(1) ANOSOV REPRESENTATIONS & DOMINATED SPLITTINGS (j.w. J. Bochi & A. Sambarino)

Γ finitely generated group & $\rho : \Gamma \to G$ representation with G Lie group ($G \subseteq GL(d,\mathbb{R})$)

Goal: Understand representations which are: faithful (inj.) & discrete ($\rho(\Gamma) \subseteq G$ discrete subset)
Particularly interesting is when the embedding $\rho : \Gamma \to G$ is quasi-isométric.

Some examples:
- $\rightarrow$ (Quasi)-Fuchsian representations
- $\rightarrow$ Hitchin representations
- $\rightarrow$ Benoist representations

$\Rightarrow$ Anosov representations (in the sense of Labourie)
- $\rightarrow$ Schottky groups ...
- $\rightarrow$ Those for which $G/\rho(\Gamma)$ is compact

Thm (Weil/Eshermann-Thurston) If $\rho : \Gamma \to G$ is faithful & discrete and $G/\rho(\Gamma)$ is compact $\Rightarrow \forall \rho'$ small deformation of ρ, $G/\rho'(\Gamma)$ is still compact and ρ' is F&D (quasi-isom).

ρ' is a deformation of ρ if $\exists \{\rho_t\}$ continuous path of deformations from ρ to ρ'.

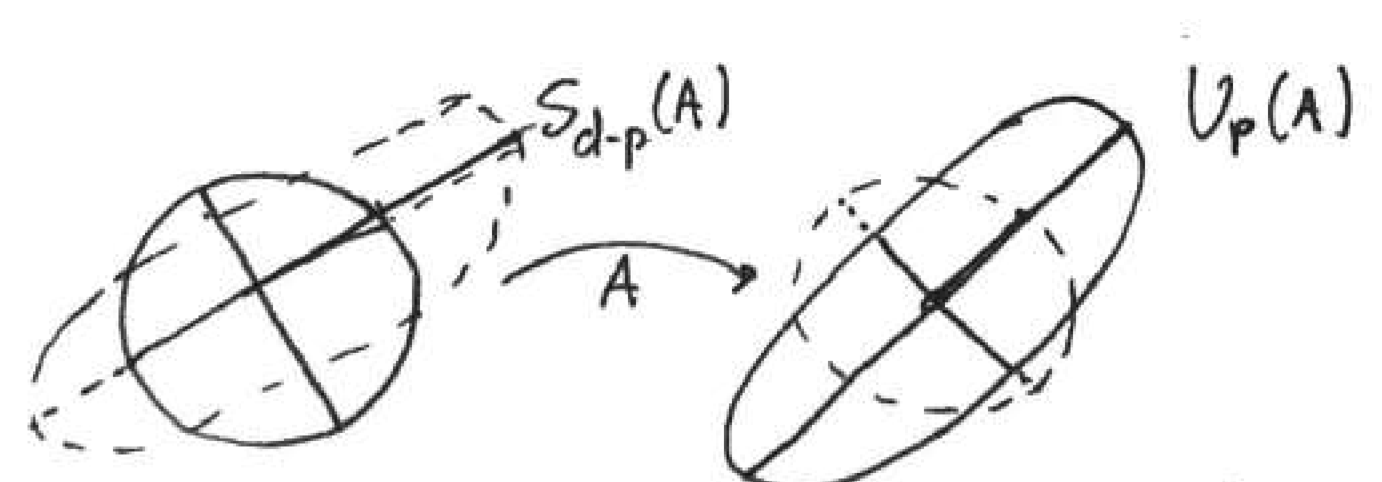
Rmk: This is a transversality result. Generalized by Labourie (for $\pi_1(M)$ with M negatively curved) and Guichard-Wienhard (for general Gromov hyp. groups).
The generalization involves the geodesic flow of Grom. hyp. groups and equivariant maps from $\partial\Gamma \to G/P$ with $P < G$ parabolic.
$\quad$ ← hard to establish its existence.

Makes sense to search for equivalent formulations (GGKW & KLP)

Def $\rho : \Gamma \to GL(d,\mathbb{R})$ is p-dominated if $\exists C, \lambda > 0$ such that if $\gamma \in \Gamma$ one has

$$\frac{\sigma_p(\rho(\gamma))}{\sigma_{p+1}(\rho(\gamma))} > C e^{\lambda|\gamma|}$$

where $\sigma_i(A)$ ith singular value of A
$|\gamma|$ word length of $\gamma \in \Gamma$ resp fixed (sym) generator S.



sing value: $\sqrt{\text{eigenvalue of } A^tA}$

(Rem: No assumption on Γ, just finite gen. set S.)
(Rem: One can do the same for general Lie groups using Cartan's projection...
$p=1$ is the most important case ← Reduced via Tits & GW.)

(2) I will explain the main ideas of our proof (j.w. J. Bochi & A. Sambarino) of the following results (that first appeared in KLP).

Thm 1 The set of p-dominated representations is open. (And all are quasi-isometric.)

Thm 2 If Γ admits a p-dominated representation $\rho : \Gamma \to G$ $\Rightarrow$ Γ is Gromov hyperbolic and ρ is P-Anosov (in the sense of Labourie/Guichard-Wienhard for an adequate parabolic P)
$\hookrightarrow$ in part $\exists$ limit maps from $\partial\Gamma$ to G/P.

Both results use DOMINATED SPLITTING (from smooth dynamics) which we will explain.

QUESTION: Does the interior of the set of discrete & faithful representations from Γ (word hyp) to $GL(d,\mathbb{R})$ contains representations which are not p-Anosov for no $p \in \{1, \dots, d-1\}$.

- GGKW constructs QI repr. from $\mathbb{F}_2$ which is not stable.
- If rank$(G) = 1$, quasi-isometry $\Rightarrow$ p-dominated (convex cocompact)
- Avila-Bochi-Yoccoz: In $SL(2,\mathbb{R})$, for $\mathbb{F}_2$, interior of faithful (and discrete) representation are 1-dominated.
- Sullivan: In $SL(2,\mathbb{C})$ for $\mathbb{F}_2$ (and other Kleinian groups), the interior of faithful (and discrete) representations are convex cocompact (p-dom for some p).
- Goldman: $PSL(2,\mathbb{R})$ for $\pi_1(\Sigma_g)$ non 1-dominated are either non faithful or non discrete.

Not much more results. Related to STABILITY CONJECTURE IN DIFF. DYNAMICS.

§§ DOMINATED SPLITTING: (Mañé ~1970, appears in works on diff eqns from 50's, 60's other name)
X compact metric space & $\{\phi^t : X \to X\}_{t\in\mathbb{T}}$ continuous dynamical system where $\mathbb{T} = \mathbb{Z}$ or $\mathbb{R}$.

Let $E \xrightarrow{\pi} X$ a vector bundle over X (with fixed Riem. metric), we denote $E_x = \pi^{-1}(x)$ the fibers.

An ~~flow~~ action $\{\psi^t : E \to E\}_{t\in\mathbb{T}}$ is called a linear flow over $\{\phi^t\}$ if $\pi \circ \psi^t = \phi^t \circ \pi$ (i.e. $\psi^t : E_x \to E_{\phi^t x}$) and $\psi^t : E_x \to E_{\phi^t x}$ is a linear automorphism.

Examples: $\to$ $E = X \times \mathbb{R}^d$ if $\mathbb{T} = \mathbb{Z}$ $\Rightarrow$ ψ^t is encoded by a choice $A : X \to GL(d,\mathbb{R})$
providing $\psi^1(v) = A(x)v$ if $v \in E_x$. (linear cocycle over $T = \phi^1$)
$\to$ If $E = TM$ and $\phi^1 = f : M \to M$ diffeo $\Rightarrow$ $Df : TM \to M$ is linear flow ...

(3) We say that $\{\psi^t : E \to E\}$ admits a dominated splitting if $\exists\, E = E^{cs} \oplus E^{cu}$ continuous ψ^t-invariant splitting and constants $C, \lambda > 0$ such that

$\forall t > 0$ one has:
$x \in X$

$$\frac{\|\psi^t(v^{cs})\|}{\|v^{cs}\|} \leq \frac{1}{2} \frac{\|\psi^t(v^{cu})\|}{\|v^{cu}\|} \quad \text{where } v^{cs} \in E^{cs}_x \setminus \{0\},\ v^{cu} \in E^{cu}_x \setminus \{0\}.$$

(Rem: Independent of the Riemannian metric.)

The following result was shown by Bochi-Gourmelon generalizing a 2-dim result of Avila-Bochi-Yoccoz

Thm (BG) The following are equivalent:

1) ψ^t admits a dominated splitting with $\dim E^{cu} = p$.

2) $\exists\, C, \lambda > 0$ s.t. $\dfrac{\sigma_p(\psi^t/_{E_x})}{\sigma_{p+1}(\psi^t/_{E_x})} > C e^{\lambda t}$ (σ_i has sense since there is a fixed Riem. metric)

3) $\exists\, \psi^t$-invariant cone field of dimension p.

Sketch: 1) $\Leftrightarrow$ 3) Classical: "Fibered Perron-Frobenius"

1) $\Rightarrow$ 2) Direct.

2) $\to$ 1) Condition 2) implies that $U_p(\psi^t_{\phi^{t}(x)})$ converges uniformly ($\&$ exp) to a continuous ψ^t-inv bundle $E^{cu}(x)$.

Also $S_{d-p}(\psi^{-t}_{\phi^t(x)}) \to E^{cs}(x)$.

One needs to show that $E^{cs}_{(x)} \oplus E^{cu}_{(x)}$ $\forall x \in X$: Use of Oseledets thm. $\boxtimes$

Rem GGKW proves similar statement without use of Oseledets thm (condition CLI) and works for p-adic groups.

§§ PROOF OF THEOREM 1

Quasi-isometry is immediate as the distance in $GL(d, \mathbb{R})$ is comparable to $\log\|A\| - \log\det(A)$. An exponential gap in singular values $\Rightarrow$ translates into quasi-isometry of the embedding.

To show that p-dominated is open we shall assume that Γ is Gromov hyperbolic and use a property of such groups which is not elementary. (discrete vers. of geod. flow)

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Given $\gamma \in \Gamma$ we define its CONE TYPE as $C^+(\gamma) = \{\eta \in \Gamma : |\eta\gamma| = |\eta| + |\gamma|\}$
(we have fixed sym. generating set S).

Fact: If C is a cone type and $a \in S \cap C \Rightarrow aC := C^+(a\gamma)$ is well defined.
(i.e. $C = C^+(\gamma)$ for some γ)

For Gromov hyperbolic groups, as a consequence of the classical Morse-lemma it follows that there exists only finitely many cone types (Cannon).

We can associate to (Γ, S) a graph $\mathcal{G}$ with labels so that:

Vertices: Cone types.
(labeled) edges If $\exists a \in S \cap C_1$ s.t. $aC_1 = C_2$ we have $C_1 \xrightarrow{a} C_2$.

One gets a finite automaton and if

$\Lambda = \{\{a_i\}_{i\in\mathbb{Z}}$ admissible sequence of labeled edges$\}$ one has that

Λ is a shift invariant subset of $S^{\mathbb{Z}}$ and contains (encodes) ALL bi-infinite geodesics passing ~~fa~~ through id. This set is "δ-dense" in the group Γ.

One considers the linear cocycle (linear flow) over $T: \Lambda \to \Lambda$ shift ~~determined~~ by $A: \Lambda \to GL(d,\mathbb{R})$, $\{a_i\} \mapsto \rho(a_0)$

Translating p-domination one gets that condition (2) of BG-Theorem is satisfied. Since condition (3) is clearly an open condition one obtains that closeby representations also give cocycles with dominated splitting. As Λ is "dense" in Γ one recovers the p-domination for closeby representations.

Remark: The conefields also allow to locate the subspaces, as an application we show that boundary maps vary analitically (result of BCLS), Hölder exponents...

§§ PROOF OF THEOREM 2 (just that Γ is Gromov hyperbolic) We take the following as a definition.

(BOWDITCH) Γ is a non-elementary Gromov hyp. group $\Leftrightarrow \exists X$ compact, perfect metric space s.t. Γ acts on X and the diagonal action of $\Gamma \curvearrowright X^{(3)}$ (distinct triples) is properly discontinuous and cocompact.

Remark: In this case $X \cong \partial\Gamma$ equivariantly.

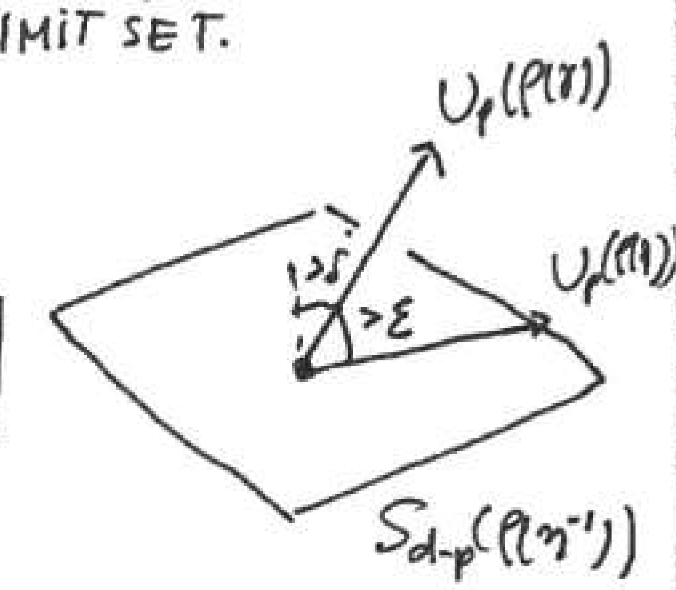
⑤ The proof is elementary but quite involved, I'll just present some key ingredients.

Define $X = \bigcap_{n>0} \overline{\{U_p(\rho(\gamma)) : |\gamma| > n\}} = \{\text{all limits of } U_p(\rho(\gamma_m)) \mid |\gamma_m| \to \infty\}$

↳ Benoist for Zariski-dense / GGKW in general called LIMIT SET.

Lemma: $\forall \varepsilon > 0\ \exists \delta > 0$ such that if $\gamma, \eta \in \Gamma$ are large enough, then:

$$d(U_p(\rho(\gamma)), U_p(\rho(\eta))) > \varepsilon \implies \measuredangle(U_p(\rho(\gamma)), S_{d-p}(\rho(\eta^{-1}))) > \delta$$



This allows to push triples together, etc, etc...

To prove this lemma we use BG again, & consider

$$\mathcal{D}^p_{C,K,\mu} = \left\{ \{A_i\}_{i\in\mathbb{Z}} : \|A_i^{\pm 1}\| \le K,\ \frac{\sigma_p(A_{n+m}\cdots A_n)}{\sigma_{p+1}(A_{n+m}\cdots A_n)} \ge C e^{\lambda m} \right\}$$

(CLI sequences in GGKW) compact & shift invariant subset

The cocycle $A_0 : \mathcal{D}^p_{C,K,\mu} \to GL(d,\mathbb{R})$, $\{A_i\}_{i\in\mathbb{Z}} \mapsto A_0$ has dominated splitting thanks to BG-Thm and this gives <u>good</u> angles for <u>good</u> sequences.

The other ingredient is a linear algebra estimate (that works FOR EVERY group Γ) which is something like:

$$d(\gamma,\eta) \ge \nu(|\gamma| + |\eta|) - c_0 - c_1 \left| \log d(U_p(\rho(\gamma)), U_p(\rho(\eta))) \right|$$

For p-dominated representations, this is enough to conclude.

Remark: Finer angle estimates also allow us to use these results to obtain a Morse-lemma in the symmetric space (recovering other results of KLP).

FINAL REMARK: All this works in general and does NOT use an important characteristic of (Anosov) representations: ONLY FINITELY MANY MATRICES ARE INVOLVED. In GGKW some results make use of this fact (AMS). Also, in ABY some results use a similar fact and contrast with results false in more general setting. Problem: Understand this better.