Partial hyperbolicity and leaf conjugacy in nilmanifolds

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Motivation

Theorem (Mañe-Franks)

In dimension 2. C^1-Robust transitivity $\Leftrightarrow$ Anosov in $\mathbb{T}^2$.

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In dimension 3:

Theorem (Diaz-Pujals-Ures)

C^1-Robust transitivity implies partial hyperbolicity.

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Topological classification:

Conjecture (Pujals)

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Progress by Bonatti-Wilkinson and Brin-Burago-Ivanov. More recently Hammerlindl.

Transitivity is necessary due to recent example of Hertz-Hertz-Ures.

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In certain cases our results imply uniqueness of attractors and repellers.

Some Definitions

Definition

$f : M^3 \to M^3$ is *strong partially hyperbolic* (SPH) if $TM = E^s \oplus E^c \oplus E^u$ (all dim $= 1$) and $\exists N > 0$:

$$\|Df^N|_{E^s(x)}\| < \|Df^N|_{E^c(x)}\| < \|Df^N|_{E^u(x)}\|$$

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Definition

A SPH diffeomorphism f is *dynamically coherent* if there exists f-invariant foliations $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$ tangent to $E^s \oplus E^c$ and $E^c \oplus E^u$.

$\Rightarrow \exists \mathcal{F}^c$ tangent to E^c.

Main Result:

Theorem (j.w. A.Hammerlindl)

$f : M \to M$ a SPH diffeomorphism. If $M = \mathbb{T}^3$ and f has no attracting nor repelling 2-torus or if $M = Nil$ then:

- *f is dynamically coherent*
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For $M = \mathbb{T}^3$ if f is isotopic to Anosov then we obtain leaf conjugacy to Anosov (in particular with 3 different eigenvalues).

Previous results

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Theorem (Hertz-Hertz-Ures)

There exist open sets of SPH diffeomorphisms of $\mathbb{T}^3$ which are NOT dynamically coherent.

Main ingredients:

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- **Existence of branching foliations and transverse foliations:** Given by Burago-Ivanov's result.
- **Classification of foliations:** Assuming there are no torus leaves, one can classify foliations on $\mathbb{T}^3$ and *Nil*.
- **Global product structure:** Different arguments depending on the isotopy class.
- **In nilmanifolds there are no invariant torus:** Since they must be *Anosov tori* (c.f. Hertz-Hertz-Ures).

Leaf conjugacy in the isotopy class of Anosov

Say $f : \mathbb{T}^3 \to \mathbb{T}^3$ is such that f_* is Anosov and $T\mathbb{T}^3 = E^{ss} \oplus E^{ws} \oplus E^u$.

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This is C^1-robust so it contradicts the work of Dolgopyat-Wilkison.

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