3.1.3 that $E(\bar{x}) \cap F(\bar{x}) = \{0\}$. The proof that $E(\bar{x})$ and $F(\bar{x})$ are subspaces of $T_{\bar{x}}M_f$ and satisfy the properties $(a')$,$(b')$, and $(c')$ is left to the readers. It can be found in [CP, Theorem 2.6].

For simplicity from now on, we will say that $\Lambda \subseteq M_f$ admits a dominated splitting for $f$ instead of saying that $\Lambda$ admits a dominated splitting for the linear cocycle $Df$.

### 3.1.3 Robust transitivity

Remember that $f \in \mathrm{End}^1(M)$ is *topologically transitive (or transitive)* iff:

$$\exists\, x \in M \text{ such that } \{f^n(x) : n \geq 0\} \text{ is dense in } M.$$

The following result relates the transitivity of $f$ on $M$ with the transitivity of the shift map $\sigma_f$ on $M_f$. Since this is a well known result in the literature we left the proof to the readers, further details may be find in [AH94, Theorem 3.5.3].

**Proposition 3.6.** *Let $f : M \to M$ be a surjective endomorphism. Then,*

*$f$ is transitive if and only if the shift map $\sigma_f : M_f \to M_f$ is transitive.*

We say that $f$ is *robustly transitive* if

$$\exists\, \mathcal{U}_f\ C^1 \text{ neighborhood of } f \text{ in } \mathrm{End}^1(M) \text{ such that every } g \in \mathcal{U}_f \text{ is transitive.}$$

The proposition below shows that $\dim(\ker(Df_x^n)) = 1$, for $x \in S_f$ and $n \geq 1$, is a necessary condition for robust transitivity.

**Proposition 3.7.** *Let $f \in \mathrm{End}^1(\mathrm{M})$, $x \in S_f$, and $n \geq 1$ such that $Df_x^n \equiv 0$. Then, given any neighborhood $\mathcal{U}_f$ of $f$ in $\mathrm{End}^1(\mathrm{M})$, there exist $g \in \mathcal{U}_f$ and a neighborhood $B$ of $x$ in $M$ such that $g^n(B) = \{g^n(x)\}$. In particular, $f$ is not robustly transitive.*

We first present a lemma, similar to Franks' Lemma (see [Fra71]), that will be used in the proof of the proposition above.

Fix $f \in \mathrm{End}^1(M)$. Given $p \in M$, we consider $\delta_0 > 0$ and $\eta_0 > 0$ such that $\exp_x : B_{\delta_0} \to B'(x, \delta_0)$ and $\exp_{f(p)} : B_{\eta_0} \to B'(f(p), \eta_0)$ are diffeomorphisms, where $B_r$ is a ball of radio $r$ and centered at the origin of the tangent space. Moreover, assume that $f(B'(x, \delta_0)) \subseteq B'(f(x), \eta_0)$.

**Lemma 3.8.** *Let $L = D(\exp_{f(p)}^{-1} \circ f \circ \exp_p)(0)$. Then, for any neighborhood $\mathcal{U}_f$ of $f$ in $\mathrm{End}^1(M)$, there exist $0 < r < \frac{\delta_0}{2}$ and $g \in \mathcal{U}_f$ such that $g(p) = f(p)$ and*

$$\exp_{f(p)}^{-1} \circ g \circ \exp_p(x, y) = L(x, y), \quad \forall\, (x, y) \in B_r.$$

*Proof.* Let $\varphi : \mathbb{R}^2 \to [0, 1]$ be a $C^\infty$-bump function with $\varphi(x, y) = 1$ if $\|(x, y)\| \leq 1$ and $\varphi(x, y) = 0$ if $\|(x, y)\| \geq 2$. For $r > 0$, let $\varphi_r : \mathbb{R}^2 \to [0, 1]$ be defined by $\varphi_r(x, y) = \varphi(\frac{x}{r}, \frac{y}{r})$. The estimates on $\varphi_r$ and its derivative depend on $r$ in the following fashion:

$$\sup\{\varphi_r(x, y)\} = \sup\{\varphi(x, y)\} = 1 \text{ and } \sup\{\|D\varphi_r(x, y)\|\} = \frac{C_0}{r},$$

where $C_0 = \sup\{\|D\varphi(x,y)\|\}$.

Using the bump function $\varphi_r$, we can define a perturbation $g_r$. First we write $f$ in terms of its linear and nonlinear terms:

$$\exp_{f(p)}^{-1} \circ f \circ \exp_p(x,y) = L(x,y) + \tilde{f}(x,y),$$

where $\tilde{f}(0,0) = (0,0)$ and $D\tilde{f}(0,0) = 0$. Thus, $\|D\tilde{f}(x,y)\|$ and $\frac{\|\tilde{f}(x,y)\|}{\|(x,y)\|}$ both go to zero as $\|(x,y)\|$ goes to zero. We only consider $r > 0$ for which $\mathrm{supp}(\varphi_r) \subseteq B_{2r}$. Let $g_r$ be equal to $f$ outside $B_{\delta_0}$, and

$$\begin{aligned} \exp_{f(p)}^{-1} \circ g_r \circ \exp_p(x,y) &= \varphi_r(x,y)L(x,y) + (1 - \varphi_r(x,y)) \exp_{f(p)}^{-1} \circ f \circ \exp_p(x,y) \\ &= L(x,y) + (1 - \varphi_r(x,y))\tilde{f}(x,y), \end{aligned}$$

for $(x,y) \in B_{\delta_0}$. Note that $\exp_{f(p)}^{-1} \circ g_r \circ \exp_p(x,y) = \exp_{f(p)}^{-1} \circ f \circ \exp_p(x,y)$ for $\|(x,y)\| \geq 2r$. On the other hand for $\|(x,y)\| \leq r$, we have $\varphi_r(x,y) = 1$ and $\exp_{f(p)}^{-1} \circ g_r \circ \exp_p(x,y) = L(x,y)$.

To check that $g_r$ is near $f$ for small $r$, we need to calculate the derivative of $g_r$:

$$\begin{aligned} D(\exp_{f(p)}^{-1} \circ g_r \circ \exp_p)(x,y) &= L + (1 - \varphi_r(x,y))D\tilde{f}(x,y) + \tilde{f}(x,y)D\varphi_r(x,y) \\ &= L + D\tilde{f}(x,y) - \varphi_r(x,y)D\tilde{f}(x,y) \\ &+ \tilde{f}(x,y)D\varphi_r(x,y). \end{aligned}$$

Therefore,

$$\begin{aligned} D(\exp_{f(p)}^{-1} \circ g_r \circ \exp_p)(x,y) &= D(\exp_{f(p)}^{-1} \circ f \circ \exp_p)(x,y) - \varphi_r(x,y)D\tilde{f}(x,y) \\ &+ \tilde{f}(x,y)D\varphi_r(x,y). \end{aligned}$$

For $\|(x,y)\| \geq 2r$, $D(\exp_{f(p)}^{-1} \circ g_r \circ \exp_p)(x,y) = D(\exp_{f(p)}^{-1} \circ f \circ \exp_p)(x,y)$, so we only need to consider $(x,y)$ with $\|(x,y)\| \leq 2r$. For $\|(x,y)\| \leq 2r$, we have

$$\begin{aligned} \|D(\exp_{f(p)}^{-1} \circ g_r \circ \exp_p)(x,y) &- D(\exp_{f(p)}^{-1} \circ f \circ \exp_p)(x,y)\| \\ &\leq |\varphi_r(x,y)|\|D\tilde{f}(x,y)\| + \|\tilde{f}(x,y)\|\|D\varphi_r(x,y)\| \\ &\leq \|D\tilde{f}(x,y)\| + \frac{C_0}{r}\|\tilde{f}(x,y)\| \\ &\leq \|D\tilde{f}(x,y)\| + 2C_0\frac{\|\tilde{f}(x,y)\|}{\|(x,y)\|}. \end{aligned}$$

In this last calculation, we used the estimates given above for $\|(x,y)\| \leq 2r$ and that $\sup\{\|D\varphi_r(x,y)\|\} = \frac{C_0}{r}$. From the estimate, the derivative of

$g_r$ approaches to the derivative of $f$ as $r$ goes to zero. Since $g_r(p) = f(p)$, the Mean Value Theorem proves that the $C^0$-distance from $g_r$ to $f$ goes to zero also. Therefore, given any neighborhood $\mathcal{U}_f$ of $f$ in $\mathrm{End}^1(M)$, we may choose $r > 0$ such that $g_r \in \mathcal{U}_f$. $\square$

Now we are able to prove Proposition 3.7.

*Proof of Proposition 3.7.* Suppose $x_0 = x$ and $x_j = f(x_{j-1})$ for $1 \leq j \leq n$. Assume also that $x_0, x_n \in S_f$ and $x_j \notin S_f$ for $1 \leq j \leq n-1$. Then, by Lemma 3.8, there exist $g \in \mathcal{U}_f$ and a neighborhood $B$ of 0 in $T_{x_0}M$ such that

$$\exp_{x_n}^{-1} \circ g^n \circ \exp_{x_0}(v) = Df^n_{x_0}(v) \equiv 0$$

for $v \in B \subseteq T_{x_0}M$. Therefore, $g^n(B') = \{x_n\} = \{g^n(x_0)\}$, where $B' = \exp_{x_0}(B)$. In particular, $g$ is not transitive. Because, $g^m(x) \in B'$ for infinitely many $m \geq 1$, then there exists $m_0 \geq 1$ such that $g^{m_0}(x)$ is a periodic point. $\square$

The next result is well known, hence we do not present its proof, for further details see for instance [Fra71].

**Theorem 3.9** (Franks' Lemma)**.** *Let $f \in \mathrm{End}^1(M)$. Given a neighborhood $\mathcal{U}_f$ of $f$ in $\mathrm{End}^1(M)$, there exist $\varepsilon > 0$ and a neighborhood $\mathcal{U}'_f$ of $f$ contained in $\mathcal{U}_f$ such that if $g \in \mathcal{U}'_f$ and $\Gamma = \{x_0, ..., x_n\}$ and*

$$L : \bigoplus_{x_j \in \Gamma} T_{x_j}M \to \bigoplus_{x_j \in \Gamma} T_{g(x_j)}M$$

*is a linear map satisfying $\|L - Dg|_{\oplus_{x_j \in \Gamma} T_{x_j}M}\| < \varepsilon$. Then, there exist $\tilde{g} \in \mathcal{U}_f$ and a neighborhood $W$ of $\Gamma$ such that $D\tilde{g}|_{\oplus_{x_j \in \Gamma} T_{x_j}M} = L$ and $\tilde{g}|_{W^c} = g|_{W^c}$.*

## 3.2 Precise statement of the Main Theorem 2

In this section, we present a technical result that is important to prove Theorem B' stated at the Introduction. Before we state the theorem, let us fix some notation.

Let $f \in \mathrm{End}^1(\mathrm{M})$ with $\dim(\ker(Df^n_x)) \leq 1$ for every $x \in M$ and $n \geq 1$. Then, define

$$\Lambda = \left\{ \bar{x} \in M_f \;\middle|\; \begin{array}{l} \bullet\ \forall j \geq 0,\ \pi_0(\sigma_f^j(\bar{x})) \notin \mathrm{Per}(f); \\ \bullet\ \exists (j_n)_n \subseteq \mathbb{Z},\ j_n \to \pm\infty \text{ when } n \to \pm\infty, \\ \quad\ \text{such that } \pi_0(\sigma_f^{j_n}(\bar{x})) \in S_f. \end{array} \right\} \tag{3.2.1}$$

Note that $\Lambda$ is a $\sigma_f$-invariant set.

Given $\bar{x} = (x_j)_j \in \Lambda$. We define $x_n = \pi_0(\sigma_f^n(\bar{x}))$, and the following splitting on $T_\Lambda M_f$:

$$E(\bar{x}) = \ker\left(Df_{x_0}^{n^+(\bar{x})}\right) \text{ and } F(\bar{x}) = \mathrm{Im}\left(Df_{x_{n^-(\bar{x})}}^{|n^-(\bar{x})|}\right),$$

where $n^-(\bar{x})$ and $n^+(\bar{x})$ are defined as $\max\{n < 0 : x_n \in S_f \text{ and } x_{n+1} \notin S_f\}$ and $\min\{n \geq 0 : x_n \in S_f\}$, respectively.

For simplicity, from now on we suppress the explicit dependence on $\bar{x}$ in $n_j^{+(-)}(\bar{x})$.

The theorem below is a technical result and shows that $\Lambda$ as it was defined above admits a dominated splitting.

**Main Theorem 2.** *Suppose $\Lambda \neq \emptyset$ as defined above and there exists a neighborhood $\mathcal{U}_f$ of $f$ in* $\mathrm{End}^1(M)$ *such that for every $g \in \mathcal{U}_f$ holds* $\dim(\ker(Dg_x^n)) \leq 1, \forall\, x \in M, n \geq 1$. *Then, $T_\Lambda M_f = E \oplus F$ is a dominated splitting for $f$.*

## 3.3 Consequences of the Main Theorem 2

The goal of this section is to prove that $M$ is covered by the torus $\mathbb{T}^2$. For this, we first present some preliminary results.

### 3.3.1 Proof of Theorem B'

Let $f \in \mathrm{End}^1(M)$ be robustly transitive with $S_f \neq \emptyset$. Consider $\mathcal{U}_f$ a neighborhood of $f$ in $\mathrm{End}^1(M)$ such that every $g \in \mathcal{U}_f$ is transitive.

Proposition 3.7 implies that

$$\forall g \in \mathcal{U}_f,\ dim(\ker(Dg_x^n)) \leq 1 \text{ for every } x \in M \text{ and } n \geq 1.$$

Furthermore, by Lemma 3.8, we may assume, after considering a perturbation, that $S_f$ has nonempty interior.

Let us show now that generically the forward and backward orbits by the shift map are dense in $M_f$. Concretely,

**Lemma 3.10.** *There exists a residual set $\mathscr{G} \subseteq M_f$ such that for every $\bar{x} \in \mathscr{G}$ the forward and backward orbits of $\bar{x}$ by $\sigma_f$ are dense in $M_f$. That is, the sets*

$$\mathcal{O}^+(\bar{x}, \sigma_f) = \{\sigma_f^j(\bar{x}) : j \geq 0\} \text{ and } \mathcal{O}^-(\bar{x}, \sigma_f) = \{\sigma_f^j(\bar{x}) : j \leq 0\}$$

*are dense in $M_f$.*

*Proof.* Since $f$ is transitive, by Proposition 3.6, $\sigma_f$ is transitive as well. Then, consider a open basis $\mathcal{C} = \{C_n\}_n$ of $M_f$ and define

$$A_n^+ = \{\bar{x} \in M_f : \exists j \geq 0 \text{ such that } \sigma_f^j(\bar{x}) \in C_n\} = \bigcup_{j \geq 0} \sigma_f^{-j}(C_n)$$

and

$$A_n^- = \{\bar{x} \in M_f : \exists j \leq 0 \text{ such that } \sigma_f^j(\bar{x}) \in C_n\} = \bigcup_{j \geq 0} \sigma_f^j(C_n).$$

Since $\sigma_f$ is a transitive homeomorphism, $A_n^+$ and $A_n^-$ are dense open sets of $M_f$. Therefore,

$$\mathcal{R}^+ = \bigcap_n A_n^+ \quad \text{and} \quad \mathcal{R}^- = \bigcap_n A_n^-$$

are residual sets in $M_f$. In particular, $\mathcal{R} = \mathcal{R}^+ \cap \mathcal{R}^-$ is so. □

Now, we are in condition to prove Theorem B'.

*Proof of Theorem B'.* Since $S_f$ has nonempty interior and Lemma 3.10, we have that $\Lambda$ defined on (3.2.1) by

$$\Lambda = \left\{ \bar{x} \in M_f \;\middle|\; \begin{array}{l} \bullet \;\; \forall j \geq 0, \pi_0(\sigma_f^j(\bar{x})) \notin \mathrm{Per}(f); \\ \bullet \;\; \exists (j_n)_n \in \mathbb{Z}, j_n \to \pm\infty \text{ when } n \to \pm\infty, \\ \quad\;\; \text{such that } \;\pi_0(\sigma_f^{j_n}(\bar{x})) \in S_f. \end{array} \right\}$$

is a dense set in $M_f$. Therefore, Main Theorem 2, implies that the splitting $T_\Lambda M_f = E \oplus F$ defined in Section 3.2 is a dominated splitting over $\Lambda$ for $f$. Furthermore, by Proposition 3.5, we can extend it to $M_f$ the closure of $\Lambda$. In other words, $M_f$ admits a dominated splitting. This concludes the proof of Theorem B'. □

### 3.3.2 Proof of Theorem C

Consider $f : M \to M$ a robustly transitive endomorphism.

*Proof of Theorem C.* The proof will be divided in two cases.

*Case* 1: The critical set $S_f$ is empty.

Then, $f$ is a local diffeomorphism. In particular, $f$ is a covering map. Let $n$ be the degree of $f$. It is well known that for any $n$-sheeted covering

$p : \tilde{N} \to N$ of the compact surface $N$ by the compact surface $\tilde{N}$, the Euler characteristic of surfaces are related by the formula $\chi(\tilde{N}) = n\chi(N)$ (see for instance [Shu74]). Then,

$$\chi(M) = n\chi(M) \implies \text{either } \ n = 1 \ \text{ or } \ \chi(M) = 0.$$

If $\chi(M) = 0$, by Classification of Surfaces Theorem, either $M$ is the torus $\mathbb{T}^2$ or the Klein bottle $\mathbb{K}^2$.

If $n = 1$, we get that $f$ is a diffeomorphism and it is well known that there exists robustly transitive diffeomorphisms on the torus $\mathbb{T}^2$ (see [Mañ82]).

*Case* 2: The critical set $S_f$ is nonempty.

Theorem B' implies that $M_f$ admits a dominated splitting, let us say $TM_f = E \oplus F$. By Remark 3.4, we may define a continuous subbundle over $M$ by

$$E : x \in M \mapsto E(\bar{x}) \subseteq T_xM$$

where $x = \pi_0(\bar{x})$. Let $(\tilde{M}, p, p^*(E))$ be the double covering of $E$ over $M$. Hence, since the subbundle $p^*(E)$ of $T\tilde{M}$ is orientable, we can define a vector field $X : \tilde{M} \to T\tilde{M}$ such that $X(x) \neq 0 \in p^*(E)$. Therefore, one gets that $\chi(\tilde{M}) = 0$ and so $\chi(M) = 0$. Thus, $M$ either is the torus $\mathbb{T}^2$ or the Klein bottle $\mathbb{K}^2$. □

## 3.4 Proof of the Main Theorem 2

Consider $f \in \text{End}^1(M)$ and a neighborhood $\mathcal{U}_f$ of $f$ in $\text{End}^1$(M) such that

$$\forall g \in \mathcal{U}_f,\ \dim(\ker(Dg_x^n)) \leq 1, \quad \text{for every } \ x \in M, n \geq 1.$$

Moreover, assume that $\Lambda$ as it was defined in (3.2.1) is nonempty. Let $T_\Lambda M_f = E \oplus F$ the splitting defined in Section 3.2.

The proposition below shows that $T_\Lambda M_f = E \oplus F$ is a splitting over $\Lambda$, and $E$ and $F$ are $Df$-invariant, as defined above. In particular, the splitting satisfies the invariance property, $(a')$, in Definition 3.1.4.

**Lemma 3.11.** *Suppose $\Lambda \neq \emptyset$. Then, the following properties hold:*

*(i) The maps $E, F : \bar{x} \in \Lambda \longmapsto E(\bar{x}), F(\bar{x})$ are well defined;*

*(ii) $T_\Lambda M_f = E \oplus F$ is a splitting over $\Lambda$;*

*(iii) $Df(E(\bar{x})) \subseteq E(\sigma_f(\bar{x}))$ and $Df(F(\bar{x})) = F(\sigma_f(\bar{x})), \forall \bar{x} \in \Lambda$.*

*Proof.* Given $\bar{x} = (x_j)_j \in \Lambda$, then $E(\bar{x})$ and $F(\bar{x})$ are well defined. In fact, if $x_0 \in S_f$, then $n^+ = 0$ and $E(\bar{x}) = \ker(Df_{x_0})$. If $x_0 \notin S_f$, then $x_j \notin S_f$ for $0 \le j \le n^+ - 1$. Thus, $Df^{n^+}_{x_0}$ is an isomorphism and, as $\dim(\ker(Df_x)) \le 1$, there exists $v \in T_{x_0}M$ such that $Df^{n^+}_{x_0}(v) \in \ker(Df_{x_{n^+}})$, showing that $E(\bar{x})$ is well defined. $F(\bar{x})$ is well defined follows from its definition. Proving item $(i)$.

Item $(ii)$ follows from observing that, as $\dim(\ker(Df^n_x)) \le 1$ for every $x \in M$ and $n \ge 1$, we have

$$\dim(E(\bar{x})) = \dim(F(\bar{x})) = 1 \quad \text{and} \quad T_{\bar{x}}M_f = E(\bar{x}) \oplus F(\bar{x}).$$

In order to prove the item $(iii)$, consider $\bar{x} = (x_j)_j$ any point in $\Lambda$ and denote $\bar{y} = \sigma_f(\bar{x}) = (y_j)_j$ where $y_j = x_{j+1}$ for $j \in \mathbb{Z}$. Note that, if $x_0 \in S_f$ then $n^+(\bar{x}) = 0$ and either $n^-(\bar{y}) = n^-(\bar{x}) - 1$ (i.e., $y_0 = \pi_0(\bar{y}) \in S_f$) or $n^-(\bar{y}) = -1$(i.e., $y_0 \notin S_f$). Hence, we have

$$E(\bar{x}) = \ker(Df_{x_0}) \quad \text{and so} \quad Df(E(\bar{x})) = \{0\} \subseteq E(\sigma_f(\bar{x}))$$

and, for $n^-(\bar{y}) = n^-(\bar{x}) - 1$ we get $y_{n^-(\bar{y})} = x_{n^-(\bar{x})}$, hence

$$\begin{aligned} F(\sigma_f(\bar{x})) = F(\bar{y}) &= \mathbb{R}\langle Df^{|n^-(\bar{y})|}_{y_{n^-(\bar{y})}}(w)\rangle \big(\text{where } w \in \ker^{\perp}(Df_{y_{n^-(\bar{y})}})\big) \\ &= \mathbb{R}\langle Df^{|n^-(\bar{x})|+1}_{x_{n^-(\bar{x})}}(w)\rangle \qquad (3.4.1) \\ &= Df_{x_0}(\mathbb{R}\langle Df^{|n^-(\bar{x})|}_{x_{n^-(\bar{x})}}(w)\rangle) = Df(F(\bar{x})), \end{aligned}$$

or, for $n^-(\bar{y}) = -1$ we get $y_0 \notin S_f$ and so

$$F(\sigma_f(\bar{x})) = \mathbb{R}\langle Df_{y_{-1}}(w)\rangle, \text{ where } w \in \ker^{\perp}(Df_{y_{-1}}), \quad \text{and} \quad E(\bar{x}) = \ker(Df_{x_0}).$$

Therefore, as $y_{-1} = x_0$, we have

$$T_{\bar{x}}M_f = \ker(Df_{x_0}) \oplus \ker^{\perp}(Df_{x_0}), \text{ and, hence, } Df(T_{\bar{x}}M_f) = Df(\ker^{\perp}(Df_{x_0})).$$

In particular,

$$Df(F(\bar{x})) = Df(\ker^{\perp}(Df_{x_0})) = F(\sigma_f(\bar{x})).$$

Finally, if $x_0 \notin S_f$ then $n^+(\bar{x}) \ge 1$ and $n^-(\bar{y}) = n^-(\bar{x}) - 1$. Thus, by equation (3.4.1), we have $Df(F(\bar{x})) = F(\sigma_f(\bar{x}))$. Furthermore, by definition, we have that $Df_{x_j}$ is an isomorphism, for $0 \le j \le n^+(\bar{x}) - 1$, and that

$$E(\sigma_f(\bar{x})) = E(\bar{y}) = \mathbb{R}\langle v\rangle \ \big(\text{where } Df^{n^+(\bar{y})}_{y_0}(v) \in \ker(Df_{x_{n^+(\bar{y})}})\big).$$

Using that $Df_{x_0}$ is an isomorphism and $n^+(\bar{y}) = n^+(\bar{x}) - 1$, we have that $y_{n^+(\bar{y})} = x_{n^+(\bar{x})}$ and there exists a unique $v' \in T_{x_0}M$ such that $Df_{x_0}v' = v$. Therefore, since

$$Df_{x_0}^{n^+(\bar{x})}(v') = Df_{y_0}^{n^+(\bar{y})}(v) \in \ker(Df_{x_{n^+(\bar{y})}}) = \ker(Df_{x_{n^+(\bar{x})}}),$$

we get $E(\bar{x}) = \mathbb{R}\langle v' \rangle$ and $E(\sigma_f(\bar{x})) = Df_{x_0}(\mathbb{R}\langle v' \rangle) = Df(E(\bar{x}))$. This complete the proof. $\square$

The next lemma states that the angle between $E$ and $F$ is uniformly bounded away from zero. In particular, it shows property $(b')$ of Definition 3.1.4.

**Lemma 3.12.** *There exists $\alpha > 0$ such that for any $\bar{x} \in \Lambda$ holds*

$$\sphericalangle(E(\bar{x}), F(\bar{x})) \geq \alpha.$$

*Proof.* Suppose by contradiction that given $\alpha_n$, there exists $\bar{x}_n \in \Lambda$ such that

$$\sphericalangle(E(\bar{x}_n), F(\bar{x}_n)) < \alpha_n.$$

By Lemma 3.9 (Franks' Lemma), there exists $\varepsilon > 0$ such that if a linear map $L : T_xM \to T_{f(x)}M$ satisfies $\|L - Df_x\| < \varepsilon$, then there exist $g \in \mathcal{U}_f$ and a neighborhood $B$ of $x$ in $M$ such that:

$$g\,|_{M\setminus B} = f\,|_{M\setminus B} \quad \text{and} \quad Dg_x = L.$$

Choosing $\alpha_n$ and $x_{-1} = \pi_0(\sigma_f^{-1}(\bar{x}_n))$ small enough such that $L = R \circ Df_{x_{-1}}$, where $R$ is the rotation of angle smaller than $\alpha_n$ for which $R(F(\bar{x}_n)) = E(\bar{x}_n)$, satisfies:

$$\|L - Df_{x_{-1}}\| = \sup\{\|Df_x\| : x \in M\}\|R - I\| < \varepsilon,$$

where $I$ is the identity. Therefore, there exist $g \in \mathcal{U}_f$ and a neighborhood $B$ of $x_{-1}$ in $M$ such that

- $x_j = \pi_0(\sigma_f^j(\bar{x}_n)) \notin B$, for $n^-(\bar{x}_n) \leq j \leq -2$ and $0 \leq j \leq n^+(\bar{x}_n)$;
- $g(x_j) = x_{j+1}$, for every $n^-(\bar{x}_n) \leq j \leq n^+(\bar{x}_n)$;
- $Dg_{x_{-1}} = L$.

Then, as $x_j \notin B$ for $n^-(\bar{x}_n) \leq j \leq -2$ and $0 \leq j \leq n^+(\bar{x}_n)$, we have $Dg_{x_j} = Df_{x_j}$. Hence,

$$\begin{aligned} Dg_{x_{n^-}}^{|n^-|}(F(\sigma_f^{n^-}(\bar{x}_n))) &= Dg_{x_{-1}}(F(\sigma_f^{-1}(\bar{x}_n)) \\ &= R(F(\bar{x}_n)) = E(\bar{x}_n). \end{aligned}$$

In particular, for $n = |n^-| + n^+$, we have $Dg_{x_{n^-}}^n \equiv 0$ which is a contradiction. $\square$

Therefore, to conclude the proof of the Main Theorem remains to prove the domination property (3.1.2), item $(c')$ of Definition 3.1.4.

We will define a splitting over a neighborhood of $S_f$ which will be useful to show some properties about the splitting $T_\Lambda M_f = E \oplus F$.

Denote by $v_x$ and $w_x$ unit vectors in $T_xM$ such that $m(Df_x) = \|Dfv_x\|$ and $\|Df_x\| = \|Dfw_x\|$ for $x \in M$. Note, in particular, that $v_x \in E(x)$ for $x \in S_f$.

It is not hard to see that the maps $x \mapsto V_x$ and $x \mapsto W_x$ are continuous, where $V_x$ and $W_x$ are the vector spaces generated by $v_x$ and $w_x$, respectively. Hence, given $\delta > 0$ and $\theta > 0$ with $\delta < \theta$, we may take a neighborhood $U$ of $S_f$ such that $m(Df_x) < \delta$ and $\|Df_x\| > \theta$ for every $x \in U$. In particular, by Remark 3.1, it follows that $v_x$ and $w_x$ are orthogonal vectors.

Consider the cone field

$$\mathscr{C} : x \in U \longmapsto \mathscr{C}_V(x,\eta) := \mathscr{C}_{V_x}(\eta) \subseteq T_xM.$$

We will prove some technical results that relate the splitting $E \oplus F$ and $V \oplus W$ in a neighborhood of $S_f$.

**Lemma 3.13.** *For every $\varepsilon > 0$ and $\eta > 0$, there exists a neighborhood $U' \subseteq U$ of $S_f$ such that for every $x \in U'$, we have $Df_x(\mathscr{C}^*_V(x,\eta))$ is contained in $\mathscr{C}_{Df_x(W_x)}(f(x),\varepsilon)$.*

*Proof.* Note that it is enough to prove for $x \in U \backslash S_f$, because for $x \in S_f$ we have

$$V_x = \ker(Df_x) \quad \text{and} \quad \mathscr{C}^*_V(x,\eta) \cap \ker(Df_x) = \{0\},$$

then

$$Df_x(\mathscr{C}^*_V(x,\eta)) = Df_x(W_x) \text{ for every } \eta > 0.$$

Thus, suppose $x \in U\backslash S_f$. In this case, we will show that $Df_x(\mathscr{C}^*_V(x,\eta))$ is contained in the following cone:

$$\left\{ u_1 + u_2 \in Df(V_x) \oplus Df(W_x) : \|u_1\| \leq \frac{\delta}{\eta\theta}\|u_2\| \right\}.$$

Indeed,

$$\frac{\|Dfv\|}{\|Dfw\|} \quad = \quad \frac{m(Df_x)\|v\|}{\|Df_x\|\|w\|} \leq \frac{\delta}{\theta}\frac{\|v\|}{\|w\|} \leq \frac{\delta}{\eta\theta},$$

for $v + w \in \mathscr{C}^*_V(x,\eta)$. Then, to conclude the proof, we take a neighborhood $U'$ of $S_f$ contained in $U$ such that $\delta > 0$ is small enough so that:

$$\mathscr{C}_{Df_x(W_x),Df_x(V_x)}(f(x),\delta\eta^{-1}\theta^{-1}) \subseteq \mathscr{C}_{Df_x(W_x)}(f(x),\varepsilon).$$

This finishes the proof. $\square$

**Proposition 3.14.** *Given $\varepsilon > 0$, there exists a neighborhood $U'$ of $S_f$ such that for every $\bar{x} \in \Lambda$ with $x_0 = \pi_0(\bar{x}) \in U'$, we have*

$$\sphericalangle(F(\sigma_f(\bar{x})), Df_{x_0}(W_{x_0})) < \varepsilon \quad \text{and} \quad \sphericalangle(E(\bar{x}), V_{x_0}) < \varepsilon.$$

*Proof.* It is sufficient to prove for $x_0 \notin S_f$. Because, if $x_0 \in S_f$, we have that $V_{x_0} = E(x_0) = \ker(Df_{x_0})$ and, consequently, $Df_{x_0}(W_{x_0}) = F(\sigma_f(\bar{x}))$. In particular, it follows the statement.

Choose $\alpha > 0$ and $\eta > 0$ small enough so that $0 < \eta < \alpha < \varepsilon$. Then, by Lemma 3.13, there exists $U'$ a neighborhood of $S_f$ such that $Df_x(\mathscr{C}^*_V(x, \eta))$ is contained in $\mathscr{C}_{Df_x(W_x)}(f(x), \alpha/2)$ for every $x \in U'$.

We prove our assertion by contradiction. The proof is divided in two parts.

*First Part*: $\sphericalangle(F(\sigma_f(\bar{x})), Df_{x_0}(W_{x_0})) < \varepsilon$.

Indeed, suppose by contradiction that $\sphericalangle(F(\sigma_f(\bar{x})), Df_{x_0}(W_{x_0})) > \alpha/2$. Then, $F(\sigma_f(\bar{x})) \notin Df_{x_0}(\mathscr{C}^*_V(x_0, \eta))$. Since $Df_{x_0} : T_{x_0}M \to T_{\pi_0(\sigma_f(\bar{x}))}M$ is an isomorphism, $F(\bar{x})$ must belong to $\mathscr{C}_V(x_0, \eta)$. Thus, by Lemma 3.12, we have that $E(\bar{x}) \notin \mathscr{C}_V(x_0, \eta)$.

Then, by Lemma 3.9(Franks' Lemma), there exist $g \in \mathcal{U}_f$, and $B$ and $B'$ disjoint neighborhoods of $\pi_0(\sigma_f^{-1}(\bar{x}))$ and $x_0$ in $M$, respectively, such that:

- $g \mid_{M \setminus (B \cup B')} = f \mid_{M \setminus (B \cup B')}$;
- $g(\pi_0(\sigma_f^j(\bar{x}))) = \pi_0(\sigma_f^{j+1}(\bar{x}))$ for every $n^- \leq j \leq n^+$;
- $Dg_{\pi_0(\sigma_f^{-1}(\bar{x}))} = R_1 \circ Df_{\pi_0(\sigma_f^{-1}(\bar{x}))}$ and $Dg_{x_0} = R_2 \circ Df_{x_0}$, where $R_1$ and $R_2$ are rotation maps of angle smaller than $\alpha$ satisfying:

$$F'(\bar{x}) = R_1(F(\bar{x})) \notin \mathscr{C}_V(x_0, \eta) \quad \text{and} \quad R_2(Df(F'(\bar{x}))) = E(\sigma_f(\bar{x})).$$

This is possible, since $F(\bar{x}) \in \mathscr{C}_V(x_0, \eta)$ and $F'(\bar{x}), E(\bar{x}) \in \mathscr{C}^*(x_0, \eta)$. Hence, $Df(F'(\bar{x})), E(\sigma_f(\bar{x})) \in \mathscr{C}_{Df_{x_0}(W_{x_0})}(f(x_0), \alpha/2)$. Therefore, there exist $n \geq 1$ and $x \in S_f$ such that $Dg^n_x \equiv 0$ which is a contradiction. Then, we conclude that $\sphericalangle(F(\bar{x}), Df_{x_0}(W_{x_0})) < \alpha/2$. In particular, this concludes the proof of first part.

*Second Part*: $\sphericalangle(E(\bar{x}), V_{x_0}) < \varepsilon$.

Suppose by contradiction that $\sphericalangle(E(x_j), V_{x_j}) \geq \eta$. Then, $E(\bar{x}) \in \mathscr{C}^*_V(x_0, \eta)$ and so $E(\sigma_f(\bar{x})) \in Df_{x_0}(\mathscr{C}^*_V(x_0, \eta))$. In particular,

$$E(\sigma_f(\bar{x})) \in \mathscr{C}_{Df_{x_0}(W_{x_0})}(f(x_0), \alpha/2).$$

In other words, $\sphericalangle(F(\sigma_f(\bar{x})), E(\sigma_f(\bar{x}))) < \alpha$. Contradicting the Lemma 3.12. Therefore, $\sphericalangle(E(\bar{x}), V_{x_0}) < \eta$ proves the second part and, consequently, the proposition. □

The next lemma is very important for proof the domination property, item $(c')$. This lemma can be found in the Potrie's thesis (see [Pot12, Appendix A]). However, the following proof is little bit different.

**Lemma 3.15.** *Given $\varepsilon > 0$ and $K > 0$, there exists $l > 0$ such that if $A_1, ..., A_l$ is a sequence in $GL(2, \mathbb{R})$ verifying:*

- $\max\limits_{1 \le i \le l}\{\|A_i\|, \|A_i^{-1}\|\} \le K$ *for each* $1 \le A_i \le l$*;*
- $\|A_l ... A_1(v)\| \ge \frac{1}{2}\|A_l ... A_1(w)\|$ *for unit vectors* $v, w \in \mathbb{R}^2$*.*

*Then, there exist rotations $R_1, ..., R_l$ of angles smaller than $\varepsilon$ such that*

$$R_l A_l ... R_1 A_1(\mathbb{R}\langle w \rangle) = A_l ... A_1(\mathbb{R}\langle v \rangle).$$

Before to start the proof, let us introduce some notations. Denote by $P^1(\mathbb{R})$ the real projective space which is the set of all one-dimensional subspaces of $\mathbb{R}^2$. Then, for any subspaces $V, W \in P^1(\mathbb{R})$ the distance between them is defined by the angle between them.

*Proof.* Define $\mathcal{A}_n = A_n \cdots A_1$, $V_n = \mathcal{A}_n(\mathbb{R}\langle v \rangle)$ and $W_n = \mathcal{A}_n(\mathbb{R}\langle w \rangle)$.

First, let us define by recurrence a sequence of intervals $(I_n)_{\ge 1}$,

$$\begin{array}{ccc} I_0 & \text{is the vector} & w; \\ I_1 & \text{is a } \varepsilon\text{-neighborhood of} & A_1(I_0); \\ I_2 & \text{is a } \varepsilon\text{-neighborhood of} & A_2(I_1); \\ \vdots & \vdots & \vdots \\ I_{n+1} & \text{is a } \varepsilon\text{-neighborhood of} & A_{n+1}(I_n), \quad \text{for } n \ge 1, \end{array}$$

where a $\varepsilon$-neighborhood of $X \subseteq P^1(\mathbb{R})$ is a set of all $V \in P^1(\mathbb{R})$ which the angle between it and some one-dimensional subspace of $\mathbb{R}^2$ in $X$ is smaller or equal than $\varepsilon$.

We introduce to follow a statement that will be useful to the proof of lemma.

*Claim 1:* If a neighborhood around $V_l$ of length $\varepsilon > 0$ intersects $A_l(I_{l-1})$ for some $l \ge 1$, then we obtain that there exist rotations $R_1, ..., R_l$ of angles smaller than $\varepsilon$ such that

$$R_l A_l ... R_1 A_1(\mathbb{R}\langle w \rangle) = A_l ... A_1(\mathbb{R}\langle v \rangle).$$

*Proof of Claim 1.* To construction of $I_n$, we will build by induction:

Suppose $z$ belongs a $\varepsilon$-neighborhood around $V_l$ intersection $I_l$, then there exist $z_{l-1} \in I_{l-1}$ and a rotation $R_l$ such that $R_l A_l(z_{l-1}) \in V_l$ and $z = A_l(z_{l-1})$. But, since $z_{l-1}$ belongs a $\varepsilon$-neighborhood around $A_{l-1}(I_{l-2})$, there exist $R_{l-1}$ rotation and $z_{l-2} \in I_{l-2}$ so that $z_{l-1} = R_{l-1}A_{l-1}(z_{l-2})$. Repeating the process, one has the wished.

□

Now, we will prove that there exists $l \geq 1$ satisfying the statement above. For this, suppose, without loss of generality, that

$$\Theta = \inf\{\sphericalangle(W_n, V_n) \mid n \geq 0\} > 0.$$

Otherwise, the lemma is clearly obtained.

Note that for each $\mathbb{R}\langle z\rangle \in P^1(\mathbb{R})$, we can right $z = \alpha v + \beta w$. Then, we have that

$$\mathcal{A}_n(z) = \alpha\|\mathcal{A}_n(v)\|v_n + \beta\|\mathcal{A}_n(w)\|w_n$$

and using the supposition, one has for very $n \geq 1$ that:

$$\begin{aligned}
\sphericalangle(W_n, \mathcal{A}_n(\mathbb{R}\langle z\rangle)) &= \frac{|\alpha|\|\mathcal{A}_n(v)\| \sin\sphericalangle(W_n, V_n)}{|\beta|\|\mathcal{A}_n(w)\| + |\alpha|\|\mathcal{A}_n(v)\|\cos\sphericalangle(W_n, V_n)} \\
&\geq \frac{|\alpha|\|\mathcal{A}_n(v)\| \sin\sphericalangle(W_n, V_n)}{2|\beta|\|\mathcal{A}_n(v)\| + |\alpha|\|\mathcal{A}_n(v)\|\cos\sphericalangle(W_n, V_n)} \\
&\geq \frac{|\alpha| \sin\sphericalangle(W_n, V_n)}{2|\beta| + |\alpha|\cos\sphericalangle(W_n, V_n)} \geq \frac{(|\alpha|/|\beta|)\sin\Theta}{2 + (|\alpha|/|\beta|)\cos\Theta}.
\end{aligned}$$

This implies that there exists $0 < \lambda < 1$ depending only of $\Theta$, the maximal contraction, so that any interval $I \subseteq P^1(\mathbb{R})$ around of $\mathbb{R}\langle w\rangle$ has its image by $\mathcal{A}_n$ contracted with rate at most $\lambda$. That is,

$$diam(\mathcal{A}_n(I)) \geq \lambda diam(I), \forall n \geq 1.$$

Hence, one has that the $diam(I_n)$ is growing as $n$ is growing. Therefore, there exists $l \geq 1$ satisfying the *Claim 1.*

□

Let us prove this auxiliary lemma before proving finally the existence of a dominated splitting.

**Lemma 3.16.** *There exists $m \geq 1$ such that for every $\bar{x} \in \Lambda$, there exists $1 \leq k(\bar{x}) \leq m$, so that*

$$\|Df^k\mid_{E(\bar{x})}\| < \frac{1}{2}\|Df^k\mid_{F(\bar{x})}\|.$$

*Proof.* Suppose by contradiction that given $m \geq 1$ there exists $\bar{x}_m \in \Lambda$ such that

$$\|Df^k |_{E(\bar{x}_m)} \| \geq \frac{1}{2}\|Df^k |_{F(\bar{x}_m)} \|, \tag{3.4.2}$$

for $1 \leq k \leq m$. Note that $1 \leq m \leq n^+(\bar{x}_m)$, otherwise $\|Df^m |_{E(\bar{x}_m)} \| = 0$.

Denote by $y_m$ the point $\pi_0(\bar{x}_m)$.

We would like to show that

$$\exists\, g \in \mathcal{U}_f \text{ such that } Dg_x^n \equiv 0 \text{ for some } x \in M \text{ and } n \geq 1, \tag{3.4.3}$$

which is a contradiction, since $\dim(\ker(Dg_x^n)) \leq 1$ for every $g \in \mathcal{U}_f$ and every $x \in M, n \geq 1$.

For this, we assume the following assertion that will be prove later.

*Claim 1:* There exist a neighborhood $U$ of $S_f$ and $m_0 \geq 1$ such that

$$y_m, f(y_m), ..., f^m(y_m) \notin U \text{ for } m \geq m_0.$$

Let $U$ be a neighborhood of $S_f$ given by Claim 1 above. By continuity of $Df$, we can choose $K > 0$ such that $\max\{\|Df_x\|, \|Df_x^{-1}\|\} \leq K$ in $M \backslash U$. Then, by Lemma 3.15, we have that for every $\varepsilon > 0$, there exist $l \geq 1$ and $R_1, \ldots, R_l$ rotations of angle smaller than $\varepsilon$ verifying that

$$\begin{aligned} R_l Df_{f^{l-1}(y_m)} \ldots R_1 Df_{y_m} F(y_m) &= Df_{f^{l-1}(y_m)} \ldots Df_{y_m} E(y_m) \\ &= Df_{y_m}^l E(y_m). \end{aligned}$$

Let $\varepsilon > 0$ given by Lemma 3.9(Franks' Lemma) such that

$$\|R_{k+1} Df_{f^k(y_m)} - Df_{f^k(y_m)}\| < \varepsilon, \text{ for } 0 \leq k \leq l-1.$$

Then, there exist $g \in \mathcal{U}_f$ and a neighborhood $B$ of $\{y_m, ..., f^{l-1}(y_m)\}$ in $M$ satisfying:

- $Dg_{f^k(y_m)} = R_{k+1} Df_{f^k(y_m)}$, for $0 \leq k \leq l-1$;
- $g(f^{k-1}(y_m)) = f^k(y_m)$, for $0 \leq k \leq l$;
- $g |_{M\backslash B} = f |_{M\backslash B}$.

Therefore, taking $n = |n^-(\bar{x}_m)| + n^+(\bar{x}_m)$, we get $Dg^n_{\pi_0(\sigma_f^{n^-}(\bar{x}_m))} \equiv 0$. This proves the statement (3.4.3).

□

Now, we can prove the Claim 1.

*Proof of Claim 1.* Suppose by contradiction that there exist two sequences, $(U_n)_n$ of neighborhoods of $S_f$ such that $U_n \subseteq U_{n+1}$ and $\bigcap_n U_n = S_f$, and $(y_n)_n$ subsequence of $(y_m)_m$ such that for any $n \geq 1$ there exists $1 \leq l_n \leq m_n$ such that $f^{l_n}(y_n) \in U_n$, where $m_n$ goes to infinity as $n$ goes to infinity.

Assume that equation (3.4.2) holds. We will show that the statement (3.4.3) holds, and so, we get a contradiction. Then, we conclude the proof.

We fix $\varepsilon > 0$, given by Lemma 3.9(Franks' Lemma), such that given $\Gamma = \{x_0, ..., x_n\}$ and

$$L : \bigoplus_{x_j \in \Gamma} T_{x_j} M \to \bigoplus_{x_j \in \Gamma} T_{f(x_j)} M$$

a linear map satisfying $\|L - Df \mid_{\oplus_{x_j \in \Gamma} T_{x_j} M} \| < \varepsilon$. Then, there exist $g \in \mathcal{U}_f$ and a neighborhood $B$ of $\Gamma$ such that $Dg \mid_{\oplus_{x_j \in \Gamma} T_{x_j} M} = L$ and $g \mid_{M \setminus B} = f \mid_{M \setminus B}$.

Let $\alpha > 0$ be the number given by Lemma 3.12. We fix $0 < \theta < \alpha$ such that any rotation $R$ of angle smaller than $\theta$ satisfies:

$$\|RDf_x - Df_x\| \leq \sup\{\|Df_x\| : x \in M\} \|R - I\| < \varepsilon.$$

By Proposition 3.14, we have that given $\varepsilon' > 0$ there exists a neighborhood $U$ of $S_f$ in $M$ such that for every $\bar{x} \in \Lambda$ which $\pi_0(\bar{x}) \in U$, we get

$$\sphericalangle(F(\sigma_f(\bar{x})), Df_{x_0}(W_{x_0})) < \varepsilon' \quad \text{and} \quad \sphericalangle(E(\bar{x}), V(x_0)) < \varepsilon'.$$

Hence, given $\beta > 0$ we can choose $\varepsilon' > 0$ small enough such that the cone

$$\mathscr{C}_{F,E}(\sigma_f(\bar{x}), \beta/2) = \left\{ v + w \in E(\sigma_f(\bar{x})) \oplus F(\sigma_f(\bar{x})) : \|v\| \leq \frac{\beta}{2} \|w\| \right\}$$

contains the cone $\mathscr{C}_{Df_x(W_x)}(f(x), \varepsilon')$ for every $\bar{x} \in \Lambda$ which $x = \pi_0(\bar{x}) \in U$. Then, we consider $0 < \beta < \theta$ small enough such that

$$\mathscr{C}_{F,E}(\bar{x}, \beta) \subseteq \mathscr{C}_F(\bar{x}, \theta),$$

or equivalently

$$\forall v \in \mathscr{C}_{F,E}(\bar{x}, \beta), \ \sphericalangle(F(\bar{x}), \mathbb{R}\langle v \rangle) < \theta.$$

By Lemma 3.13, for $0 < \eta < \theta/2$ there exists $U' \subseteq U$ such that $Df_x(\mathscr{C}_V^*(x, \eta))$ is contained in $\mathscr{C}_{Df_x(W_x)}(f(x), \varepsilon')$ for $x \in U'$. In particular, $Df_x(\mathscr{C}_V^*(x, \eta))$ is contained in $\mathscr{C}_F(\sigma_f(\bar{x}), \theta)$ for every $\bar{x} \in \Lambda$ which $x = \pi_0(\bar{x}) \in U'$.

Note that $E(\bar{x}) \in \mathscr{C}_V(x, \eta)$ for every $x = \pi_0(\bar{x}) \in U'$. Indeed, if $E(\bar{x}) \in \mathscr{C}_V^*(x, \eta)$ then

$$E(\sigma_f(\bar{x})) = Df(E(\bar{x})) \in \mathscr{C}_F(\sigma_f(\bar{x}), \theta).$$

Contradicting that $\sphericalangle(E(\bar{x}), F(\bar{x})) \geq \alpha$ for every $\bar{x} \in \Lambda$.

Now, we can prove the statement (3.4.3).

Assume, without loss of generality, that $U_n \subseteq U'$ for every $n$. We consider $(\bar{x}_n)_n$ a subsequence of $(\bar{x}_m)_m$ such that $y_n = \pi_0(\bar{x}_n)$. Then, we have that

$$Df^k(\mathscr{C}^*_{F,E}(\bar{x}_n, \beta)) \subseteq \mathscr{C}^*_{F,E}(\sigma_f^k(\bar{x}_n), \beta/2)$$

for $1 \leq k \leq m_n$. Since

$$\mathscr{C}^*_{F,E}(\bar{x}_n, \beta) = \{v + w \in E(\bar{x}_n) \oplus F(\bar{x}_n) : \|w\| \leq \beta^{-1}\|v\|\},$$

we get, by equation 3.4.3, that

$$\begin{aligned} \|Df^k(w)\| &= \|Df^k|_{F(\bar{x}_n)}\|\|w\| \leq 2\|Df^k|_{E(\bar{x}_n)}\|\|w\| \\ &\leq 2\beta^{-1}\|Df^k|_{F(\bar{x}_n)}\|\|v\| = 2\beta^{-1}\|Df^k(v)\|. \end{aligned}$$

In other words, $Df^k(\mathscr{C}^*_{F,E}(\bar{x}_n, \beta)) \subseteq \mathscr{C}^*_{F,E}(\sigma_f^k(\bar{x}_n), \beta/2)$. In particular, one has $\mathscr{C}_{F,E}(\sigma_f^k(\bar{x}_n), \beta/2)$ is contained in $Df^k(\mathscr{C}_{F,E}(\bar{x}_n, \beta))$.

Note that, as $Df_x(\mathscr{C}^*_V(x, \eta))$ is contained in $\mathscr{C}_{Df_x(W_x)}(f(x), \varepsilon')$, and $\mathscr{C}_{Df_x(W_x)}(f(x), \varepsilon')$ is contained in $\mathscr{C}_{F,E}(\bar{x}, \beta/2)$ for every $x = \pi_0(\bar{x}) \in U'$, we have

$$Df_x(\mathscr{C}^*_V(x, \eta)) \subseteq \mathscr{C}_{F,E}(\bar{x}, \beta/2)$$

for every $x = \pi_0(\bar{x}) \in U'$. Therefore, we have

$$Df^{l_n}(\mathscr{C}^*_{F,E}(\bar{x}_n, \beta)) \subseteq \mathscr{C}_V(f^{l_n}(y_n), \eta)$$

where $f^{l_n}(y_n) = \pi_0(\sigma_f^{l_n}(\bar{x}_n)) \in U_n \subseteq U'$. Since,

$$Df^{l_n+1}(\mathscr{C}^*_{F,E}(\bar{x}_n, \beta)) = \mathscr{C}^*_{F,E}(\sigma_f^{l_n+1}(\bar{x}_n), \beta/2)$$

and

$$Df_{f^{l_n}(y_n)}(\mathscr{C}^*_V(f^{l_n}(y_n), \eta)) \subseteq \mathscr{C}_{F,E}(\sigma_f^{l_n+1}(\bar{x}_n), \beta/2).$$

Therefore, we get

$$E(\sigma_f^{l_n}(\bar{x})) \in \mathscr{C}_V(f^{l_n}(y_n), \eta) \quad \text{and} \quad Df^{l_n}(\mathscr{C}^*_{F,E}(\bar{x}_n, \beta)) \subseteq \mathscr{C}_V(f^{l_n}(y_n), \eta).$$

In particular, there exist $v \in \mathscr{C}_{F,E}(\bar{x}_n, \beta)$ such that

$$\sphericalangle(F(\bar{x}_n), \mathbb{R}\langle v \rangle) < \theta \quad \text{and} \quad \sphericalangle(E(\bar{x}_n), \mathbb{R}\langle Df^{l_n}(v) \rangle) < \eta$$

Then, we take $\Gamma = \{\pi_0(\sigma_f^{-1}(\bar{x}_n)), \pi_0(\sigma_f^{l_n-1}(\bar{x}_n))\}$ and the following linear maps

$$L_1 : T_{\pi_0(\sigma_f^{-1}(\bar{x}_n))}M \to T_{y_n}M, \text{ defined by } L_1 = R_1 Df_{\pi_0(\sigma_f^{-1}(\bar{x}_n))},$$

and

$$L_2 : T_{f^{l_n-1}(y_n)}M \to T_{f^{l_n}(y_n)}M, \text{ defined by } L_2 = R_2 Df_{f^{l_n-1}(x_n)},$$

where $R_1$ and $R_2$ are rotations of angle smaller that $\theta$ such that

$$R_1(F(\bar{x}_n)) = \mathbb{R}\langle v\rangle \quad \text{and} \quad R_2(\mathbb{R}\langle Df^{l_n}(v)\rangle) = E(\bar{x}_n).$$

By Franks' Lemma, there exist $g \in \mathcal{U}_f$ and a neighborhood $B$ of $\Gamma$ in $M$ satisfying:

- $g \mid_{M\setminus B} = g \mid_{M\setminus B}$;
- $g(\pi_0(\sigma_f^j(\bar{x}_n))) = \pi_0(\sigma_f^{j+1}(\bar{x}_n))$ for $n^-(\bar{x}_n) \le j \le n^-(\bar{x}_n)$;
- $Dg_{\pi_0(\sigma_f^{-1}(\bar{x}_n))} = L_1$ and $Dg_{\pi_0(\sigma_f^{l_n}(\bar{x}_n))} = L_2$

Therefore, for $n = |n^-| + n^+$, one has $Dg^n_{\pi_0(\sigma_f^{n^-}(\bar{x}_n))} \equiv 0$, which is a contradiction. □

Finally, we are able to prove the domination property (3.1.2).

**Lemma 3.17.** *There exists $l \ge 1$ such that for any $\bar{x} \in \Lambda$,*

$$\|Df^l \mid_{E(\bar{x})}\| < \frac{1}{2}\|Df^l \mid_{F(\bar{x})}\|.$$

*Proof.* Consider $m \ge 1$ given by Lemma 3.16. Then, for $\bar{x} \in \Lambda$, there exist $k_0 := k(\bar{x})$, with $1 \le k_0 \le m$, such that if $l \gg m$, then

$$\begin{aligned}
\|Df^l \mid_{E(\bar{x})}\| &\le \|Df^{l-k_0} \mid_{E(\sigma_f^{k_0}(\bar{x}))}\| \|Df^{k_0} \mid_{E(\bar{x})}\| \\
&\le \frac{1}{2}\|Df^{l-k_0} \mid_{E(\sigma_f^{k_0}(\bar{x}))}\| \|Df^{k_0} \mid_{F(\bar{x})}\|.
\end{aligned}$$

Assuming $l - k_0 \gg m$ and $\bar{x}_1 = \sigma_f^{k_0}(\bar{x})$, there exists $k_1 := k(\sigma_f^{k_0}(\bar{x}))$, with $1 \le k_1 \le m$, such that

$$\begin{aligned}
\|Df^l \mid_{E(\bar{x})}\| &\le \frac{1}{2}\|Df^{l-k_0} \mid_{E(\bar{x}_1))}\| \|Df^{k_0} \mid_{F(\bar{x})}\| \\
&\le \frac{1}{2}\|Df^{l-(k_0+k_1)} \mid_{E(\sigma_f^{k_1}(\bar{x}_1))}\| \|Df^{k_1} \mid_{E(\sigma_f^{k_1}(\bar{x}_1))}\| \|Df^{k_0} \mid_{F(\bar{x})}\| \\
&\le \left(\frac{1}{2}\right)^2 \|Df^{l-k_0-k_1} \mid_{E(\bar{x}_2)}\| \|Df^{k_1+k_0} \mid_{F(\bar{x})}\|.
\end{aligned}$$

where $\bar{x}_2 = \sigma_f^{k_1}(\bar{x}_1)$.

We consider $n^+(\bar{x}) \gg l$. Then, repeating the process, we get $1 \leq L_r \leq m$, where
$L_r = l - \sum_{i=0}^{r-1} k_i$, such that

$$\begin{aligned} \|Df^l \mid_{E(\bar{x})} \| &\leq \left(\frac{1}{2}\right)^r \|Df^{L_r} \mid_{E(\sigma_f^{k_{r-1}}(\bar{x}_{r-1}))} \| \|Df^{l-L_r} \mid_{F(\bar{x})} \| \\ &\leq \left(\frac{1}{2}\right)^r C_0 \|Df^l \mid_{F(\bar{x})} \|. \end{aligned}$$

where $C_0$ is chosen so that

$$\max\{\|Df^i \mid_{E(\bar{x})} \| : i = 1, 2, \ldots, m\} \leq C_0 \max\{\|Df^i \mid_{F(\bar{x})} \| : i = 1, 2, \ldots, m\}$$

for every $\bar{x} \in \Lambda$. Therefore, taking $l \gg 1$ such that $(1/2)^r C_0 \leq 1/2$, for every $\bar{x} \in \Lambda$ we get that:

- If $n^+(\bar{x}) \geq l$, we have

$$\|Df^l \mid_{E(\bar{x})} \| < \frac{1}{2} \|Df^l \mid_{F(\bar{x})} \|.$$

- If $n^+(\bar{x}) < l$, we have $\|Df^l \mid_{E(\bar{x})} \| = 0$. In particular,

$$\|Df^l \mid_{E(\bar{x})} \| = 0 < \frac{1}{2} \|Df^l \mid_{F(\bar{x})} \|.$$

This concludes the proof. □

Finally, we prove Main Theorem 2.

*Proof of Main Theorem 2.* Therefore, by Lemma 3.12 and 3.17, we have that items $(b')$ and $(c')$ hold, and, by item $(iii)$ of Lemma 3.11, one has that item $(a')$ of Definition 3.1.4 holds. This concludes the proof of Main Theorem 2. □

## 3.5 Proof of Theorem B

Finally, we can prove the Theorem B. Let $f \in \mathrm{End}^1(M)$ be a robustly transitive endomorphism with nonempty critical set. By Proposition 3.6 and Lemma 3.10, we may choose a point $\bar{x} = (x_j) \in M_f$ such that

$$\mathcal{O}^+(\bar{x}, \sigma_f) = \{\sigma_f^j(\bar{x}) : j \geq 0\} \quad \text{and} \quad \mathcal{O}^-(\bar{x}, \sigma_f) = \{\sigma_f^j(\bar{x}) : j \leq 0\}$$

are dense in $M_f$.

In order to show that $M$ admits a dominated splitting for $f$, we will prove that the orbit

$$\mathcal{O}(\bar{x}, \sigma_f) = \{\sigma_f^n(\bar{x}) : j \in \mathbb{Z}\}$$

exhibits a dominated splitting. And so, by Proposition 3.5, we can extend it to the whole $M_f$.

Before starting the proof, let us introduce some useful results.

Let $p \in S_f$. By Lemma 3.9, there exist three sequences, two sequences $(B_n)$ and $(B'_n)$ of neighborhoods of $p$, with $\bar{B}_n \subset B'_n$, and a sequence $(f_n)$ of $C^1$-endomorphisms satisfying:

$(i)$ $f_n|_{B_n} = Df_p$, for each $n \geq 1$, and $\operatorname{diam}(B'_n) \to 0$;

$(ii)$ $f_n|_{M \setminus B'_n} = f$ and $f_n(p) = f(p)$, for each $n \geq 1$;

$(iii)$ $f_n$ converges to $f$ in $\operatorname{End}^1(M)$.

*Claim 1:* There exists a sequence $(\bar{x}^n)$, $\bar{x}^n \in M_{f_n}$ for each $n \geq 1$, such that $\bar{x}^n \to \bar{x}$ as $n \to \infty$ in $M^{\mathbb{Z}}$.

Note that

$$\bar{x}^n \longrightarrow \bar{x} \text{ in } M^{\mathbb{Z}} \iff \left\{ \begin{array}{c} \forall \varepsilon > 0,\ N \geq 0,\ \exists n_0 \geq 1 \text{ such that} \\ d(x_j^n, x_j) < \varepsilon, \quad \text{for } |j| \leq N \text{ and } n \geq n_0. \end{array} \right. \qquad (\circledast)$$

*Proof of Claim 1.* First, we build the sequence $(\bar{x}^n)$, $\bar{x}^n \in M_{f_n}$. For this, note that, given $N_1 \geq 0$, we may choose $n_1 \geq 1$ such that

$$x_j \notin B_n \backslash \{p\}, \text{ for } |j| \leq N_1 \text{ and } n \geq n_1.$$

This is possible because the orbits $\mathcal{O}^+(\bar{x}, \sigma_f)$ and $\mathcal{O}^-(\bar{x}, \sigma_f)$ are dense in $M_f$. Thus, we may define $\bar{x}^n \in M_{f_n}$ so that $x_j^n = x_j$ for $|j| \leq N_1$ and $n \geq n_1$.

Given $\varepsilon > 0$ and $N > 0$, we may choose $N_1 > N$ large enough so that

$$\frac{\sup\{d(x, y) : x, y \in M\}}{2^{|j|}} < \frac{\varepsilon}{2}, \text{ for every } |j| = N_1 + 1.$$

Therefore, choosing $n_1 \geq 1$ as above, we have for every $n \geq n_1$ that

$$\bar{d}(\bar{x}^n, \bar{x}) = \sum_{j \in \mathbb{Z}} \frac{d(x_j^n, x_j)}{2^{|j|}} = \sum_{|j| > N_1} \frac{\varepsilon}{2^{|j| - N_1}} < \varepsilon,$$

since $x_j^n = x_j$ for $|j| \leq N_1$. Therefore, by $(\circledast)$, one has $\bar{x}^n \to \bar{x}$ in $M^{\mathbb{Z}}$. $\square$

It follows from Theorem B' that $M_{f_n}$ admits a dominated splitting for each $n \geq 1$. Suppose, $TM_{f_n} = E_n \oplus F_n$ is the dominated splitting for $f_n$.

**Lemma 3.18.** *There exists $\alpha > 0$ such that the angle between $E_n$ and $F_n$ is greater than or equal to $\alpha$, for every $n \geq 1$.*

*Proof.* By the proof of Lemma 3.12 follows that if

$$\sphericalangle(E_n(\bar{y}), F_n(\bar{y})) < \alpha,$$

for some $\alpha > 0$ small enough, then there exist a point $y_0 \in M$ and a number $k \geq 1$ such that

$$\dim(\ker(Df^k_{y_0})) = 2.$$

Contradicting the fact that $f_n$ is a robustly transitive endomorphism. $\square$

For the next lemma, remember that

$$\Lambda_n = \left\{ \bar{x} \in M_{f_n} \;\middle|\; \begin{array}{l} \bullet \;\; \forall j \geq 0,\; \pi_0(\sigma^j_{f_n}(\bar{x})) \notin \mathrm{Per}(f_n); \\ \bullet \;\; \exists (j_k)_k \subseteq \mathbb{Z},\; j_k \to \pm\infty \text{ when } k \to \pm\infty, \\ \quad\;\; \text{such that } \; \pi_0(\sigma^{j_k}_f(\bar{x})) \in S_f. \end{array} \right\} \tag{3.5.1}$$

**Lemma 3.19.** *There exists $m \geq 1$ so that for every $\bar{y} \in \Lambda_n$, there exists $k := k(\bar{y}) \in \mathbb{N}$, $1 \leq k \leq m$, such that*

$$\|Df^k_n \mid_{E_n(\bar{y})} \| < \frac{1}{2} \|Df^k_n \mid_{F_n(\bar{y})} \| \quad \textit{for every} \;\; n \geq 1.$$

*Proof.* Suppose by contradiction that for each $m \geq 1$, there exist $n \geq 1$ and $y \in \Lambda_{f_n}$ such that

$$||Df^k_n \mid_{E_n(\bar{y})} \| \geq \frac{1}{2} \|Df^k_n \mid_{F_n(\bar{y})} \|,$$

for every $1 \leq k \leq m$. Then, repeating the proof of Lemma 3.16, one obtain a contradiction. $\square$

**Lemma 3.20.** *$\mathcal{O}(\bar{x}, \sigma_f)$ admits a dominated splitting.*

*Proof.* Note that $\sigma^j_{f_n}(\bar{x}^n) \to \sigma^j_f(\bar{x})$ in $M^{\mathbb{Z}}$. Since $\sigma_{f_n} = \sigma \mid_{M_{f_n}}$ and $\sigma_f = \sigma \mid_{M_f}$, where $\sigma : M^{\mathbb{Z}} \to M^{\mathbb{Z}}$ is the shift map which is a continuous map. Then, by Proposition 3.5, we define

$$E(\sigma^j_f(\bar{x})) = \lim E_n(\sigma^j_{f_n}(\bar{x}^n)) \;\; \text{and} \;\; F(\sigma^j_f(\bar{x})) = \lim F_n(\sigma^j_{f_n}(\bar{x}^n)).$$

It follows from Lemma 3.18 that $E$ and $F$ satisfy the property $(b')$ of Definition 3.1.4. Moreover, by construction of $f_n$, we have that:

$$\forall j \in \mathbb{Z}, \exists N > 0 \text{ such that } (Df_n)_{x_j^n} = Df_{x_j} \text{ for } n \geq N.$$

Hence, $E$ and $F$ are $Df$-invariant, i.e., they satisfy the property $(a')$ of Definition 3.1.4.

Finally, to conclude the proof we will show that $f$ satisfies the property $(c')$ of Definition 3.1.4. And so, we conclude the proof.

Indeed, let $m \in \mathbb{N}$ be given by Lemma 3.19. Then, given $\bar{y} = \sigma_f^j(\bar{x})$ for some $j \in \mathbb{Z}$, we may choose $n_0 \geq 1$ large enough such that $\sigma_{f_n}^j(\bar{x}^n) = (x_{l+j}^n)_l \in M_{f_n}$ satisfies $x_{l+j}^n = y_l$ and $(Df_n)_{x_{l+j}^n} = Df_{y_l}$ for $|l| \leq m$ and $n \geq n_0$. Thus, there exists $k := k(\bar{y})$, $1 \leq k \leq m$, such that

$$\begin{aligned} \|Df^k\,|_{E(\bar{y})}\,\| &= \lim \|Df_n^k\,|_{E_n(\sigma_{f_n}^j((\bar{x}^n))}\,\| \\ &\leq \|Df^k\,|_{F(\bar{y})}\,\| = \lim \|Df_n^k\,|_{F_n(\sigma_{f_n}^j((\bar{x}^n))}\,\|. \end{aligned}$$

Therefore, for every $\bar{y} \in \mathcal{O}(\bar{x}, \sigma_f)$ there exists $1 \leq k \leq m$ such that

$$\|Df^k\,|_{E(\bar{y})}\,\| < \frac{1}{2}\|Df^k\,|_{F(\bar{y})}\,\|.$$

It follows from Lemma 3.17 that $f$ satisfies the property $(c')$ of Definition 3.1.4.

□

*Proof of Theorem B.* Since $\mathcal{O}(\bar{x}, \sigma_f)$ admits a dominated splitting and it is a dense subset of $M_f$, it follows from Proposition 3.5 that $M$ admits a dominated splitting for $f$. □

## 3.6 Isotopy classes

### 3.6.1 Precise statement of the Main Theorem 3

**Main Theorem 3.** *If $f \in \mathrm{End}^1(M)$ is a transitive endomorphism admitting a dominated splitting. Then $f$ is homotopic to a linear map having at least one eigenvalue with modulus greater than one.*

The section below is based on [PS07] . For completeness we give here the details adapting the proof of [PS07] in our setting.

### 3.6.2 Topological expanding direction

Let $f \in \text{End}^1(\text{M})$ be a transitive endomorphism and $M$ is either the torus $\mathbb{T}^2$ or the Klein bottle $\mathbb{K}^2$. Assume that $M$ admits a dominated splitting for $f$. Suppose that $TM = E \oplus \mathscr{C}_E^*$ is the dominated splitting, where $\mathscr{C}_E^* : x \in M \mapsto \mathscr{C}_E^*(x) := \mathscr{C}_E^*(x, \eta)$ is a cone field and $E$ a subbundle over $M$, both $Df$-invariants.

An $E$-arc is an injective Lipschitz curve $\gamma : [0, 1] \to M$ such that $\gamma' \in \mathscr{C}_E^*$, where $\gamma'$ denote the set of tangent vectors of $\gamma$. We denote by $\ell(\gamma)$ the length of $\gamma$.

**Definition 3.6.1.** *We say that an $E$-arc $\gamma$ is a $\delta$-$E$-arc provided the next condition holds:*

$$\ell(f^n(\gamma)) \leq \delta, \quad \textit{for every} \;\; n \geq 0.$$

In other words, a $\delta$-$E$-arc is a Lipschitz curve that does not grows in length for the future and always remains transversal to the $E$ subbundle, since $\mathscr{C}_E^*$ is $Df$-invariant.

The following result give us an interesting property of a $\delta$-$E$-arc.

**Lemma 3.21.** *There exist $0 < \lambda < 1, \delta > 0, C > 0$ and $n_0 \geq 1$ such that given any $\delta$-$E$-arc $\gamma$ follows that for every $x \in f^{n_0}(\gamma)$ holds*

$$\|Df^j \mid_{E(x)} \| < C\lambda^j, \quad \textit{for every} \;\; j \geq 1. \tag{3.6.1}$$

*Proof.* By dominated splitting, we have that there exists $m \geq 1$ such that

$$\|Df^m \mid_{E(x)} \| \leq \frac{1}{2}\|Df^m(v)\|, \forall v \in \mathscr{C}_E^*(x), \|v\| = 1, \;\; \text{and} \;\; x \in M.$$

Furthermore, given $a > 0$, there exist $\delta_1 > 0$ and $\theta_1 > 0$ such that for every $x, y$ with $d(x, y) < \delta_1$ and $v \in \mathscr{C}_E^*(x), w \in \mathscr{C}_E^*(y)$, with $\measuredangle(w, v) < \theta_1$ [1], one has

$$\|Df(v)\| > (1 - a)\|Df(w)\|,$$

and

$$\|Df \mid_{E(y)} \|, \|Df \mid_{E(x)} \| < a, \text{ if } \; x, y \in B(S_f, \delta_1/2),$$

or

$$\|Df \mid_{E(x)} \| \leq (1 + a)\|Df \mid_{E(y)} \|, \text{ if } \; x \notin B(S_f, \delta_1/2),$$

where $B(S_f, \delta_1/2) = \{x \in M : d(x, S_f) < \delta_1/2\}$.

Fix $0 < \delta < \delta_1$ and $n_0 \geq 1$ such that $f^{n_0}(\mathscr{C}_E^*(x)) \subseteq \mathscr{C}_E^*(f^{n_0}(x), \theta_1)$ for every $x \in M$. Then for $\gamma' \subseteq \mathscr{C}_E^*(x)$, we have $\gamma'_{n_0} \subseteq \mathscr{C}_E^*(x, \theta_1)$, where $\gamma_n$

[1] The angle between $v$ and $w$ is calculated using the local identification $TM \mid_U = U \times \mathbb{R}^2$.

denotes the curve $f^n(\gamma)$. We now apply the observation above, for $\beta > 0$ such that $1 < (1-a)(1+\beta) < 2$ , we have for every $t \in (0,1)$ that $\|Df^k\mid_{\mathbb{R}\langle\gamma'_{n_0}(t)\rangle}\| \leq (1+\beta)^k$ for every $k$ sufficiently large.

In fact, assume that $\gamma := \gamma_{n_0}$ is parametrized by arc length. Suppose by contradiction that there exists a sequence $(k_j)_j$ going to infinity as $j$ goes to infinity such that

$$\|Df^{k_j}\mid_{\mathbb{R}\langle\gamma'(t_j)\rangle}\| > (1+\beta)^{k_j}.$$

Then, we have that

$$\|Df^{k_j}(\gamma'(t))\| \geq (1-a)^{k_j}\|Df^{k_j}(\gamma'(t_j))\| \geq ((1-a)(1+\beta))^{k_j}.$$

In particular, $\ell(f^{k_j}(\gamma)) \geq ((1-a)(1+\beta))^{k_j}\ell(\gamma) > \delta$. Contradicting that $\gamma$ is a $\delta$-$E$-arc.

To finish the proof, choose $\beta > 0$ small enough such that $1 < ((1-a)(1+\beta))^m < 2$. Hence,

$$\|Df^{km}\mid_{E(x)}\| \leq \lambda^k, \quad \text{for all} \quad k \geq 1, x \in \gamma_{n_0},$$

where $\frac{((1-a)(1+\beta))^m}{2} < \lambda < 1$. This shows that (3.6.1) holds. $\square$

We can suppose that, up to taking an iterated, $\gamma$ satisfies Lemma 3.21. The next lemma ensures the existence of the local stable manifold for points belonging to a $\delta$-$E$-arc. Fix $0 < \lambda < 1$ given by Lemma 3.21 and $\lambda' > 0$ such that $(1+a)\lambda < \lambda' < 1$.

**Lemma 3.22.** *Let $\gamma$ be a $\delta$-$E$-arc given by Lemma 3.21. Then, there exists $\alpha > 0$ such that for every $x \in \gamma$, there is a unique curve $\sigma_x : (-\alpha, \alpha) \to M$ orientation preserving satisfying:*

$$\begin{cases} \sigma'_x(t) \in E(\sigma(t)) \quad \text{with} \quad \|\sigma'_x(t)\| = 1; \\ \sigma_x(0) = x. \end{cases} \tag{$*$}$$

*Proof.* We prove that the equation $(*)$ has a unique solution for $x_0 \in \gamma$. There exists $\alpha_0 > 0$ such that it has at least one solution defined on $(-\alpha_0, \alpha_0)$. Suppose by contradiction that $\sigma_1, \sigma_2 : (-\alpha_0, \alpha_0) \to M$ are solutions of $(*)$. For every $0 < \alpha < \alpha_0$, consider $\gamma_\alpha$ the set of points $x \in M$ such that there exists a solution $\sigma_x$ of $(*)$ with $\sigma_x(\alpha) = x$. Note that $\gamma_\alpha$ is a closed connected set in $M$ (see [Sot79],Appendix). Now, consider for $a > 0$ small enough given in Lemma 3.21,

$$\|Df\mid_{E(y)}\|, \|Df\mid_{E(x)}\| < a, \text{ if } \; x, y \in B(S_f, \alpha_0)$$

or

$$\|Df\mid_{E(x)}\| \leq (1+a)\|Df\mid_{E(y)}\|, \text{ if } \; x \notin B(S_f, \alpha_0).$$

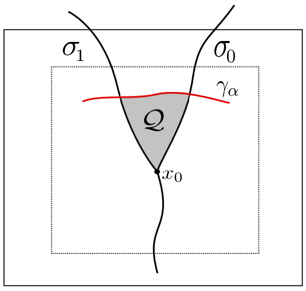


Figure 3.1: Bounded region $\mathcal{Q}$

We fix $0 < \alpha < \alpha_0$ and $\gamma_\alpha$ as above, then we denote by $\mathcal{Q}$ the region bounded by $\sigma_0, \sigma_1$ and $\gamma_\alpha$ as in figure 3.1.
Then, the diameter of the set $f^n(\mathcal{Q})$ goes to zero as $n$ goes to infinity. For any point $x \in \mathcal{Q}$, $d(f^n(x), f^n(x_0))$ goes to zero uniformly, because

$$\begin{aligned} d(f^n(x), f^n(x_0)) &\leq \ell(f^n \circ \sigma_x) \leq \|Df^n \mid_E (x_0)\|(1+a)^n \ell(\sigma_x) \\ &\leq C\lambda^n(1+a)^n\alpha \leq C\lambda'^n\alpha. \end{aligned}$$

Now, we can take an open set $B$ contained in $\mathrm{int}(\mathcal{Q})$ such that the distance between the closure of $B$ and $M \backslash \mathrm{int}(\mathcal{Q})$ is larger than $\varepsilon > 0$. Then, by observation above and the transitivity of $f$, it follows that we can choose $n \geq 1$ such that the diameter of $f^n(\mathrm{int}(\mathcal{Q}))$ is smaller than $\varepsilon > 0$ and $f^n(\mathrm{int}(\mathcal{Q})) \cap B \neq \emptyset$. Thus, $f^n(\mathrm{int}(\mathcal{Q})) \subseteq \mathcal{Q}$. Contradicting the transitivity. $\square$

Denote by $W^s_\varepsilon(x)$ the set $\{\sigma_x(t) : t \in (-\varepsilon, \varepsilon)\}$. In particular, note that

$$W^s_\varepsilon(x) = \{y \in B(x, \varepsilon) \subseteq M : d(f^n(x), f^n(y)) \to 0, \text{ as } n \to +\infty\}.$$

We can consider the box

$$W^s_\varepsilon(\gamma) = \bigcup_{x \in \gamma} W^s_\varepsilon(x).$$

It is an open set.

The next result shows that the existence of $\delta$-$E$-arc is an obstruction for transitivity.

**Theorem 3.23.** *There exists $\delta_0 > 0$ such that if $\gamma$ is a $\delta$-$E$-arc with $0 < \delta \leq \delta_0$, then one of the following properties holds:*

(1) $\omega(\gamma) \subseteq \tilde{\beta}$, *where $\tilde{\beta}$ is a periodic simple closed curve normally attracting.*

(2) *There exists a normally attracting periodic arc* $\tilde{\beta}$ *such that* $\gamma \subseteq W^s_\varepsilon(\tilde{\beta})$.

(3) $\omega(\gamma) \subseteq \mathrm{Per}(f)$, *where* $\mathrm{Per}(f)$ *is the set of periodic points of* $f$. *Moreover, one of the periodic points is either a semi-attracting periodic point or an attracting one (i.e., the set of points* $y \in M$ *such that* $d(f^n(p), f^n(y)) \to 0$ *contains an open set in* $M$*).*

*Proof.* Define $\gamma_n := f^n(\gamma)$. Since $f$ is transitive, we have that there exists $n_0 \geq 1$ such that

$$W^s_\varepsilon(\gamma) \cap W^s_\varepsilon(\gamma_{n_0}) \neq \emptyset.$$

If $\ell(\gamma_{kn_0})$ goes to zero as $k$ goes to infinity, then $\omega(\gamma)$ consist of a periodic orbit.

Indeed, if $\ell(\gamma_{kn_0}) \to 0$ , then $\ell(\gamma_n) \to 0$ as $n \to \infty$. Let $p$ be an accumulation point of $\gamma_{kn_0}$. That is, there exist a subsequence $(k_j)_j$ and $x \in \gamma$ such that $f^{k_j n_0}(x) \to p$. In particular, as $\ell(\gamma_n) \to 0$, one has $\gamma_{k_j n_0} \to p$ as $j \to \infty$, and by the property $W^s_\varepsilon(\gamma) \cap W^s_\varepsilon(\gamma_{n_0}) \neq \emptyset$, it follows that the limit is independent of the subsequence $(k_j)_j$, and so, we have $\gamma_{kn_0} \to p$ as $k \to \infty$. Hence, $\gamma_{kn_0+r} \to f^r(p)$ for $0 \leq r \leq n_0 - 1$, implying that $p$ is a periodic point. Thus, for any $x \in \gamma$ we have that $\omega(x)$ consist only of the periodic orbit of $p$. This proves item (3).

If $\ell(\gamma_{kn_0})$ does not go to zero as $k$ goes to infinity, then there exists $(k_j)_j$ such that $\gamma_{k_j n_0} \to \beta$, where $\beta$ is an arc which is at least $C^1$ and tangent to $\mathscr{C}^*_E$, since

$$\gamma'(t^-) = \lim_{s \to 0 (s<0)} \frac{\gamma(t+s) - \gamma(t)}{s} \quad \text{and} \quad \gamma'(t^+) = \lim_{s \to 0 (s>0)} \frac{\gamma(t+s) - \gamma(t)}{s}$$

belong to $\mathscr{C}^*_E$, hence $\lim_{j \to \infty} Df^{k_j n_0}(\gamma'(t^-)) = \lim_{j \to \infty} Df^{k_j n_0}(\gamma'(t^+))$. Note that $\beta' = f^{n_0}(\beta)$ is the limit of $f^{n_0(1+k_j)}(\gamma)$. Moreover, $\beta \cup \beta'$ is a $C^1$-curve. Let

$$\tilde{\beta} = \bigcup_{k \geq 0} f^{kn_0}(\beta).$$

Then, there exist two possibilities: either $\tilde{\beta}$ is an arc or a simple closed curve. To prove this, notice that $f^{kn_0}(\beta)$ is a $\delta$-$E$-arc for every $k \geq 0$. In particular, for each $x \in \tilde{\beta}$ there exists $\varepsilon(x) > 0$ such that $W^s_{\varepsilon(x)}(x)$ is the local stable manifold for $x$. Thus,

$$W^s_{\varepsilon(x)}(\tilde{\beta}) = \bigcup_{x \in \tilde{\beta}} W^s_{\varepsilon(x)}(x)$$

is a neighborhood of $\tilde{\beta}$. We only have to show that, given $x \in \tilde{\beta}$, there exists a neighborhood $B(x)$ of $x$ in $M$ such that $B(x) \cap \tilde{\beta}$ is an arc. This implies that $\tilde{\beta}$ is a simple closed curve or an interval. Thus, take $x \in \tilde{\beta}$, in particular $x \in f^{k_1 n_0}(\beta)$. Take $I$ an open interval in $f^{k_1 n_0}(\beta)$ containing $x$ and let $B(x)$ be a neighborhood of $x$ such that $B(x) \subseteq W^s(\tilde{\beta})$ and $B(x) \cap \beta_1 \subseteq I$, where $\beta_1$ is any interval containing $f^{k_1 n_0}(\beta)$ and $\ell(\beta_1) \leq 2\delta_0$ ($\delta_0$ small). Now let $y \in \tilde{\beta} \cap B(x)$. We prove that $y \in I$. There is $k_2$ such that $y \in f^{k_2 n_0}(\beta)$. Since

$$f^{k_1 n_0}(\beta) = \lim_{j \to \infty} f^{k_j n_0 + k_1 n_0}(\gamma), \;\; f^{k_2 n_0}(\beta) = \lim_{j \to \infty} f^{k_j n_0 + k_2 n_0}(\gamma),$$

and both have nonempty intersection with $B(x)$, we conclude that for some $j$ follows that $f^{k_j n_0 + k_1 n_0}(\gamma)$ and $f^{k_j n_0 + k_2 n_0}(\gamma)$ are linked by a local stable manifold. Hence $f^{k_1 n_0}(\beta) \cap f^{k_2 n_0}(\beta)$ is an arc $\beta'$ tangent to $\mathscr{C}_E^*$ with $\ell(\beta') \leq 2\delta_0$. Therefore $y \in B(x) \cap \beta' \subseteq I$ as we wish, completing the proof that $\tilde{\beta}$ is an arc or simple closed curve. Moreover, since $f^{n_0}(\tilde{\beta}) \subseteq \tilde{\beta}$, it follows that for any $x \in \gamma$, $\omega(x)$ is the $\omega$-limit of a point in $\tilde{\beta}$, hence (1) or (2) holds, completing the proof. $\square$

**Corollary 3.24.** *There is no $\delta$-$E$-arc provided $\delta$ small.*

*Proof.* From Theorem 3.23 follows that the $\omega$-limit of a $\delta$-$E$-arc is either a periodic simple closed curve normally attracting, or a semi-attracting periodic point or there exists a normally attracting periodic arc. In any case, it contradicts that $f$ is transitive. $\square$

**Lemma 3.25.** *Given $\delta > 0$, there exists $n_0 \geq 1$ such that for every $E$-arc $\gamma$ with $\delta/2 \leq \ell(\gamma) \leq \delta$, one has that the length of $f^n(\gamma)$ is at least $2\delta$ for some $0 \leq n \leq n_0$.*

*Proof.* Fix $\delta_0 > 0$ given by Theorem 3.23. Suppose by contradiction that there exists $0 < \delta < \delta_0/2$ such that for every $n \geq 0$, there exists an $E$-arc $\gamma_n$ with $\gamma_n' \subseteq \mathscr{C}_E^*$ so that

$$\ell(f^k(\gamma_n)) \leq 2\delta \;\; \text{for every} \;\; 0 \leq k \leq n.$$

As $\gamma_n' \subseteq \mathscr{C}_E^*$, one has that the Lipschitz constant of $\gamma_n$ is uniformly bounded. In particular, the family $\{\gamma_n\}_n$ is uniformly bounded and equicontinuous. That is,

- $d(\gamma_n(t), \gamma_n(0)) \leq \delta$ for every $t \in [0,1]$ and $n \geq 1$;
- $\forall \varepsilon > 0, \exists \nu > 0$ such that for every $n \geq 1$,

$$\forall\, t, s \in [0,1], |t - s| < \nu \Longrightarrow d(\gamma_n(t), \gamma_n(s)) < \varepsilon.$$

Then, by Arzelà-Ascoli's Theorem, up to take a subsequence, $\gamma_n$ converges uniformly to the $2\delta$-$E$-arc $\gamma$, since $\gamma$ is a Lipschitz curve with $\ell(f^k(\gamma)) \leq 2\delta$ and $\gamma' \subseteq \mathscr{C}_E^*$. Contradicting the Corollary 3.24. $\square$

**Lemma 3.26.** *Let $\delta > 0$ and $n_0 \geq 1$ be given by Lemma 3.25. Then, there exists $\varepsilon > 0$ such that for every $E$-arc $\gamma$ with $\ell(\gamma) \geq \delta/2$, one has $\ell(f^k(\gamma)) \geq \varepsilon$ for every $1 \leq k \leq n_0$.*

*Proof.* Suppose that for every $\varepsilon_n > 0$ there exist $E$-arc $\gamma_n$ with $\ell(\gamma_n) \geq \delta/2$ and $1 \leq k_n \leq n_0$ such that $\ell(f^{k_n}(\gamma_n) < \varepsilon_n$. Then, up to take a subsequence, we have that there exists $E$-arc $\gamma$, $\gamma = \lim_{n\to\infty} \gamma_n$, with $\ell(\gamma) \geq \delta/2$ so that $\ell(f^k(\gamma)) = 0$ for some $1 \leq k \leq n_0$. Therefore, there exists $t \in (0,1)$ and $\gamma'(t) \in \mathscr{C}_E^*$ such that $\|Df^k(\gamma'(t)\| = 0$ which contradicts the fact that the dominated splitting. $\square$

To prove Main Theorem 3 we use similar arguments as in [BBI09] and [PT72].

### 3.6.3 Proof of the Main Theorem 3

Let $\mathbb{R}^2$ be the universal covering of $M$ and let $\widetilde{E}$ be the lift of the subbundle $E$ on $\mathbb{R}^2$. In particular, one has that $\widetilde{E}$ is orientable. Considering the metric on $\mathbb{R}^2$ which is the lift of the metric on $M$, we have that the cone field $\mathscr{C}_{\widetilde{E}}^*$ is the lift of the cone field $\mathscr{C}_E^*$ on $\mathbb{R}^2$.

**Lemma 3.27.** *There exist $\varepsilon > 0$ and a constant $C > 0$ such that for any curve $\tilde{\gamma} : [0,1] \to \mathbb{R}^2$ of class $C^1$ such that $\tilde{\gamma}' \subseteq \mathscr{C}_{\widetilde{E}}^*$, we have*

$$area(B(\tilde{\gamma}, \varepsilon)) \geq C\ell(\tilde{\gamma}),$$

*where $B(\tilde{\gamma}, \varepsilon) = \{\tilde{x} \in \mathbb{R}^2 : d(\tilde{x}, \tilde{\gamma}) < \varepsilon\}$.*

*Proof.* We prove first that $\tilde{\gamma}$ is injective. Moreover, there exists $\varepsilon > 0$ such that the ball $B(x, \varepsilon)$, centered in $x \in \tilde{\gamma}$ and radio $\varepsilon$, intersects $\tilde{\gamma}$ just once.

Indeed, suppose $\tilde{\gamma}(0) = \tilde{\gamma}(1)$. Let $D$ be a disk such that its boundary $\partial D$ is the curve $\tilde{\gamma}$. Since $\widetilde{E}$ is orientable and transversal to $\partial D$, we can define a non-vanishing field on $D$. Then, by Poincaré-Bendixson Theorem, it has a singularity in $D$ which is a contradiction. Therefore, $\tilde{\gamma}$ is injective.

Fix $\varepsilon > 0$ small enough such that the tangent curve to $\widetilde{E}$ passing through the point $x$ divides $B(\tilde{x}, \varepsilon)$ in two connected components. It is possible, because $\widetilde{E}$ induces a continuous vector field on $\mathbb{R}^2$ and it is bounded. Now, suppose that $\tilde{\gamma}(t_1) \in B(\gamma(t_0), \varepsilon)$ for some $0 \leq t_0 < t_1 \leq 1$. Since $\tilde{\gamma}' \subseteq \mathscr{C}_{\widetilde{E}}^*$, we can take a disk $D$ such that the distribution $\widetilde{E}$ induce a continuous vector

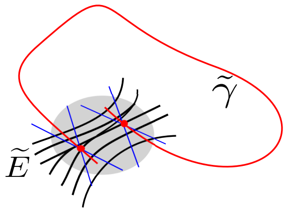


Figure 3.2: Distribution $\widetilde{E}$ and the disk $D$

field on $D$ ($D$ is a disk whose boundary is the union of a tangent curve to $\widetilde{E}$ from $\tilde{\gamma}(t_1)$ to $\tilde{\gamma}(t_0)$ and $\tilde{\gamma}$). Then, repeating the same arguments one gets, by Poincaré-Bendixson Theorem, that such vector field has a singularity in $D$, which is a contradiction. Therefore, we conclude that there exists $\varepsilon > 0$ such that $\tilde{\gamma}$ intersects $B(x, \varepsilon)$ at most once. Up to changing $\varepsilon$, we can assume that any $C^1$-curve tangent to the cone field with length larger than $\ell_0$ is not contained in a ball of radio $\varepsilon$.

Finally, we can prove the lemma. Assume $\ell(\tilde{\gamma}) \gg \ell_0$. Then, we consider $k \geq 1$ the largest integer less than or equal to $\ell(\tilde{\gamma})/\ell_0$ and the set $\{\tilde{x}_1, ..., \tilde{x}_k\}$ contained in $\tilde{\gamma}$ such that the curve $\tilde{\gamma}_j$ in $\tilde{\gamma}$ that passes through $\tilde{x}_j$ has length $\ell_0$ and $\{B(\tilde{x}_j, \varepsilon/2)\}_j$ are two-by-two disjoints. Thus, we have

$$area(B(\tilde{\gamma}, \varepsilon)) \geq \sum_{1 \leq j \leq k} area(B(\tilde{x}_j, \varepsilon/2)) \geq C_0 \frac{\ell(\tilde{\gamma})}{2\ell_0},$$

where $C_0$ is the area of the ball of radio $\varepsilon/2$. Therefore, taking $C = \frac{C_0}{2\ell_0}$, one has

$$area(B(\tilde{\gamma}, \varepsilon)) \geq C\ell(\tilde{\gamma}).$$

$\square$

*Proof of Main Theorem* 3. Let $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ be a lift of $f$. Then, there exists a unique square matrix $L$ with integers entries such that $\tilde{f} = L + \phi$, where $\phi$ is $\pi_1(M)$-periodic map (that is, $\phi(\tilde{x} + v) = \phi(\tilde{x})$ for every $v \in \pi_1(M)$ and $\tilde{x} \in \mathbb{R}^2$). Assume by contradiction that the absolute value of all eigenvalues of $L$ are less than or equal to one. Thus, the diameter of the images of any compact set under the iterates of $\tilde{f}$ grows polynomially.

We now apply this observation to the ball $B_n$ of center $\tilde{x}_n \in \tilde{\gamma}_n$ and radio the diameter of $\tilde{\gamma}_n$ plus $\varepsilon$, where $\tilde{\gamma}_n$ is the image by $\tilde{f}^n$ of a $C^1$-curve $\tilde{\gamma}$ with $\tilde{\gamma}' \subseteq \mathscr{C}^*_{\widetilde{E}}$ and $\varepsilon > 0$ is given by Lemma 3.27. Note that $B_n$ contains the neighborhood $B(\tilde{\gamma}_n, \varepsilon)$ of $\tilde{\gamma}_n$. Then, the area of $B_n$ grows polynomially, implying that the diameter of $\tilde{\gamma}_n$ grows polynomially. This is a contradiction, because, by Lemmas 3.25 and 3.26, we have that $\ell(\tilde{\gamma}_n)$ grows exponentially and, by Lemma 3.27, we have that

$$area(B(\tilde{\gamma}_n, \varepsilon)) \geq C\ell(\tilde{\gamma}_n).$$

Therefore, $L$ has at least one eigenvalue with modulus larger than one. $\square$

### 3.6.4 Proof of Theorem D

Let $f \in \mathrm{End}^1(\mathrm{M})$ be a robustly transitive endomorphism. So, we have two possibilities, either $f$ admits a dominated splitting or not. Suppose $f$ admits a dominated splitting, then, Main Theorem 3 implies that $f$ is homotopic to a linear map having at least one eigenvalue with modulus larger than one, proving our assertion. Now, assume $f$ does not admit a dominated splitting. If $S_f$ the critical set of $f$ is nonempty, from Lemma 3.8 follows that there exists $g$ sufficiently close to $f$ such that $S_g$ has nonempty interior. Hence, Theorem A implies that $g$ admits a dominated splitting, and so, by the same argument before, $g$ is homotopic to a linear map that has at least one eigenvalue with modulus larger than one; since $g$ and $f$ are close, they are homotopic and therefore $f$ is also homotopic to a linear map having at least one eigenvalue with modulus larger than one. Finally, if the critical set $S_f$ is empty, then $f$ is a local diffeomorphism, and [LP13] proved that $f$ is volume expanding. Thus, using the same arguments as in the proof of Main Theorem 3, we have that if the absolute value of all eigenvalues of $L$ are less or equal to one, then the area of a ball $B$ grows polynomially, contradicting the fact that $f$ is volume expanding. $\square$

# Chapter 4

# Examples of robustly transitive endomorphisms

This chapter is another joint work with C. Lizana. In context of diffeomorphisms the first examples of non-hyperbolic robustly transitive diffeomorphisms were given by M. Shub ([Shu71]) in $\mathbb{T}^4$ and by R. Mañé ([Mañ78]) in $\mathbb{T}^3$. In the endomorphisms context, the first examples appear in [LP13] and [HG13]. In both cases, the endomorphisms does not admit critical points. For endomorphisms admitting critical points the first example was constructed in [BR13] and later in [ILP16] both on $\mathbb{T}^2$.

In This chapter, we present new examples of robustly transitive endomorphisms with critical points. We construct a new example on the torus complementing the classes of example constructed in [BR13] and [ILP16].

## 4.1 Preliminaries

### 4.1.1 Iterated function systems-(IFS)

In this section we introduce a very useful tool known as Iterated Function System that we will apply in the following sections.

More concretely, given $f_1, \ldots, f_n : \mathbb{S}^1 \to \mathbb{S}^1$ orientation preserving maps, not necessarily invertible, we defined as *Iterated Function System*, IFS for short, the set $<f_1, \ldots, f_n>$ of all possible finite compositions of $f_i's$, that is,

$$<f_1, \ldots, f_n> := \{h : h = f_{i_m} \circ \cdots \circ f_{i_k} \circ \cdots \circ f_{i_1},\, i_k \in \{1, \ldots, n\},\, m \in \mathbb{N}\}.$$

The *orbit* of $x$ is given by $\mathcal{O}(x) := \{h(x) : h \in <f_1, \ldots, f_n>\}$. A subset $I$ of $\mathbb{S}^1$ is *minimal* if $\overline{\mathcal{O}(x)} \supset I$ for every $x \in I$. Note that this is equivalent to say that for every $x \in I$ and $J \subset I$ open interval, there exists $h \in <f_1, \ldots, f_n>$ such that $h(x) \in J$.

**Example 4.1.1.** *Fix $\epsilon, \delta_0 > 0$ small enough and $\mu \in \mathbb{N}$. Consider the $C^1$ orientation preserving local homeomorphisms $f_0 : \mathbb{S}^1 \to \mathbb{S}^1, f_0(x) = \mu x \pmod 1$ and $f_1, f_2 : \mathbb{S}^1 \to \mathbb{S}^1$ of topological degree $\mu$ defined as follows (see figure 4.1):*

(1) *There exist $p_0^i$ attractor fixed points and $p_1^i, p_2^i$ repeller fixed points of $f_i$, for $i = 1, 2$;*

(2) *There exist $I_i = (a_i, b_i)$, $i = 1, 2$, open intervals contained in $(-\frac{3\epsilon}{2}, \frac{3\epsilon}{2})$ such that*

- *$(p_1^1, p_2^1) \cup (p_1^2, p_2^2)$ is contained $(-2\epsilon, 2\epsilon)$;*
- *$I = (p_0^1, p_0^2) \subset I_1 \cap I_2$ and $f_i'(a_i) = f_i'(b_i) = 1, i = 1, 2$;*
- *$f_i \mid_{I_i}$ is contracting, $i = 1, 2$;*
- *$a_2 < -\epsilon < p_0^1 < 0 < p_0^2 < \epsilon < b_1$ and $p_0^1 < f_2(p_0^1) < 1 < f_1(p_0^2) < p_0^2$;*

(3) *$1 - \delta_0 < |f_i'(x)| < 1 - \delta_0/2$, with $x \in I$, $i = 1, 2$;*

(4) *$\mu - 2\epsilon < f_i' < \mu + 2\epsilon$ in $\mathbb{S}^1 \backslash (-2\epsilon, 2\epsilon)$, $i = 1, 2$.*

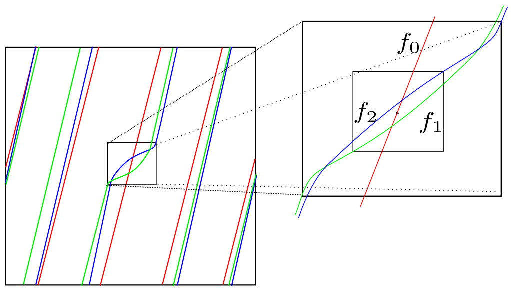


Figure 4.1: The graphs of $f_1$ and $f_2$ with topological degrees $\mu > 1$.

**Remark 4.1.** *We may take $f_i$ $2\epsilon$ $C^0$-close to $f_0(x) := \mu x \pmod 1$, $i = 1, 2$.*

Let us consider $f_{i,0} : \mathbb{S}^1 \to \mathbb{S}^1$ the inverse branch of $f_i$ such that $f_{i,0} \mid_{I_i}$ is expanding. Let us study the IFS generated by them. Note that $p_0^i$ is a repeller fixed point for $f_{i,0}$, with $i = 1, 2$.

### 4.1.2 Homotopy

**Proposition 4.2.** *Given $\delta > 0$, there exists $\epsilon_0 > 0$ such that if $f, g : \mathbb{S}^1 \to \mathbb{S}^1$ are $\epsilon_0$ $C^0$-close of topological degree $\mu \in \mathbb{N}$, there exists an homotopy $H_t$ between $f$ and $g$ such that $|\partial_t H_t| < \delta$.*

*Proof.* Let $\pi : \mathbb{R} \to \mathbb{S}^1$ be the universal covering map. Let $\tilde{f}$ and $\tilde{g}$ be the lift of $f$ and $g$ such that $\tilde{f}$ and $\tilde{g}$ are $\epsilon$ $C^0$-close each other. Then we define the homotopy from $\tilde{f}$ to $\tilde{g}$ by

$$\tilde{H}_t(\tilde{x}) = (1-t)\tilde{f}(\tilde{x}) + t\tilde{g}(\tilde{x}),$$

for every $\tilde{x} \in \mathbb{R}, t \in [0,1]$. Since

$$\begin{aligned} \tilde{H}_t(\tilde{x}+m) &= (1-t)\tilde{f}(\tilde{x}+m) + t\tilde{g}(\tilde{x}+m) \\ &= (1-t)(\tilde{f}(\tilde{x}) + \mu m) + t(\tilde{g}(\tilde{x}) + \mu m) = \tilde{H}_t(\tilde{x}) + \mu m, \end{aligned}$$

for every $\tilde{x} \in \mathbb{R}$, $m \in \mathbb{Z}$, we have that the homotopy $H_t(x) = \pi \circ \tilde{H}_t(\tilde{x})$ between $f$ and $g$ is well defined. Furthermore,

$$|\partial_t H_t| \leq (\max_{\tilde{x}\in\mathbb{R}} |D\pi(\tilde{x})|)|\partial_t \tilde{H}_t| \leq (\max_{\tilde{x}\in\mathbb{R}} |D\pi(\tilde{x})|)\epsilon.$$

Since $|\partial_t \tilde{H}_t| = |\tilde{f}(\tilde{x}) - \tilde{g}(\tilde{x})| < \epsilon_0$ and $\max_{\tilde{x}\in\mathbb{R}} |D\pi(\tilde{x})|$ is bounded, we may chose $\epsilon_0 > 0$ small enough such that $|\partial_t H_t| < \delta$ finishing the proof. □

## 4.2 Construction of the examples

### 4.2.1 Construction of the skew-product map on $\mathbb{T}^2$

Let $L$ be a matrix with spectrum $\sigma(L) = \{\lambda, \mu\}$, $\lambda, \mu \in \mathbb{Z}$ and $|\lambda| > |\mu| > 1$. After a change of coordinates, if necessary, we may assume that

$$L = \begin{pmatrix} \mu & 0 \\ 0 & \lambda \end{pmatrix}.$$

Consider $\epsilon, \delta_0$ and $f_1, f_2$ as the Example 4.1.1. Since the map $f_0(x) = \mu x \,(\mathrm{mod}\, 1)$ is transitive, there exists a residual set of points with dense forward orbit. Hence, given $\alpha_i \in (\mu - \delta_0, \mu + \delta_0)$, $i = 1, 2$, there exists $\beta_i$ such that

- $\mathcal{O}^+_\mu(\beta_i) = \{\mu^k \beta_i : k \in \mathbb{N}\}$ is dense in $\mathbb{S}^1$; and
- $f_i$ may be defined in $\mathbb{S}^1 \setminus (p^i_1, p^i_2)$ by the affine map $f_i(x) = \alpha_i x + \beta_i$.

Remember that $f_1$ and $f_2$ have unique attractor fixed points $p_0^1$ and $p_0^2$, respectively, which are contained in $(-\epsilon, \epsilon)$.

The following results show that every orbit of the IFS $< f_0, f_1, f_2 >$ contain the open interval $I = (p_0^1, p_0^2)$. Before, remember that $f_{i,0} : \mathbb{S}^1 \to \mathbb{S}^1$ the inverse branch of $f_i$ such that $f_{i,0} \mid_{I_i}$ is expanding.

**Lemma 4.3.** *For every open interval $J \subset I$, there exists $h \in < f_{1,0}, f_{2,0} >$ such that $p_0^1 \cup p_0^2 \subset h(J)$.*

*Proof.* It is enough to prove that there exists $h \in < f_{1,0}, f_{2,0} >$ such that $h(J)$ contain either $p_0^1$ or $p_0^2$. Because if there exists $h \in < f_{1,0}, f_{2,0} >$ such that $p_0^1 \in h(J)$, since $p_0^1$ is a repeller for $f_{1,0}$ there exists $n \in \mathbb{N}$ such that $f_{1,0}^n(h(J)) \supset I$.

Since $f_{1,0}(p_0^1, f_1(p_0^2)) = I$ and $f_{2,0}(f_2(p_0^1), p_0^2) = I$, given $J$ an open interval contained in $I$ follows that there exists $h \in < f_{1,0}, f_{2,0} >$ such that $h(J)$ contain $f_2(p_0^1)$ or $f_1(p_0^2)$, since $diam(h(J)) > (1 - \frac{\delta_0}{2})^{-m} diam(J)$, where $m \in \mathbb{N}$ is such that $h = f_{i_m} \circ \ldots \circ f_{i_1}$ and $f_{i_k} \in \{f_{1,0}, f_{2,0}\}$. Hence, if $f_2(p_0^1) \in h(J)$ then apply $f_{2,0}$ and we get that $p_0^1 \in f_{2,0} \circ h(J)$. $\square$

The following result shows that the interval $I$ is contained in the closure of the orbit of IFS $< f_0, f_1, f_2 >$ given by example above.

**Proposition 4.4.** *For every $x \in I$, the set $\overline{\mathcal{O}(x)}$ contain the interval $I$.*

*Proof.* Note that by construction of $f_i$, $f_i \mid_{\mathbb{S}^1 \setminus (p_1^i, p_2^i)} = \alpha_i + \beta_i$, one has that for every $x \in \mathbb{S}^1$, there exists $h \in < f_0, f_i >$ such that $h(x) \in (p_1^i, p_2^i)$.

Indeed, suppose that there exists $x \in \mathbb{S}^1$ such that $h(x) \in \mathbb{S}^1 \setminus (p_1^i, p_2^i)$ for every $h \in < f_0, f_i >$. This contradicts the construction, because

$$\begin{aligned} \underbrace{f_0 \circ \cdots \circ f_0}_{n-k} \circ f_i \circ \underbrace{f_0 \circ \cdots \circ f_0}_{k-1}(x) &= f_0^{n-k}(f_i(f_0^{k-1}(x)) \\ &= \mu^{n-1}\alpha_i x + \mu^k \beta_i, \end{aligned}$$

for $1 \leq k \leq n-1$, belongs to the set $\{h(x) : h \in < f_0, f_i >\} \subset \mathcal{O}(x)$. Since, by construction, $\{\mu^k \beta_i : k \in \mathbb{N}\}$ is dense in $\mathbb{S}^1$, we can choose $n \geq 1$ such that $\mu^{n-1}\alpha_i x + \mu^k \beta_i \in (p_1^i, p_2^i)$, and so we can take $h \in < f_0, f_i >$ such that $h(x) \in I$.

Therefore, it is enough to show that the interval $I$ is minimal for the IFS $< f_1, f_2 >$. In order to prove minimality for $< f_1, f_2 <$, it is sufficient to prove that the associated IFS $< f_{1,0}, f_{2,0} >$ has the property that for every $J$ open interval contained in $I$ follows that

$$I \subset \bigcup_{h \in < f_{1,0}, f_{2,0} >} h(J).$$

From this property follows immediately that given any point $x \in I$ and any open interval $J \subset I$, there exists $h \in < f_{1,0}, f_{2,0} >$ such that $x \in h(J)$. Hence, there exists $h' \in < f_1, f_2 >$ such that $h'(x) \in J$. In particular, $\mathcal{O}(x)$ for $< f_1, f_2 >$ is dense in $I$. By Lemma 4.3 we may consider an open interval $J$ containing $p_0^i$ for $i = 1, 2$. Since $f_{i,0} \mid_{I_i}$ is expanding and remembering that $p_0^i$ is a repeller fixed point of $f_{i,0}$, there exists $n \in \mathbb{N}$ such that $f_{i,0}^n(J)$ contain $I$. Thus, we have

$$I \subset \bigcup_{h \in \langle f_{1,0}, f_{2,0} \rangle} h(J).$$

□

Now, we can define the skew-product map on $\mathbb{T}^2$.

Fix $y_1 < y_0 = 0 < y_2 \in \mathbb{S}^1$, $J$ an open interval in $\mathbb{S}^1$ containing $y_i$ for $i = 0, 1, 2$, and $r > 0$ small enough such that $I_r(y_i) = [y_i - r, y_i + r]$ are pairwise disjoint sets for $i = 0, 1, 2$, $I_r(y_i) \subset J \subset \lambda(I_r(y_i)) (\mathrm{mod}\, 1)$ for $i = 0, 1, 2$. Denote by $R_r(y_i) = \mathbb{S}^1 \times I_r(y_i)$ and $R_J = \mathbb{S}^1 \times J$ horizontal strips.

Let us define $F : \mathbb{T}^2 \to \mathbb{T}^2$ by

$$F(x, y) = (f_y(x), \lambda\, y),$$

where $f_y$ is the homotopy, given by Proposition 4.2, satisfying

$$f_y(x) = \begin{cases} f_i(x), & y \in I_r(y_i), i = 0, 1, 2; \\ \\ f(x), & y \in J^c, \end{cases}$$

where we denote $f_0(x) = \mu\, x \,(\mathrm{mod}\, 1)$ by $f$ in $J^c$ for simpleness.

Pick a point $z = (z_1, z_2) \in \mathbb{T}^2 \setminus R_J$. Fix small neighborhood $B_z$ of $z$ in $\mathbb{T}^2 \setminus R_J$ and consider $\theta_0 > 0$ such that $I_{\theta_0}(0) = (-\theta_0, \theta_0) \subset I$ and $F^n(V)$ and $B_z$ are disjoint for $n = 0, \ldots, N_0$, where $N_0$ is such that $\lambda^{N_0} r > 3/2$ and $V := (-\theta_0, \theta_0) \times \mathbb{S}^1$.

Let $\varepsilon$, $\delta > 0$ such that $I_\delta(z_1) \times I_\varepsilon(z_2) \subset B_z$. Consider $\psi : \mathbb{R} \to [0, \mu + \frac{1}{2}]$ $C^\infty$ map such that $\psi(y) = \psi(-y)$, $y = 0$ the unique maximum point, $\mu < \psi(0) < \mu + \frac{1}{2}$ and $\psi(y) = 0$ for $|y| \geq \varepsilon$. Let $\varphi : \mathbb{R} \to [-1, 1]$ be a $C^\infty$ odd map such that $\varphi(0) = 0, \varphi(x) = 0$ for $x \notin [-\delta, \delta]$ defined by $\varphi(x) = \int_{-\delta}^{x} \varphi'(t) dt$ where $\varphi'$ is as in Figure 4.2, and $\min\{\varphi'\} > -\frac{\lambda - \mu}{\mu + 1}$ and $\max\{\varphi'\} = \varphi'(0) = 1$. By slight abuse of notation we will denote $\psi(y)$ and $\varphi(x)$ by $\psi(y - z_2)$ and $\varphi(x - z_1)$ respectively. Now, we define $F_{\varepsilon,\delta} : \mathbb{T}^2 \to \mathbb{T}^2$ by

$$F_{\varepsilon,\delta}(x, y) = (f_y(x) - \Phi(x, y), \lambda y),$$

where

$$\Phi(x,y) = \begin{cases} \varphi(x)\psi(y), & (x,y) \in I_\delta(z_1) \times I_\varepsilon(z_2); \\ 0, & otherwise. \end{cases}$$

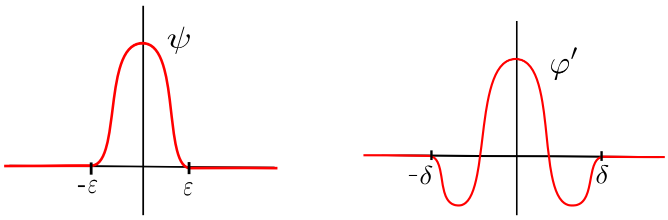


Figure 4.2: The graphs of $\psi$ and $\varphi'$, respectively.

**Remark 4.5.** *Note that $|\varphi|$ goes to zero as $\delta$ goes to zero. Hence, $F_{\varepsilon,\delta}$ goes to $F$ in the $C^0$ topology, when $\varepsilon$ and $\delta$ go to zero. In particular, since $F$ was constructed $C^0$ close to $L$, it follows that $F_{\varepsilon,\delta}$ is $C^0$ close to $L$. In consequence, $F_{\varepsilon,\delta}$ is homotopic to $L$.*

By slight abuse of notation, we will denote from now on by $F$ the map $F_{\varepsilon,\delta}$ defined above.

**Proposition 4.6.** *There exists a neighborhood $\mathcal{W}_F$ such that for every $G \in \mathcal{W}_F$, the critical set $S_G$ of $G$ is nonempty. Moreover, $G(R_r(y_i)) \supset R_J$*

*Proof.* The critical set of $F$ is

$$S_F = \{(x,y) : \mu - \varphi'(x)\psi(y) = 0\} \subset I_\delta(z_1) \times I_\varepsilon(z_2).$$

Since $\det(DF(z_1,z_2)) = \lambda(\mu - \varphi'(z_1)\psi(z_2)) < 0$ and far from $z = (z_1, z_2)$ the map is expanding, we guarantee $S_F$ is non empty. Then there exists a $C^1$ neighborhood $\mathcal{W}_F$ of $F$ such that $S_G \neq \emptyset$, it is contained in $I_\delta(z_1) \times I_\varepsilon(z_2)$ for all $G \in \mathcal{W}_F$. Moreover, picking a smaller $\mathcal{W}_F$, if necessary, we may assume that $G(R_r(y_i)) \supset R_J$. $\square$

**Proposition 4.7.** *There exists a $C^1$ neighborhood $\mathcal{W}_F$ of $F$ such that $G^{N_0}(R_r(y_0)) = \mathbb{T}^2$ for every $G \in \mathcal{W}_F$.*

*Proof.* It is clear. Since $\lambda^{N_0}r > 3/2 > 1$ and the image by $G$ of any horizontal strip is a horizontal strip. $\square$

Given $a \in \mathbb{R}$ positive and $p \in \mathbb{T}^2$, we consider $\mathscr{C}^u_a(p) \subset T_p(\mathbb{T}^2)$ *the family of unstable cones* defined by

$$\mathscr{C}^u_a(p) = \{(v_1,v_2) \in T_p(\mathbb{T}^2) : \ |v_1|/|v_2| < a\}.$$

The following lemma shows that it is possible to construct a family of unstable cones for the map $F$. For the statement of the following lemma we use the fact that $|\lambda| > \sqrt{2}$, this follows from $|det(L)| \geq 2$.

**Lemma 4.8** (Existence of unstable cones for $F$)**.** *Given $\varepsilon$, $\delta$, $a > 0$ and $\lambda'$ with $\sqrt{2} \leq \lambda' < |\lambda|$, there exist $a_0 > 0$ and $\delta_1 > 0$ with $0 < a_0 < a$ and $0 < \delta_0 < \delta$ such that if $F = F_{\varepsilon,\delta_0}$, then the following properties hold:*

*(i)* $\overline{DF_p(\mathscr{C}^u_{a_0}(p))} \setminus \{(0,0)\} \subset \mathscr{C}^u_{a_0}(F(p))$, *for every* $p \in \mathbb{T}^2$*;*

*(ii) if* $v \in \mathscr{C}^u_{a_0}(p)$, *then* $|DF_p(v)| \geq \lambda'|v|$*;*

*(iii) if* $\gamma$ *is a curve such that* $\gamma'(t) \subset \mathscr{C}^u_{a_0}(\gamma(t))$, *then*

$$diam(F(\gamma)) \geq \lambda' diam(\gamma).$$

*Proof.* Proof of *(i)*: Given $p = (x, y) \in \mathbb{T}^2$ and $a > 0$, pick $a_0$ such that $0 < a_0 < a$. Let $v = (v_1, v_2) \in \mathscr{C}^u_{a_0}(p)$, then

$$\begin{aligned} DF_p(v_1, v_2) &= \begin{pmatrix} \partial_x f_y(x) - \partial_x\Phi(x,y) & \partial_y f_y(x) - \partial_y\Phi(x,y) \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ &= ((\partial_x f_y(x) - \partial_x\Phi(x,y))v_1 + (\partial_y f_y(x) - \partial_y\Phi(x,y))v_2, \lambda v_2) \\ &= (u_1, u_2). \end{aligned}$$

▶ If $(x, y) \in I_\delta(z_1) \times I_\varepsilon(z_2)$ remembering that $\min\{\varphi'\} > -\dfrac{\lambda-\mu}{\mu+1}$, $\max\{\varphi'\} = \varphi'(z_1) = 1$ and $f_y(x) = f_0(x)$. Then, observing that $\partial_y f_y(x) = 0$, we get that

$$\begin{aligned} \frac{|u_1|}{|u_2|} &= \frac{|(\mu - \varphi'(x)\psi(y))v_1 - \varphi(x)\psi'(y)v_2|}{|\lambda v_2|} \\ &\leq \left|\frac{\mu - \varphi'(x)\psi(y))}{\lambda}\right| \frac{|v_1|}{|v_2|} + \left|\frac{\varphi(x)\psi'(y)}{\lambda}\right| \\ &\leq \left|\frac{\max\{1, \mu - min\{\varphi'\}(\mu + \frac{1}{2})\}}{\lambda}\right| \frac{|v_1|}{|v_2|} + \left|\frac{\varphi(x)\psi'(y)}{\lambda}\right| \\ &\leq a\left|1 - \frac{(\lambda-\mu)}{2\lambda(\mu+1)}\right| + a\left|\frac{\varphi(x)\psi'(y)}{a\lambda}\right|. \end{aligned}$$

By Remark 4.5, $|\varphi| \to 0$ when $\delta \to 0$, and since $|\psi'|$ is bounded, we may choose $\delta$ small enough such that

$$\frac{|u_1|}{|u_2|} < a.$$

▶ If $y \in J \setminus \cup_{i=0}^{2} I_r(y_i)$, $f_y$ is the homotopy between $f_i$ and $f_0$ given by Proposition 4.2. Without loss of generality we prove the case that $f_y$ is the homotopy between $f_1$ and $f_0$, the other case is analogous.

Given $\delta > 0$ fix $\epsilon_0$ given by Proposition 4.2, so that $f_1$ is $2\epsilon_0$ $C^0$ close to $f_0$, $|\partial_x f_y(x)| < \mu + 2\epsilon_0$ and $|\partial_y f_y(x)| < \delta$. Hence,

$$\begin{aligned}\frac{|u_1|}{|u_2|} = \frac{|\partial_x f_y(x)v_1 + \partial_y f_y(x)v_2|}{|\lambda v_2|} \quad &\leq \quad \left|\frac{\partial_x f_y(x)}{\lambda}\right| \frac{|v_1|}{|v_2|} + \left|\frac{\partial_y f_y(x)}{\lambda}\right| \\ &\leq \quad \left|\frac{\mu + 2\epsilon_0}{\lambda}\right| \frac{|v_1|}{|v_2|} + \frac{\delta}{|\lambda|}.\end{aligned}$$

Fixing $\delta > 0$ small enough such that

$$\left|\frac{\mu + 2\epsilon_0}{\lambda}\right| + \frac{\delta}{a|\lambda|} < 1$$

follows that

$$\frac{|u_1|}{|u_2|} < a.$$

▶ If $y \in I_r(y_i)$, for $i = 1,\ 2$, we have $f_y(x) = f_i(x)$. Then, $|f_i'(x)| < \mu + 2\epsilon_0$.

$$\frac{|u_1|}{|u_2|} = \frac{|\partial_x f_y(x)v_1 + \partial_y f_y(x)v_2|}{|\lambda v_2|} \quad \leq \quad \left|\frac{f_i'(x)}{\lambda}\right| \frac{|v_1|}{|v_2|} \leq \left|\frac{\mu + 2\epsilon_0}{\lambda}\right| \frac{|v_1|}{|v_2|} < a.$$

▶ In the other cases, noting that $F(x, y) = (\mu x, \lambda y)$, it follows the result by straightforward calculation. This prove $(i)$.

Proof of $(ii)$: Let $\lambda'$ be such that $\sqrt{2} \leq \lambda' < |\lambda|$. Note that

$$\begin{aligned}\left(\frac{|DF_{(x,y)}(v_1, v_2)|}{|\lambda'(v_1, v_2)|}\right)^2 \quad &= \quad \frac{|(\partial_x f_y(x) - \partial_x \Phi(x, y))v_1 + (\partial_y f_y(x) - \partial_y \Phi(x, y))v_2, \lambda v_2)|^2}{(\lambda')^2 |(v_1, v_2)|^2} \\ &= \quad \frac{\lambda^2 + \left((\partial_x f_y(x) - \partial_x \Phi(x, y))\frac{v_1}{v_2} + (\partial_y f_y(x) - \partial_y \Phi(x, y))\right)^2}{(\lambda')^2 \left(1 + \left(\frac{v_1}{v_2}\right)^2\right)} \\ &\geq \quad \frac{\lambda^2}{(\lambda')^2(1 + a^2)}.\end{aligned}$$

Since $\dfrac{|v_1|}{|v_2|} < a$. Therefore, taking $a_0 > 0$ small enough, we have

$$\left(\frac{|DF_{(x,y)}(v_1, v_2)|}{|\lambda'(v_1, v_2)|}\right)^2 \geq \frac{\lambda^2}{(\lambda')^2(1 + a_0^2)} > 1.$$

Proof of $(iii)$. It follows from the previous items. □

**Lemma 4.9.** *There exists $\mathcal{W}_F$ a $C^1$ neighborhood of $F$ such that for every $G \in \mathcal{W}_F$ the properties $(i)$, $(ii)$ and $(iii)$ of* Lemma 4.8 *hold.*

*Proof.* The proof follows from observing that $(i)$, $(ii)$ and $(iii)$ of Lemma 4.8 are open properties. □

Given $x \in \mathbb{S}^1$, we define a *vertical curve is defined by $\gamma_x := \{x\} \times J$, where $J$ is fixed at the beginning of the section. Let $J_V$ be the set $V \cap R_J$.*

*In the following results we prove that every vertical curve in $R_J$ has dense forward orbit in $J_V$ and this is an open property. In fact,*

**Lemma 4.10.** *For every vertical curve $\gamma_x$ in $R_J$ with $x \in I_{\theta_0}(0)$ holds that $J_V \subset \overline{\bigcup_{h \in <f_1, f_2>} \gamma_{h(x)}}$. In particular, $J_V \subset \overline{\bigcup_{n \geq 0} F^n(\gamma_x)}$.*

*Proof.* Observing that $\gamma_{h(x)} \subset F^n(\gamma_x)$, where $h(x) = f_{i_n} \circ \cdots \circ f_{i_1}(x)$ and $i_j \in \{1, 2\}$, we get that $\bigcup_{h \in <f_1, f_2>} \gamma_{h(x)} \subset \bigcup_{n \geq 0} F^n(\gamma_x)$. By Lemma 4.3, we have that $\overline{\mathcal{O}(x)} \supset I_{\theta_0}(0)$ following the density of the iterates of the vertical curves. □

*As a consequence of* Lemma 4.10, *we show that every vertical curve in $R_J$ has dense forward orbit in $J_V$, that is, $\gamma_x$ has dense forward orbit in $J_V$ for every $x \in \mathbb{S}^1$.*

**Corollary 4.11.** *Given any vertical curve $\gamma_x$ in $R_J$ holds that $J_V \subset \overline{\cup_{n \geq 0} F^n(\gamma_x)}$.*

*Proof.* It is sufficient to see that $h_0 \in < f_0, f_1, f_2 >$ such that $h_0(x) \in I_{\theta_0}(0)$ to prove our result. This follows by Proposition 4.4. □

*Let us set some notation that we will use for next results. Considering $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$, denote by $\pi_i : \mathbb{T}^2 \to \mathbb{T}^2$ the natural projection onto the ith-coordinate. Observe that* Corollary 4.11 *implies that $J_V \cap \cup_{n \geq 0} F^n(\gamma_x) \neq \emptyset$ for every $x \in \mathbb{S}^1$, hence the projection of $\cup_{n \geq 0} F^{-n}(J_V)$ onto the first coordinate contain $\mathbb{S}^1$, that is,*

$$\pi_1\big(\cup_{n \geq 0} F^{-n}(J_V)\big) = \mathbb{S}^1, \quad \textit{for } i = 1, 2. \tag{4.2.1}$$

*Hence, there exists* $\mathcal{W}'_F \subset \mathcal{W}_F$ *such that* $G \in \mathcal{W}_F$,

$$\pi_1(\cup_{j\leq N} G^{-j}(J_V)) = \mathbb{S}^1 \quad \text{for} \quad i = 1,2. \tag{4.2.2}$$

*Now we are in condition to prove that the property in* Corollary 4.11 *is persistent under perturbations, that is, the density of the forward orbit of every vertical curve in* $R_J$ *in* $J_V$ *is an open property. Concretely,*

**Proposition 4.12.** *There exists* $\mathcal{W}'_F$ *a* $C^1$ *neighborhood of* $F$ *such that for every* $G \in \mathcal{W}'_F$ *holds that* $J_V \subset \overline{\cup_{n\geq 0} G^n(\gamma_x)}$ *for every vertical curve* $\gamma_x$ *in* $R_J$.

*Proof.* Let us assume first the following claim that will be proved after:

*Claim 1:* For every $0 < \theta < \frac{\theta_0}{2}$, there exists $n \in \mathbb{N}$ such that $\bigcup_{i=0}^n F^i(\gamma_x)$ is $\theta$-dense in $J_V$, for every $x \in \mathbb{S}^1$; that is, $\bigcup_{i=0}^n F^i(\gamma_x)$ intersect every open set $U$ in $\mathbb{T}^2$ contained in $J_V$ with length of $\pi_1(U)$ greater than $\theta$.

Fix $0 < \theta < \frac{\theta_0}{2}$ and $n$ given by Claim 1. Then there exists $\mathcal{W}'_F$ a $C^1$ neighborhood of $F$ such that every $G \in \mathcal{W}'_F$ has the property $\bigcup_{i=0}^n G^i(\gamma_x)$ is $2\theta$-dense in $J_V$, for every $x \in \mathbb{S}^1$. We may write $G = (g^1, g^2)$, where $g^i : \mathbb{T}^2 \to \mathbb{S}^1$ satisfy:

- $g^1(x,y) = g^1_y(x)$ is $C^1$ close to $f_i(x)$ for $(x,y) \in R_r(y_i)$ and $i = 0,1,2$;
- $g^2(x,y) = g^2_x(y)$ is $C^1$ close to $\lambda y \,(mod\, 1)$ for $x \in \mathbb{S}^1$.

We have also that the repeller fixed points of $g^1_y$ are close to the repeller fixed points of $f_i$ for $y \in I_r(y_i)$ such that $g^1_y \mid_{(p^i_1 - 2\theta, p^i_2 + 2\theta)}$ is a contracting map for $i = 1,2$.

Let $U$ be an open set in $\mathbb{T}^2$ contained in $J_V$.Consider backward iterates of $U$ by $G$, since the length of $\pi_1(G^{-n}(U))$ grows as $n$ grows, there exists $m$ sufficiently large such that $\pi_1(G^{-m}(U))$ has length greater than $2\theta$. Therefore, by the assumption of $G$ we get that $\cup_{i=0}^n G^i(\gamma_x) \cap G^{-m}(U) \neq \emptyset$, so $\cup_{n\geq 0} G^n(\gamma_x) \cap U \neq \emptyset$. □

*Proof of the Claim 1.* Suppose there exists $\theta > 0$ such that for every $n \in \mathbb{N}$, there exists $x_n \in \mathbb{S}^1$ and $w_n \in J_V$ such that $\cup_{i=0}^n F^i(\gamma_{x_n}) \cap B_\theta(w_n) = \emptyset$. Let $w \in J_V$ and $x \in \mathbb{S}^1$ be accumulation points of $\{w_n\}_n$ and $\{x_n\}_n$ respectively. Take $n_0$ sufficiently large such that $B_\theta(w_n) \supset B_{\theta/2}(w)$ for every $n \geq n_0$. Therefore, $\cup_{n\geq 0} F^n(\gamma_x) \cap B_{\theta/2}(w) = \emptyset$. On the other hand, Lemma 4.10 implies there exist $N$ sufficiently large and $h \in < f_1, f_2 >$ such that $|h| = N$ and $\gamma_{h(x)} \cap B_{\theta/2}(w) \neq \emptyset$, then we may choose $n > N$ and $x_n$ close to $x$ such that $\gamma_{h(x_n)} \cap B_{\theta/2}(w) \neq \emptyset$. Since $\gamma_{h(x_n)} \subset \bigcup_{i=0}^n F^i(\gamma_{x_n})$ follows that $\bigcup_{i=0}^n F^i(\gamma_{x_n}) \cap B_\theta(w_n) \neq \emptyset$, which contradict our assumption, finishing the proof. □

*We say that a curve $\alpha$* cut across $R_J$ *if $\alpha$ intersects both components of the boundary $\partial R_J$ of $R_J$. We are going to prove that this type of curves has dense forward orbits in $J_V$ and this is an open property as well, as we did for the vertical curves. Before proving these assertions, let us prove an auxiliary result that is a little bit technical but it is an analogous version of the Inclination-Lemma(diffeomorphisms setting).*

**Proposition 4.13.** *There exists $\mathcal{W}_F \subset \mathcal{W}'_F$ ($\mathcal{W}'_F$ given* Proposition 4.12*) a $C^1$ neighborhood of $F$ such that for every $G \in \mathcal{W}_F$, $\varepsilon' > 0$, and any curve $\alpha$ cutting across $R_J$ with $\alpha' \subset \mathscr{C}^u_a$, there exist $k \geq 1$, and a family $\{\alpha_n\}_{n\geq 0}$ of curves with $\alpha_n \subset G^n(\alpha)$ cutting across $R_J$ such that the projection of $\alpha_n$ onto the first coordinate is contained in an interval of diameter $\varepsilon'$ for $n \geq k$ (i.e., $\alpha$ is "almost" a vertical curve).*

*Proof.* Let $\mathcal{W}_F$ be a $C^1$ neighborhood of $F$ such that if $G \in \mathcal{W}_F$, $G = (g^1, g^2)$ then $|\partial_y g^1|, |\partial_x g^2| < \eta$ and $|\partial_x g^1| < \mu_0, |\partial_y g^2| > \lambda_0$ where $\mu < \mu_0 < \lambda_0 < \lambda$ in $R_r(y_0)$. Moreover, suppose $G(R_r(y_i)) \supset R_J$ for $i = 0, 1, 2$. Then taking $\alpha_0 = \alpha \cap R_r(0)$ and $\alpha_n$ the connected component of $G(\alpha_{n-1}) \cap R_r(y_0)$, we get that there exists $z_0 \in \alpha_0$ such that $z_n = G^n(z_0) \in \alpha_n$, $z_n = (z^1_n, z^2_n)$, for every $n \geq 0$.

Note that for $(x, y) \in R_r(y_0)$, we have

$$DG(x,y) = \begin{pmatrix} \partial_x g^1 & \partial_y g^1 \\ \partial_x g^2 & \partial_y g^2 \end{pmatrix}.$$

Thus, given $v_0 = (v^c_0, v^u_0)$ an unit vector in $\mathscr{C}^u_a$ with $\rho_0$ the slope of $v_0$, $\rho_0 = \|v^c_0\|/\|v^u_0\|$. Let $v_n = DG(z_n)v_0$ and $\rho_n$ its slope. Then

$$\begin{aligned} \rho_1 &= \frac{\|v^c_1\|}{\|v^u_1\|} = \frac{\|\partial_x g^1 v^c_0 + \partial_y g^1 v^u_0\|}{\|\partial_x g^2 v^c_0 + \partial_y g^2 v^u_0\|} \leq \frac{|\partial_x g^1||v^c_0| + |\partial_y g^1||v^u_0|}{|\partial_y g^2||v^u_0| - |\partial_x g^2||v^c_0|} \\ &\leq \frac{\mu_0\rho_0 + \eta}{\lambda_0 - \eta\rho_0} = \frac{\mu_0\rho_0}{\lambda_0 - \eta\rho_0} + \frac{\eta}{\lambda_0 - \eta\rho_0}. \end{aligned}$$

More general,

$$\begin{aligned} \rho_n &= \frac{\|v^c_n\|}{\|v^u_n\|} = \frac{\|\partial_x g^1 v^c_{n_1} + \partial_y g^1 v^u_{n_1}\|}{\|\partial_x g^2 v^c_{n_1} + \partial_y g^2 v^u_{n_1}\|} \leq \frac{|\partial_x g^1||v^c_{n_1}| + |\partial_y g^1||v^u_{n_1}|}{|\partial_y g^2||v^u_{n_1}| - |\partial_x g^2||v^c_{n_1}|} \\ &\leq \frac{\mu_0\rho_{n-1} + \eta}{\lambda_0 - \eta\rho_{n-1}} = \frac{\mu_0\rho_{n-1}}{\lambda_0 - \eta\rho_{n-1}} + \frac{\eta}{\lambda_0 - \eta\rho_{n-1}}. \end{aligned}$$

Since $\overline{DG(\mathscr{C}^u_a)}\backslash\{(0,0)\} \subset \mathscr{C}^u_a$, we have $\rho_n < a$ and

$$\rho_n \;\leq\; \frac{\mu_0\rho_{n-1}+\eta}{\lambda_0-\eta\rho_{n-1}} \leq \frac{\rho_{n-1}}{b}+\frac{\eta}{\mu_0 b}$$

where $b = (\lambda_0 - a\eta)/\mu_0$. Then,

$$\begin{aligned}\rho_n \;&\leq\; \frac{\mu_0\rho_{n-1}+\eta}{\lambda_0-\eta\rho_{n-1}} \;\leq\; \frac{\rho_{n-1}}{b}+\frac{\eta}{\mu_0 b} \;\leq\; \frac{\rho_{n-2}}{b^2}+\frac{\eta}{\mu_0}\sum_{j=1}^{2}\frac{1}{b^i}\\ &\leq\; \cdots \leq \frac{\rho_0}{b^n}+\frac{\eta}{\mu_0}\sum_{j=1}^{n}\frac{1}{b^i} \;=\; \frac{\rho_0}{b^n}+\frac{\eta(1-b^{-n})}{\mu_0(b-1)}.\end{aligned}$$

Since $R_r(y_0)$ is a compact set, $v_0$ we can be chosen so that $\rho_0$ is the maximum possible slope of unit vectors in $\mathscr{C}^u_a$. Then, changing the neighborhood $\mathcal{W}_F$, if necessary, we may suppose that $b > 1$ and $\rho_n < \delta'$ for every $n \geq k_0$. Thus, all the nonzero tangent vectors to $\alpha_n$ have slope less than $\delta'$ for $n \geq k_0$.

Let us compare the norm of a tangent vector of $\alpha_n$ with its image by $DG$:

$$\frac{\|DGv_n\|}{\|v_n\|} = \frac{\|v_{n+1}\|}{\|v_n\|} = \frac{\sqrt{|v^c_{n+1}|^2+|v^u_{n+1}|^2}}{\sqrt{|v^c_n|^2+|v^u_n|^2}} = \frac{|v^u_{n+1}|}{|v^u_n|}\sqrt{\frac{1+\rho^2_{n+1}}{1+\rho^2_n}}.$$

Then, by the item $(ii)$ of Lemma 4.9, there exists $\lambda' > 1$ such that

$$\frac{\|DGv\|}{\|v\|} \geq \lambda', \forall v \in \mathcal{C}^u_a.$$

Hence, we have that

$$\begin{aligned}\frac{\|v_{n+1}\|}{\|v_n\|} \;&=\; \frac{\|DGv_n\|}{\|v_n\|}\sqrt{\frac{1+\rho_n^2}{1+\rho^2_{n+1}}} \geq \lambda'\sqrt{\frac{(1+\rho_n^2)(\lambda_0-\eta\rho_n)^2}{(\lambda_0-\eta\rho_n)^2+(\mu_0\rho_n+\eta)^2}}\\ &\geq\; \lambda'\sqrt{\frac{(1+\rho_n^2)(\lambda_0-\eta\rho_n)^2}{(\lambda_0)^2+(\mu_0\rho_n+\eta)^2}}.\end{aligned}$$

We may take $W_F$ and $\delta' > 0$ enough small such that

$$\frac{\|v_{n+1}\|}{\|v_n\|} \geq \lambda'\sqrt{\frac{(\lambda_0-\delta')^2}{(\lambda_0)^2}} = \lambda'\left(\frac{\lambda_0-\delta'}{\lambda_0}\right) > 1. \tag{4.2.3}$$

for $n \geq k_0$. Since $\delta'$ depends on $n, \mu_0, \lambda_0$ and $\eta$, we may choose $\delta' > 0$ such that the curves $\alpha_n$ is $\varepsilon'$ $C^1$-close to $\gamma_n$ for $n \geq k_0$ where $\gamma_n$ is the connected component of $\gamma_{z_n^1} \subset R_r(y_0)$ and $(\gamma_{z_n^1})_{n\geq 0}$ is a sequence of vertical curves in $R_J$. $\square$

*As a consequence of the* (Proposition 4.13) *follows that any curve cutting across $R_J$ has dense forward orbit in $J_V$.*

**Lemma 4.14.** *Let $\mathcal{W}_F$ given by proposition above. For every curve $\alpha$ cutting across $R_J$ such that $\alpha' \subset \mathscr{C}_a^u$ holds that $J_V \subset \overline{\cup_{n\geq 0} G^n(\alpha)}$.*

*Proof.* It sufficient to prove that given any open set $U \subset J_V$, there exists $n \geq 1$ such that $G^n(\alpha)$ intersects to $U$. In order to prove that, note that by Proposition 4.12 given an open set $U$ contained in $J_V$ one has that the projection of $G^{-n}(U)$ onto the first coordinate is a interval $U_n \subset \mathbb{S}^1$ such that $\cup_{n\geq 0} U_n$ is a covering of $\mathbb{S}^1$. Then, we can choose $\varepsilon 1 > 0$ the Lebesgue's number of the covering $\{U_n\}$. Finally, by Proposition 4.13, there exist $k \geq 1$ and a family $\{\alpha_n\}_{n\geq 0}$ of curves with $\alpha_n \subset G^n(\alpha)$ cutting across $R_J$ such that the projection of $\alpha_n$ onto the first coordinate is contained in an interval of diameter at most $\varepsilon'$, for $n \geq k$. Therefore, $\alpha_n$ intersects to family $\{U_j\}_j$. That is, $G^n(\alpha)$ intersects to $U$, and so, $\cup_{n\geq 0} G^n(\alpha)$ is dense in $J_V$.

$\square$

*So far we have proved that the forward orbit of any curve cutting across $R_J$ is robustly dense in $J_V$. Now let us prove that $J_V$ is robustly eventually onto, that is, there exists $n_0 \geq 1$ such that $G^{n_0}(J_V) = \mathbb{T}^2$ for $G$ sufficiently close to $F$. We prove first for $F$ and then for an open neighborhood of $F$.*

**Lemma 4.15.** *There exists $N \geq 1$ such that given $G \in \mathcal{W}'_F$, one has $G^n(J_V) = \mathbb{T}^2$.*

*Proof.* Since $\mathbb{S}^1 \times \{y_0\}$ is $F-$invariant and $F$ is expanding in both direction in $R_r(y_0)$, we get that there exists $n_0 \geq 1$ such that $R_r(y_0) \subset F^{n_0}(J_V)$. Hence, by Lemma 4.7 holds that $F^{N_0+n_0}(J_V) = \mathbb{T}^2$. Analogously, we can suppose that for $G \in \mathcal{W}'_F$, $G$ is expanding in both direction in $R_r(y_0)$. Then, there exists $n_1 \geq 1$ such that $R_r(y_0) \subset G^{n_1}(J_V)$, and so, $G^{N_0+n_1}(J_V) = \mathbb{T}^2$. $\square$

*Finally, we can prove that $F$ is robustly transitive.*

*Proof of the robust transitivity.* Given $G \in \mathcal{W}_F$, and $U$ and $B$ open sets in $\mathbb{T}^2$, by Lemma 4.15 there exists $N \geq 0$ such that $J_V \cap G^{-N}(B) \neq \emptyset$. Let $B'$ be an open set of $\mathbb{T}^2$ contained in the interior of $J_V \cap G^{-N}(B)$.

Now consider a curve $\alpha$ in $U$ such that $\alpha' \subset \mathscr{C}_a^u$. Then, by Lemma 4.9, there exists an iterate of $\alpha$ cutting across $R_J$, that is, there exists $m_0 \in \mathbb{N}$ such that $G^{m_0}(\alpha)$ intersect both boundaries of $R_J$. Let $\tilde{\alpha}$ be a connected component of this intersection, so $\tilde{\alpha}$ cuts across $R_J$ and $\tilde{\alpha}' \subset \mathscr{C}_a^u$ by Lemma 4.9, and Lemma 4.14 implies $\overline{\cup_{n\geq 0} G^n(\tilde{\alpha})} \supset J_V$. Then, there exists $k_0 \geq 1$ such that $G^{k_0}(\tilde{\alpha}) \cap B' \neq \emptyset$. Hence, $G^{k_0+N}(\tilde{\alpha}) \cap B \neq \emptyset$. Therefore, it follows that $G^{k_0+N+m_0}(U) \cap B \neq \emptyset$ proving that $G$ is transitive. $\square$

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