CLASSIFICATION OF PARTIALLY HYPERBOLIC DIFFEOMORPHISMS IN DIMENSION 3

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Abstract. These are notes for a minicourse given at UFRJ in the conference "A week on dynamical systems at the UFRJ" about the results obtained in [Pot, HP, HP_2]. They are a preliminary version which have not had enough proof-reading, in particular there can be errors of any type (orthographical, typos, and mainly serious mistakes). There is without a doubt a lot of references missing and, as usually happens, the last parts I wrote (which are probably the most important) were written in a hurry and are probably much worse (luckily I did not write in order, so that the worse parts are mixed). Finally, there is another important omission: Figures, I did not include all the figures I would have liked.

The goal of the notes is to present recent work joint with A. Hammerlindl on the topological classification of partially hyperbolic diffeomorphisms on certain 3-dimensional manifolds. We try to present a quite complete panorama of these results as well as the results we use for proving such classification, notably, the work of Brin-Burago-Ivanov and Bonatti-Wilkinson. We do not enter here in other important aspects of the study of partially hyperbolic systems such as stable ergodicity, robust transitivity or absolute continuity and rigidity (we refer the reader to [BuPSW, $RHRHU_1$, BDV, Wi_2]), the course by Keith Burns will certainly treat some of those aspects.

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Contents

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1. Introduction

1.1. **Why study partially hyperbolic systems?** Partially hyperbolic systems have received a lot of attention in the last few decades. We attempt here to explain some of the reasons for this from a personal (and very partial) point of view. In particular, historical claims are not based on any evidence.

In the 60's and 70's the study of dynamics was more or less divided by the point of view of the Russian school (Kolmogorov, Arnold, Sinai,Anosov,Katok,Brin,Pesin....) and Smale's school (Smale, Pugh, Palis, Franks, Newhouse, Bowen, Shub, Mañe...). Of course, both "schools" had intersection of interests, and more importantly, their interests did not cover the whole panorama of dynamical systems. Partial hyperbolicity appeared naturally in both contexts.

1.1.1. *Normal hyperbolicity.* Uniformly hyperbolic systems arose naturally in the study of both ergodicity and structural stability. Anosov systems represent an important class: they are named after Anosov who proved both ergodicity and structural stability of such systems. When studying Lie-group actions (in particular, flows) one can extend the notion of hyperbolicity to have that the action has a transverse hyperbolic structure. In such a way *normally hyperbolic* foliations appear. The same happens when one considers products of diffeomorphisms, or even certain skew-products: If one of the factors is uniformly hyperbolic, one obtains a normally hyperbolic foliation for the diffeomorphism.

In [HPS] these objects were studied in a deep way, the name *partial hyperbolicity* appeared also in [BP] where ergodicity of certain partially hyperbolic diffeomorphisms was studied.

1.1.2. *From a "conservative" point of view, the Russian school.* When studying the space of smooth conservative (which preserve a volume form) dynamics one of the first questions that appear is the question of ergodicity. When does a diffeomorphism of a manifold which preserve a volume form is ergodic? Is this phenomena abundant?. In the negative side, there is what is now called KAM theory (after Kolmogorov, Arnold and Moser) showing that there are open sets of conservative systems which are not ergodic (in high regularity). These non-ergodic examples have in common the lack of hyperbolicity: All *Lyapunov exponents* vanish in a set of positive Lebesgue measure.

Thus, the concept of *non-uniform hyperbolicty* arises naturally, and this was studied strongly by the Russian school, particularly by Pesin who proved for example that in the non-uniformly hyperbolic setting the volume measure has at most countably many ergodic components. The search for using similar techniques as in the Anosov case led them to introduce a semi-uniform version of non-uniform hyperbolicity which was called partial hyperbolicity (zero exponents are allowed, but there are uniform expansions and uniform contractions). See [BP]. These systems also allowed to treat some examples that appeared naturally in other contexts and for which the question of ergodicity was particularly relevant: These examples include frame flows on negatively curved manifolds, Algebraic systems in certain nilmanifolds, time one maps of Anosov flows, etc...

It turned out that some of the members of Smale's school got interested in this kind of problems and in particular, Pugh and Shub ([PuSh] and references therein) developed in the 90's a program to understand the problem of ergodicity in the partially hyperbolic context. This program has been very successful and a lot of development has appeared since but we will not focus in this aspect: See [Wi_2] for a recent survey with this point of view.

1.1.3. *From a non-conservative viewpoint, Smale's school.* In the non-conservative setting the first important question that arises is that of structural stability: Under what conditions the dynamics persists under perturbations?. It is in this setting that hyperbolicity appeared naturally in the work of Smale and soon after Palis and Smale conjectured that hyperbolicity was necessary for structural stability. However, it was soon realized that hyperbolic systems are not dense in the space of diffeomorphisms, and that there exist some obstructions for it. In this sense, partial hyperbolicity appears naturally in two ways:

- Some of the first examples of robustly non-hyperbolic systems were partially hyperbolic: [AS, Sh, Ma] (in contraposition with those of Newhouse [New] which rely on a different mechanism).
- Other robust dynamical behavior, such as robust transitivity always present some similar structures as partial hyperbolicity. This is the aspect we will be more interested in and we will expand on this in the next subsection.

Weak forms of hyperbolicity also appear naturally in the work of Mañé on the C^1-stability conjecture ([Ma_4]).

1.1.4. *More personal reasons.* Partially hyperbolic systems present a Df-invariant geometric structure which is defined globally. Naturally, their study promises a rich interaction between the global topology of the manifold with the underlying dynamics of the system. In the study of partially hyperbolic systems many geometric and topological properties appear naturally as well as a lot of interaction with related subjects such as: foliation theory, topology of manifolds, topological dynamics, ergodic theory, group representations, differential geometry, cocycles, measure theory, etc. The fact that a subject has such rich interactions is usually an indication of its interest.

1.2. **Robust dynamical behavior and partial hyperbolicity.** In dimension 2 it is possible to characterize C^1-robustly transitive diffeomorphisms (i.e. having a C^1-neighborhood such that every diffeomorphism in the neighborhood has a dense orbit). Mañe has shown in [Ma_2] that on any 2-dimensional manifold, a C^1-robustly transitive diffeomorphism must be Anosov. By classical results of Franks ([Fr]) we know that an Anosov diffeomorphism of a surface must be (robustly) conjugated to a linear Anosov automorphism of $\mathbb{T}^2$. Since these are transitive, we obtain that:

Theorem (Mañe-Franks). *If M is a closed two dimensional manifold, then a diffeomorphism f is C^1-robustly transitive if and only if it is Anosov and conjugated to a linear hyperbolic automorphism of $\mathbb{T}^2$.*

In a certain sense, this result shows that in order to obtain a robust dynamical property out of the existence of an invariant geometric structure it may be a good idea to develop some theory on the possible topological properties such a diffeomorphism must have.

More precisely, we identify this result as relating the following three aspects of a diffeomorphism f:

- Robust dynamical properties (in this case, transitivity).
- Df-invariant geometric structures (in this case, being Anosov).
- Topological classification (in this case, M is the two-torus and f is conjugated to a linear Anosov automorphism).

In higher dimensions, the understanding of this relationship is quite less advanced, and we essentially only have results in the sense of showing the existence of Df-invariant geometric structures when certain robust dynamical properties are present (see [DPU, BDP]). In particular, in dimension 3 it was proved in [DPU] that a C^1-robustly transitive diffeomorphism must be partially hyperbolic in a certain sense.

Similar conclusions exist for stably ergodic diffeomorphisms (see [BFP]).

One could hope that a better understanding of partially hyperbolic systems in dimension 3, in particular a topological classification result could shed light into the question of characterizing robust dynamical behavior, much as in the 2-dimensional case. This has yet to be explored although there are some results already, particularly in the conservative setting (see for example [RHRHU_2, HU]).

1.3. **Classification results in the Anosov setting.** In the Anosov case a beautiful theory has been developed which is however far from being finished. There are essentially 3 set of results:

- Franks-Newhouse ([Fr, New_2]): An Anosov diffeomorphism $f : M \to M$ of codimension 1 (either the stable or the unstable bundle is one-dimensional) is transitive and conjugated to a linear Anosov automorphism of $\mathbb{T}^d$. In particular, $M = \mathbb{T}^d$.
- Franks-Manning ([Fr, Man]): If $f : N \to N$ is an Anosov diffeomorphism and N is an *infranilmanifold* (this includes tori) then f is conjugated to a linear automorphism of N.
- Brin-Manning ([BM]): Under some *pinching conditions* all Anosov diffeomorphisms occur in infranilmanifolds.

These results cover completely the classification of Anosov diffeomorphisms in dimensions ≤ 3. However, in higher-dimensional manifolds, the understanding of the topology of manifolds admitting Anosov diffeomorphisms is very vague. For example, I am not aware if the following is known:

Question 1. *Are there any Anosov diffeomorphisms*[1] *in $S^\ell \times S^\ell$? In some simply connected manifold?*

Even if the previous question was already known (for example, it is known that S^ℓ, or $S^\ell \times S^m$ with $\ell \neq m$ do not admit Anosov diffeomorphisms) the fact that already in dimension 4 we have no clue on how to classify Anosov systems in all generality suggests that for the partially hyperbolic case we should first concentrate on the case of dimension 3 if we hope some success.

Example (Non existence of Anosov diffeomorphisms in S^3)**.** We explain here briefly two different ways to see that there are no Anosov diffeomorphisms on the sphere S^3.

[1] I am not really sure this is not known, it is well possible that there is an *ad-hoc* proof for these particular manifolds. For example, a Lefshetz index argument shows that (if the stable and unstable bundles are orientable) the manifold must have some Betti number ≥ 2. In fact, very recently a preprint by Gogolev and Rodriguez Hertz has appeared that shows that $S^2 \times S^2$ among other higher dimensional manifolds do not admit Anosov diffeomorphisms by showing that even if $S^2 \times S^2$ has its second Betti number equal to 2 the action of f in homology cannot have eigenvalues larger than 1.

(i) The first proof, paraphrasing Bowen goes as follows: "One counts periodic points dynamically and topologically and compare the results". Since S^3 is simply connected, the bundles E^s and E^u are orientable. By considering an iterate of f one can assume that Df preserves the orientation of the bundles. One deduces that every fixed point of f^n has the same Lefshetz index (with modulus = 1). On the one hand, we know that the cardinal of $\mathrm{Fix}(f^n)$ goes to infinity with n (from the fact that Anosov diffeomorphisms have exponential growth of periodic points[2]). On the other hand, since f^n must act as the identity in homology (since the only non-trivial homology groups are one-dimensional and f^n is a diffeomorphism) we get that the number of fixed points of f^n remains bounded, a contradiction. This proof generalizes to higher dimensions under certain restrictions on the homology groups of the manifold.

(ii) The second proof involves studying the foliations such manifold can have and the properties of strong stable and strong unstable manifolds. Assume then that $f : S^3 \to S^3$ is an Anosov diffeomorphism with $\dim E^s = 2$ (otherwise, consider f^{-1}). We know that the stable foliation consists of leaves homeomorphic to planes. Due to Novikov's theorem we know that S^3 does not admit such foliation, so, S^3 cannot admit Anosov diffeomorphisms. This argument generalizes to classify codimension one Anosov diffeomorphisms (see the Franks-Newhouse classification theorem above): If a closed manifold M admits a codimension one foliation by planes then M is a torus.

In the partially hyperbolic context in dimension 3, we will follow the philosophy of the second approach.

$\diamond$

Let us mention that in the higher dimensional setting, there has been some classification results under additional hypothesis: See the works by Bonhet [Bo], Carrasco [Car], Gogolev [Go] or Hammerlindl [H].

1.4. **Structure of this notes.** This notes are structured as follows. First, in section 2 we introduce partially hyperbolic systems in dimension 3 giving the most important properties for our purposes as well as plenty of examples. We discuss the fundamental concepts of dynamical coherence and leaf conjugacy and state our main result.

In section 3 we go through the breakthrough results of Brin-Burago-Ivanov trying to explain what is behind their proofs. In particular, we give a proof of the first (and only known) topological obstructions for admitting partially hyperbolic systems and we

[2]In fact, it is also useful in Franks-Manning classification to estimate the exact growth of periodic points. This is done quite cleanly in [KH] Chapter 18.6.

sketch in some detail the fundamental result of [BI] stating the existence of branching foliations tangent to the bundles of the partially hyperbolic splitting.

Section 4 is devoted to study foliations in dimension 3, mainly on torus bundles over the circle. These results are of capital importance in our results but can be used as a black box if the reader is not familiar with foliations.

After this is done, the rest of the notes is devoted to give a proof of our main result. The emphasis is made on the solvmanifold case which is, in my opinion, the most difficult one and the one in which the proof differs most from the previous work.

The notes do not present original material except for, possibly, the way to treat some of the results. In particular, I profited to treat the nilmanifold case with a different point of view than the one of [H_2, HP]. In those papers the treatment is mainly algebraic (viewing nilmanifolds as quotients of the Heissenberg group) while here we treat this case in a more geometric way. I hope this can be useful to someone.

Many of the things that differ from the way they are done in the literature are based on many discussions with a number of people. I tried to recall the names of all of them and wrote them in the footnote in the first page of these notes, however, it is well possible that I am forgetting someone and I apologize for that.

2. Partial hyperbolicity in dimension 3

Here the real notes begin. We will try to focus as soon as possible into concrete settings and not in the maximum generality. Some of the results we present in this set of notes are valid in more general settings, particularly in the case of $\mathbb{T}^3$ but we will not cover this case, see [Pot, Pot_2] for more information and a more general introduction.

2.1. First definitions.

From now on, M will denote a closed[3] 3-dimensional manifold. We will assume that TM is endowed with a Riemannian metric, but we shall usually only use the metric $\|\cdot\|$.

Definition 2.1 (Partial hyperbolicity)**.** Given $f : M \to M$ a C^1-diffeomorphism, we say that f is *partially hyperbolic* if there exists a continuous Df-invariant splitting $TM = E^s \oplus E^c \oplus E^u$ into one-dimensional subbundles which verifies that there exists $N > 0$ such that:

(i) $\|Df^N|_{E^s(x)}\| < \|Df^N|_{E^c(x)}\| < \|Df^N|_{E^u(x)}\|$.
(ii) $\|Df^N|_{E^s(x)}\| < 1 < \|Df^N|_{E^u(x)}\|$

$\diamondsuit$

[3]Compact, connected and without boundary.

In general, such diffeomorphisms are called *strong partially hyperbolic* since both extremal subbundles are non-trivial. Since we shall not work in the general context, we simplify the nomenclature to eliminate a word (see [Pot_2] for a more general introduction).

Condition (i) in the definition is a domination condition. It is important to remark here that this notion of domination is weaker that the one appearing in other literature. Sometimes this concept is called *pointwise* (or *relative*) *domination* in contraposition to *absolute domination* (see [HPS]). We say that a diffeomorphism is *absolutely partially hyperbolic* if moreover there exists constants λ, μ such that

$$\|Df^N|_{E^s(x)}\| < \lambda < \|Df^N|_{E^c(x)}\| < \mu < \|Df^N|_{E^u(x)}\|$$

The pointwise definition is more suitable in the context of studying robust transitivity and stable ergodicity since it is the one given by the results of [DPU, BDP] (pointwise domination also appears naturally in the work on the C^1-stability conjecture [Ma, Ma_2, Ma_4]). The absolute definition can be compared to certain pinching conditions (a condition in the global spectrum of Df), in particular recall Brin-Manning result on the classification of Anosov systems.

These properties are C^1-robust, the following proposition can be found in [BDV] appendix B (see also section 2.3):

Proposition 2.2. *If f is partially hyperbolic then there exist a C^1-neighborhood $\mathcal{U}$ of f such that every $g \in \mathcal{U}$ is partially hyperbolic.*

2.2. **Examples.**

2.2.1. *Anosov diffeomorphisms on $\mathbb{T}^3$.* Consider $A \in SL(3, \mathbb{Z})$ a matrix with 3 different real eigenvalues such that none of them have modulus 1. It is well known that A defines a diffeomorphism of $\mathbb{T}^3$ which is Anosov.

The eigenspaces of A give three invariant bundles over each point which verify the partially hyperbolic condition since they are different. In fact, A is even absolutely partially hyperbolic.

The following is also verified:

- There are three invariant (linear) foliations which are obtained by projection to $\mathbb{T}^3$ of lines parallel to the eigenspaces of A.
- Each of these foliations has all its leaves dense.

Exercise. Consider a matrix $A \in SL(3, \mathbb{Z})$.

(i) Prove that if A has no eigenvalues of modulus 1, then all eigenvalues are different.

(ii) Prove that if A has an eigenvalue of modulus larger than 1 and one eigenvalue of modulus 1, then, there is a vector v such that $Av = \pm 1$. Moreover, show that v has rational slope. Deduce that A is conjugated to a matrix which leaves two invariant subspaces and when considering the corresponding diffeomorphism on the torus is the product of an Anosov diffeomorphism of $\mathbb{T}^2$ with either the identity or minus the identity in S^1.

Similarly, if we consider any Anosov in $\mathbb{T}^3$ such that it preserves a splitting into 3-subbundles, we will be in the conditions of partial hyperbolicity. More interestingly, Mañé ([Ma]) made a deformation of a linear Anosov automorphism in order to obtain a partially hyperbolic diffeomorphism f of $\mathbb{T}^3$ such that the following is verified:

- f is not Anosov (nor Axiom A).
- f is C^1-robustly transitive.
- f preserves three foliations $\mathcal{W}^s$, $\mathcal{W}^c$ and $\mathcal{W}^u$ tangent to the Df-invariant bundles E^s, E^c and E^u respectively. Each of the foliations consists of leaves which are all lines and moreover, the foliation $\mathcal{W}^c$ has all its leaves dense.
- f can be made non-absolutely partially hyperbolic.

These examples are constructed by making a careful modification inside a small region and changing the index of a periodic point without loosing the partial hyperbolicity. The fact that the modification is made in a small region allows to guarantee that transitivity is not lost. Since an Axiom A diffeomorphism verifies that every periodic point in a transitive piece has the same index the resulting diffeomorphism cannot be Axiom A. The change of index, being in a small region, can be made to change dramatically the eigenvalues of the fixed poin in order to have a very strong expansion in the periodic point but still preserve some cone-fields (which guarantee partial hyperbolicity). In this way one can create non-absolutely partially hyperbolic diffeomorphisms[4]. The rest of the results follow from appropriate use of the results in [HPS].

In fact, to this point it is not well understood if, in Mañé's example, the fact that the modification is made in a small region is really necessary. The following question is unknown:

Question 2. *Is there a partially hyperbolic diffeomorphism of $\mathbb{T}^3$ in the isotopy class of an Anosov which is not transitive?*

In fact, the good question should be if it is possible to have an *attracting open set* (i.e. an open set $\emptyset \neq U \neq \mathbb{T}^d$ such that $f(\overline{U}) \subset U$). It may well be possible that there is a

[4]It is even possible to construct Anosov diffeomorphisms in $\mathbb{T}^3$ which split in 3-bundles and are not absolutely partially hyperbolic.

non-transitive diffeomorphism but which is not robustly non-transitive (for example, it could involve Denjoy type phenomena). Let us mention that Abdenur and Crovisier [AbC] have proved that among partially hyperbolic diffeomorphisms having robustly no such open sets, those which are C^1-robustly transitive form a C^1-open and dense subset.

We mention also that in [BV] it is shown that Mañé's example, if done conservative, is also stably ergodic (notice that in the conservative case it is not possible to bifurcate the periodic orbit keeping the center foliation unchanged).

2.2.2. *Skew-products.* A similar example is to consider a matrix $A \in SL(3, \mathbb{Z})$ which has an eigenvalue with modulus equal to 1 but the other two are of different modulus (since the determinant is equal to 1 one should be of modulus larger than one and the other smaller than one). It is not hard to prove that in fact this corresponds to a product: A decomposes as the product of a linear Anosov automorphism of $\mathbb{T}^2$ (given by a hyperbolic matrix in $SL(2, \mathbb{Z})$) and the identity on S^1 (see the exercise above).

Clearly, such examples are not transitive nor ergodic (clearly they preserve Lebesgue measure). However, it has been proved that they can be perturbed to be either robustly transitive or stably ergodic ([BD, BuW_1]).

These examples can be considered in a more general form, still on $\mathbb{T}^3$. Consider a diffeomorphism $F : \mathbb{T}^2 \times S^1 \to \mathbb{T}^2 \times S^1$ of the form:

$$F(x, t) = (f(x), g_x(t))$$

If $f : \mathbb{T}^2 \to \mathbb{T}^2$ is Anosov and the expansion and contraction of $g_x : S^1 \to S^1$ is smaller than the one of f for every $x \in \mathbb{T}^2$ we obtain a partially hyperbolic diffeomorphisms. Such diffeomorphisms are known as *skew-products*. Classical results imply that this structure is preserved by perturbations of F, only that the foliation by circles may cease to be differentiable ([HPS]), in fact, it is remarkable that in the conservative setting a perturbation can make the foliation to be completely "pathological": Each circle leaf intersects a full measure set of $\mathbb{T}^3$ in a finite set of points (see [ShW]; this is now a hot topic of research, see [Wi_2] and references therein for more information on this very interesting subject). In any case, we obtain that F and its perturbates (which are still partially hyperbolic) preserve a foliation by circles tangent to the center direction.

Remark 2.3. Other than the fact that the derivative of g_x is dominated by the contraction and expansion for f, there are no restrictions in the map g_x one can choose in order to get a partially hyperbolic diffeomorphism. This implies that if we want to classify these diffeomorphism from the topological viewpoint, we should treat them all as the same class of example even if their dynamics can be very different.

◊

Exercise. Obtain precise criteria for f and g_x which imply that F is partially hyperbolic. ◊

Skew-products can be further generalized. Instead of working in $\mathbb{T}^3$ one can work on more general circle bundles over the torus. These are called nilmanifolds since their fundamental group is nilpotent. These manifolds can also be seen as torus bundles over the circle where the monodromy is a Dehn-twist. There is a third way to think about these manifolds which is more algebraic: they are quotients of a nilpotent lie group (the Heissenberg group) by some co-compact lattice (we will not work so much with this interpretation, we refer the reader to [H$_2$] for a very complete and self-contained introduction to this point of view).

In fact, if we have such a bundle, given by a projection $p : N \to \mathbb{T}^2$ such that $p^{-1}(\{x\}) \simeq S^1$ we get that a *skew-product* will be any diffeomorphism $f : N \to N$ such that $f(p^{-1}(\{x\})) = p^{-1}(\{g(y)\})$ for some $g : \mathbb{T}^2 \to \mathbb{T}^2$.

In this case obtaining conditions for partial hyperbolicity gets more complicated, but we will present an explicit and very simple example borrowed from [BoW]. For algebraic examples and more information from the algebraic viewpoint we refer the reader to [H$_2$].

Example. Consider a non-trivial bundle $p : N \to \mathbb{T}^2$ over the torus. It is not hard to see that if you take U an open ball in $\mathbb{T}^2$ and V an open set such that $U \cup V = \mathbb{T}^2$ we can find differentiable charts of N (see the Exercise below) such that $\varphi_1 : U \times S^1 \to N$ and $\varphi_2 : V \times S^1 \to N$ verify the following properties:

- $\varphi_1(U \times S^1) \cup \varphi_2(V \times S^1) = N$. Moreover, $p(\varphi_1(x,t)) = x$ for every $x \in U$ and $p(\varphi_2(x,t)) = x$ for every $x \in V$.
- The change of coordinates is by a rotations in the fibers: This means, for $x \in U \cap V$, if $\pi_2 : V \times S^1$ is the projection in the second coordinate we have that the map

$$\psi_x : S^1 \to S^1 \qquad \psi(t) = \pi_2 \varphi_2^{-1}(\varphi_1(x,t))$$

  is a rigid rotation.

If $A : \mathbb{T}^2 \to \mathbb{T}^2$ is an Anosov diffeomorphism then we can write $A = g_2 \circ g_1$ where g_1 is the identity in V and g_2 the identity in U. For this, U and V must be properly be chosen (see the Exercise below).

We can thus define the following maps $G_1 : N \to N$ is defined to be the identity in $V \times S^1$ and $\varphi_1 \circ (g_1 \times Id) \circ \varphi_1^{-1}$ in $U \times S^1$ and similarly $G_2 : N \to N$ as the identity in $U \times S^1$ and $\varphi_2 \circ (g_2 \times Id) \circ \varphi_2^{-1}$ in $V \times S^1$. One can check that $F = G_2 \times G_1$ is a partially hyperbolic diffeomorphism.

$\diamondsuit$

Exercise. (i) Show that every circle bundle over the torus can be decomposed as above. In particular, V can be chosen such that $U \cap V$ is an annulus and the bundle is determined up to homeomorphism by the degree of the map $x \mapsto \psi_x$ from $U \cap V$ to $\mathrm{Homeo}(S^1)$ (which is homotopy equivalent to a circle).

(ii) Show that one can choose two open sets U and V of $\mathbb{T}^2$ with U contractible such that $A = g_2 \circ g_1$ as above.

(iii) Check that the diffeomorphism F defined above is (absolutely) partially hyperbolic.

2.2.3. *Anosov flows.* Given a flow $\varphi_t : M \to M$ of a 3-dimensional manifold, we say it is an *Anosov flow* if (modulo changing the metric) there exists a splitting $TM = E^s \oplus E^0 \oplus E^u$ into 1-dimensional bundles such that $\|D\varphi_t|_{E^0}\| = 1$ and there exist constants $C > 0$ and $\lambda < 1$ such that:

$$\|D\varphi_t|_{E^s}\| < C\lambda^t \qquad ; \qquad \|D\varphi_{-t}|_{E^u}\| < C\lambda^t \quad \forall t \geq 0$$

As the reader can easily notice, the definition is very much related with the definition of partial hyperbolicity. In fact, it is very easy to see that if φ_t is an Anosov flow, then, the diffeomorphism φ_1 is (absolutely) partially hyperbolic. Moreover, it has been proved in [HPS] that there is a C^1-open neighborhood of φ_1 consisting on partially hyperbolic diffeomorphisms which fix a foliation homeomorphic to the foliation by the orbits of the flow φ_t (though the foliation will have C^1-leaves, it may cease to be differentiable, in fact, much worse can happen, see [AVW]).

Some well known examples of Anosov flows are algebraic ones:

- The suspension of a linear Anosov automorphism of $\mathbb{T}^2$ is obtained as follows: Consider in $\mathbb{T}^2 \times [0, 1]$ the constant vector field given by vectors tangent to the second coordinate (i.e. whose integral lines are of the form $\varphi_t((x, s)) = (x, t + s)$). Now, we can identify $\mathbb{T}^2 \times \{0\}$ with $\mathbb{T}^2 \times \{1\}$ as follows: $(x, 1) \sim (Ax, 0)$. The manifold one obtains by this process will be denoted as S_A and it is sometimes called the *mapping torus* of A. In Thurston's geometries, it belongs to the Solv case and it is possible to see this flow as an algebraic flow. Anosov flows are structurally stable, so, beyond the algebraic case, one can construct examples by perturbing those flows.
- Given a (closed) surface S whose curvature is everywhere negative, it is a well known result that the geodesic flow on T^1S (the unitary tangent bundle of S) is an Anosov flow (see [KH]). It is possible to make an easy proof of this, at least in the case of constant negative curvature, where T^1S can be identified with

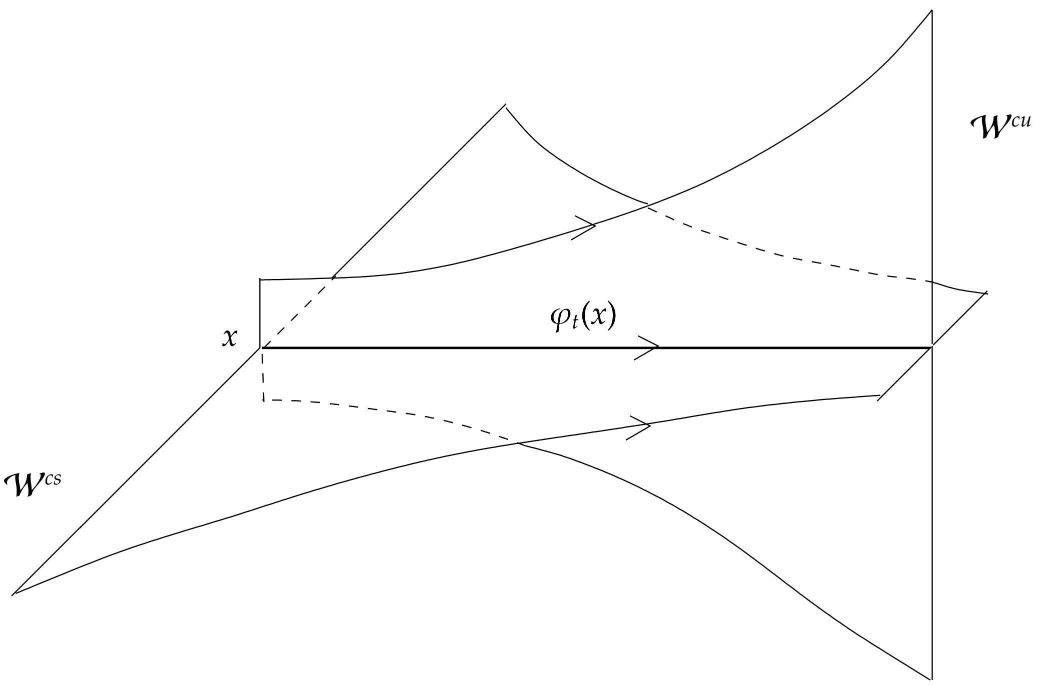

FIGURE 1. Local picture of an Anosov flow.

a quotient of $PSL(2,\mathbb{R})$ by some representation of the fundamental group of S and the geodesic flow is given by the action of diagonal matrices.

Exercise. Show that the above defined flows are indeed Anosov flows.

A remarkable result of Ghys ([Ghy]) shows that if the bundles E^s and E^u are of class C^2, then the Anosov flow must be conjugated to one of the examples presented above. One could infer from this result that *every* Anosov flow should be of this form, after all, it could be possible isotope a given Anosov flow to render its bundles differentiable (as it is the case in Anosov diffeomorphisms in dimensions ≤ 3). However, this is far from being true and there is a huge Zoo of examples of Anosov flows which we still do not completely understand. In particular, the following is an open question:

Question 3. *Which* 3*-dimensional manifolds admit Anosov flows?*

The first *anomalous* examples of Anosov flows were constructed by Franks and Williams, which presented examples of non-transitive Anosov flows in 3-manifolds by doing a surgery between the suspension of a DA-attractor on $\mathbb{T}^2$ and a DA-repeller along some periodic orbit. The construction is very simple and beautiful and we recommend reading the original paper [FW] which is quite simple (and in fact one can be convinced after seeing the pictures!).

Some development of the theory of Anosov systems has been obtained mainly by T. Barbot and S. Fenley (see [BaFe] and specially references therein). By improving the surgery techniques of [FW] many pathological examples generalizing those in [BL] have been obtained by C. Bonatti, F. Beguin and B. Yu ([BBY]), their constructions give hope that a classification in terms of transverse tori could be possible in the lines of previous work of Brunella and the above cited results, however, to my understanding, such classification would not involve describing[5] explicitly the 3-manifolds admitting Anosov flows, at least not more than in the JSJ-sense (see [Hat]).

In the direction of Question 3 there is one remarkable obstruction which is related with Novikov's Theorem on codimension one foliations in dimension 3 (for higher dimensional flows, Verjovsky managed to use instead Haefliger's argument which is weaker in a very clever way to show similar results for codimension one Anosov flows [V]). In fact, since we know that a 3-manifold admitting an Anosov flow must admit a codimension one foliation all of whose leaves are planes or cylinders, one obtains some topological obstructions for admitting Anosov flows[6] . These are essentially the only known-obstructions other than the ones contained in the following results we state now (look for references inside those results for previous related results):

Theorem 2.4 (Margulis-Thurston-Plante-Verjovsky [V])**.** *If the fundamental group of a 3-dimensional manifold M is (virtually) solvable and $\varphi_t : M \to M$ is an Anosov flow, then $M = S_A$ for some linear Anosov automorphism A and φ_t is orbit equivalent to the suspension of A.*

The previous Theorem implies in particular that M must have fundamental group with exponential growth. A classification is possible on manifolds which are circle bundles over surfaces:

Theorem 2.5 (Ghys [Ghy_2])**.** *Let M be a 3-dimensional bundle over a closed surface S and $\varphi_t : M \to M$ an Anosov flow. Then, $M = T^1 S$ and φ_t is orbit equivalent to the geodesic flow of (any) metric of negative curvature on S. In particular, all metrics with Anosov geodesic flow on S give rise to topologically equivalent flows.*

We will discuss more in section 2.4 the concept of topological equivalence, orbit equivalence, conjugacy, etc. We have not been precise above, it will be clear later.

We close this discussion by mentioning some results related with robust transitivity and stable ergodicity. It was proved in [BD] that if φ_t is a transitive Anosov flow then there is a perturbation of φ_1 which is robustly transitive. In a similar way, if φ_t is

[5] I am far from being expert, I would suggest waiting for [BBY] to come out to understand better this claims.

[6] Let us mention as a digression that it is known that some hyperbolic 3-manifolds admit Anosov flows, and more remarkably that some do not [RSS].

volume preserving and mixing[7] with respect to the volume measure then φ_1 itself is stably ergodic [BuPW]. A very interesting open question is the following:

Question 4. *Given a topologically mixing Anosov flow $\varphi_t : M \to M$ is it true that φ_1 is C^1-robustly transitive? What about for the time one map of the geodesic flow in constant negative curvature?*

2.2.4. *Pujals' Conjecture.* In 2001 in a Conference, E. Pujals informally conjectured (asked?) that the above mentioned examples were the complete list of transitive partially hyperbolic diffeomorphisms. This has to be understood as allowing finite lifts of the presented examples or allowing taking iterates.

In [BoW] the conjecture was given a more concrete form, also, some examples were given showing that the finite lifts and iterates are necessary. That paper also discusses some particular cases showing that in some cases, a semilocal property imposes a global form of the partially hyperbolic diffeomorphism. Also, there is a result there that shows that even if the classification of Anosov flows is not completed, the classification of partially hyperbolic diffeomorphisms modulo this classification is possible. We refer the reader to the original paper [BoW] for more details on their results. Some of the ideas appearing there will appear in this notes (particularly in section 8).

In [BBI] the first set of topological obstructions were identified, particularly those related with Novikov's theorem. We will explain the results in [BBI] (and their generalizations in [BI, H_2, Par]) in more detail later.

I consider the papers [BoW] and [BBI] as foundational papers on the classification problem of partially hyperbolic systems in dimension 3.

2.3. **Some well known properties.**

2.3.1. *Cone fields and persistence.* A *cone-field* $\mathcal{C}$ in M is a function from M to the power set of TM given by a continuous decomposition of $TM = E \oplus F$ and a continuous function $\alpha : M \to \mathbb{R}_{>0}$.

Given a continuous sub bundle $E \subset TM$ and a transverse subbundle F such that $TM = E \oplus F$ and a function $\alpha : M \to \mathbb{R}$ we can define the cone[8] field $\mathcal{C}^E_\alpha : M \to \mathcal{P}(TM)$ such that $\mathcal{C}^E_\alpha(x)$ is the set of vectors $v \in T_xM$ such that if $v = v_E + v_F$ with $v_E \in E$ and $v_F \in F$ then $\|v_E\| > \alpha(x)\|v_F\|$. The closure of the cone-field $\overline{\mathcal{C}}^E_\alpha$ is given by the set of vectors $v \in T_xM$ satisfying $\|v_E\| \geq \alpha(x)\|v_F\|$. The dimension of the cone-field $\mathcal{C}^E_\alpha$ is by

[7] If a conservative Anosov flow is volume preserving it is ergodic. The problem is that if it is not mixing, then the time one map itself is not ergodic (think about the suspension of A).

[8] The set $\mathcal{P}(TM)$ denotes the set of subsets of the set TM. There is a slight abuse of notation in that we will assume that every cone (open or closed) contains 0 even if it is not explicitly said.

definition the dimension of E. Notice that the set of vectors that do not belong to $\overline{C}^E_\alpha$ for each point is also a cone-field of complementary dimension which we call the *complementary cone-field*. The dimension of a cone-field is the dimension of E.

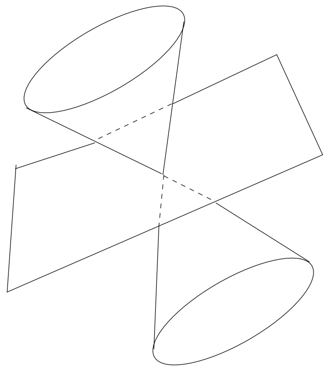

FIGURE 2. A cone in a vector space. The plane represents E and the angle α is the angle between E and the cone. Notice that the complement is also a cone with center a line.

See [BoGo] for more general definitions. We will only use this concept.

We get the following classical characterization of partial hyperbolicity:

Proposition 2.6 (Cone Criterium)**.** *Let $f : M \to M$ be a diffeomorphism such that there exists cone fields $\mathcal{C}^{cs}$ and $\mathcal{C}^{cu}$ of dimension 2 and values $N > 0$ and $\lambda > 1$ such that:*

- $Df^N(\overline{\mathcal{C}}^{cu}(x)) \subset \mathcal{C}^{cu}(f^N(x))$
- $Df^{-N}(\overline{\mathcal{C}}^{cs}(x)) \subset \mathcal{C}^{cu}(f^{-N}(x))$
- *For every vector $v \notin \mathcal{C}^{cu}$ we have that $\|Df^{-N}v\| > \lambda\|v\|$*
- *For every vector $v \notin \mathcal{C}^{cs}$ we have that $\|Df^{N}v\| > \lambda\|v\|$*

Then, f is partially hyperbolic. Moreover, if f is partially hyperbolic there exist conefields $\mathcal{C}^{cu}$ and $\mathcal{C}^{cs}$ verifying those properties.

Exercise. Prove the previous criterium. See also [BDV, Appendix B]. Develop a similar criterium for absolutely partially hyperbolic diffeomorphisms.

$\diamond$

An easy consequence of Proposition 2.6 is that being partially hyperbolic is a C^1-open property (Proposition 2.2).

2.3.2. *Invariant foliations.* As in Anosov diffeomorphisms, invariant foliations play a fundamental role in the study of dynamics and topological classification of partially hyperbolic systems. In this notes, *foliation* will mean a continuous foliation (with C^0-charts) which has C^1-leaves which are tangent to a continuous distribution (see section 4.1 for more details and references). For a foliation $\mathcal{F}$ we denote as $\mathcal{F}(x)$ to the leaf of $\mathcal{F}$ containing x.

We present now a fundamental result on partially hyperbolic diffeomorphisms which is now quite classical (see [HPS]). There are many proofs of this result which can be found in the literature. Of course it is valid in much more generality, but we state what we will need and present a proof since in our context it is quite simple and helps understand the difficulties in more general cases. The reader is invited to fill in the details.

Theorem 2.7 (Strong Foliations)**.** *Let $f : M \to M$ a partially hyperbolic diffeomorphism. Then, there exist f-invariant foliations $\mathcal{W}^s$ and $\mathcal{W}^u$ tangent to the bundles E^s and E^u. Moreover, the bundles E^s and E^u are uniquely integrable.*

Proof. Since we are in dimension 3 we know that the bundles E^s and E^u are one-dimensional. We will work in the universal cover $\tilde{M}$ of M and with a lift $\tilde{f}$ in order to have that E^s and E^u are orientable. Modulo considering an iterate we can assume that $D\tilde{f}$ preserves the orientations of E^s and E^u. We will show that E^s is uniquely integrable: f-invariance follows as a direct consequence (and unique integrability, since it is a local property shows that this is true in M).

We can consider by taking a further iterate that we have the following:

- There exist a cone-field C^{cu} such that for every $x \in \tilde{M}$ and any unit vectors v^s, v^{cu} tangent respectively to $E^s(x)$ and to $C^{cu}(x)$ we have that

$$\|Dfv^s\| < \min\{\frac{1}{2}, \frac{1}{2}\|Dfv^{cu}\|\}$$

By Peano's existence theorem there exist integral curves tangent to E^s. So, we assume by contradiction that there exist two different curves γ_1 and γ_2 tangent to E^s which start at a point x and are different.

This implies that there is a differentiable curve η whose tangent vectors are contained in a center-unstable cone-field C^{cu} and intersects γ_1 and γ_2 in different points[9]. We obtain two points $z \neq w$ which belong to γ_1 and γ_2 respectively and such that both belong to η.

For a given $n > 0$ we can consider η_n to be the shortest curve joining $\tilde{f}^n(z)$ and $\tilde{f}^n(w)$ and whose tangent vectors are contained in $D\tilde{f}^n(C^{cu})$. The fact that there is at least one such curve is given by the fact that $f^n(\eta)$ is one of such curves.

[9]Consider for example a curve which is tangent first to E^u and then to E^c. One can smooth this without loosing the uniform angle with E^s

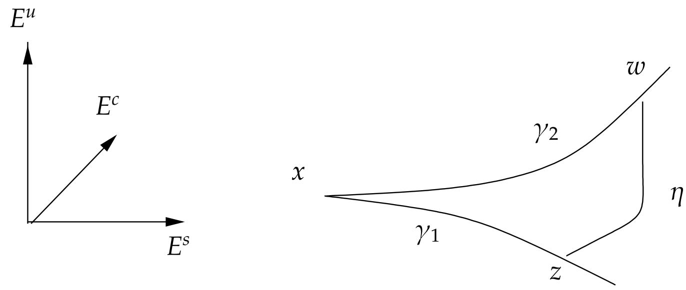

FIGURE 3. The curves γ_1, γ_2 and η.

Now, we know that $d(z,w)$ is smaller or equal to the length of $f^{-n}(\eta_n)$ which is of the order of $\|D_{f^n(z)}\tilde{f}^{-n}v\|d(\tilde{f}^n(z),\tilde{f}^n(w))$ where v is a vector tangent to η_n at $f^n(z)$ (in particular, it stays in C^{cu} for n iterates).

On the other hand we have that

$$d(\tilde{f}^n(z),\tilde{f}^n(w)) \leq d(\tilde{f}^n(z),\tilde{f}^n(x)) + d(\tilde{f}^n(x),\tilde{f}^n(w)) \approx 2\|D_{\tilde{f}^n(x)}\tilde{f}^n|_{E^s(\tilde{f}^n(x))}\|d(x,z)$$

Since we can do this at very small scales and points remain close for all future iterates (this is essential and it is where the fact that E^s is uniformly contracted, or at least that the curves remain of small length is used crucially), we conclude that

$$\begin{aligned} d(z,w) \leq \ell(f^{-n}(\eta_n)) \approx \|D_{f^n(z)}\tilde{f}^{-n}v\|d(\tilde{f}^n(z),\tilde{f}^n(w)) \leq \\ \leq \|D_{f^n(z)}\tilde{f}^{-n}v\|(d(\tilde{f}^n(z),\tilde{f}^n(x)) + d(\tilde{f}^n(x),\tilde{f}^n(w))) \approx \\ \approx 2\|D_{f^n(z)}\tilde{f}^{-n}v\|\|D_{\tilde{f}^n(x)}\tilde{f}^n|_{E^s}\|d(x,z) \to 0 \end{aligned}$$

which is a contradiction showing the unique integrability of the bundles.

□

Remark 2.8. The fact that one remains at small scales is crucial for two (independent) reasons:

- Because of being at small scales it is possible to compare the distance between points with the (smallest) length of curves tangent to the center unstable cone-field joining the points. Notice that the local uniform transversality is lost globally and this estimate falls apart. Later, we will see how the concept of *quasi-isometry* allows to avoid this problem in certain cases.
- The pointwise domination property only allows one to compare the differential along the stable and the center-unstable when points are close enough. In the absolutely dominated case we get that $\|D_{f^n(z)}\tilde{f}^{-n}v\|\|D_{\tilde{f}^n(x)}\tilde{f}^n|_{E^s}\|$ goes to zero uniformly even if the points $f^n(x)$ and $f^n(z)$ are very far apart, so, this problem can be ignored.

We recommend the reader which is not familiar with this classical argument to really fill in the details of the proof.

$\diamondsuit$

Exercise. Show that if J is an arc tangent to E^s then the length of $f^n(J)$ decreases exponentially fast.

2.4. **Dynamical coherence and leaf conjugacy.** As we have shown in the examples, foliations tangent to the center direction allow sometimes to distinguish between different classes of partially hyperbolic diffeomorphisms. This was observed in [BoW] where Pujals' conjecture was given a more precise form. Integrability of the center bundle is then at the heart of the classification problem of partially hyperbolic systems.

Because of some technical reasons, as well as from its use in the study of stable ergodicity, we will ask for something slightly stronger than integrability of E^c into a foliation.

Definition 2.9 (Dynamical Coherence)**.** We say a partially hyperbolic diffeomorphism $f : M \to M$ is *dynamically coherent* if there exist f-invariant foliations $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ tangent to the bundles $E^{cs} = E^s \oplus E^c$ and $E^{cu} = E^c \oplus E^u$.

$\diamondsuit$

As a consequence, a dynamically coherent partially hyperbolic diffeomorphism f posses an f-invariant foliation $\mathcal{W}^c$ tangent to E^c obtained by intersecting the two transverse foliations above. However, it is not clear if the existence of a f-invariant foliation $\mathcal{W}^c$ implies dynamical coherence (it does when E^c is uniquely integrable but this is far from trivial, see [BBI] and Section 3 bellow).

In general, there are two reasons for non-integrability of a distribution: the failure of the Frobenius bracket condition ([CC]) which only applies in the higher dimensional case and lack of smoothness. In the latter case, the bundle has integral lines, but they may fail to be unique. When the dimension of the central bundle is higher dimensional (which cannot happen in dimension 3) it was noticed by Wilkinson that one can construct examples for which the distributions are even analytic but where dynamical coherence fails ([Wi$_1$, BuW$_2$]).

In dimension 3, recently an example was presented by Hertz-Hertz-Ures ([RHRHU$_4$]) of a non-dynamically coherent example of partially hyperbolic diffeomorphism in $\mathbb{T}^3$. Their proof is based on the following criterium:

Theorem 2.10 (Hertz-Hertz-Ures [RHRHU$_4$])**.** *Let $f : M \to M$ be a dynamically coherent partially hyperbolic diffeomorphism, then $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ have no torus leafs.*

We will explain the proof of this result later when we have developed some tools in order to show its proof. They have conjectured that this is the unique obstruction for integrability:

Conjecture (Hertz-Hertz-Ures [RHRHU$_4$]). *Let $f : M \to M$ be a partially hyperbolic diffeomorphism such that there does not exist a f-periodic two dimensional torus T tangent to either E^{cs} nor E^{cu}. Then, f is dynamically coherent.*

Notice that the existence of such torus, being normally hyperbolic, implies the existence of an open set U such that $f(\overline{U}) \subset U$. This cannot happen if the non-wandering set of f is the whole manifold, in particular, when f is conservative or transitive.

Example (Non-dynamically coherent examples [RHRHU$_4$]). We will construct a partially hyperbolic diffeomorphism of $\mathbb{T}^3$ verifying that it has a fixed torus T^{cu} tangent to E^{cu} and such that if there exists a foliation tangent to E^{cu} then it must have T^{cu} as a leaf. Theorem 2.10 then implies that such an example cannot be dynamically coherent. The construction is quite flexible (it allows also to construct examples of dynamically coherent partially hyperbolic diffeomorphisms with non-uniquely integrable bundles, but we will not concentrate on this, we refer the reader to [RHRHU$_4$]). We consider $A \in SL(2,\mathbb{Z})$ a hyperbolic matrix, we denote as λ the stable eigenvalue of A and v^s, v^u denote unit vectors in the eigenspaces associated respectively to λ and λ^{-1}. We further consider $\psi : S^1 \to S^1$ a diffeomorphism of S^1 which we view as $[-1,1]/_{(-1)\sim 1}$ such that it verifies the following properties:

- $\psi(0) = 0$ and $\psi(1) = 1$ are the only fixed points of ψ.
- $0 < \psi'(0) < \lambda < 1 < \psi'(1) < \lambda^{-1}$.

We will define a diffeomorphism $f : \mathbb{T}^3 \to \mathbb{T}^3$ which we view as $\mathbb{T}^3 = \mathbb{T}^2 \times S^1$ defined as:

$$f(x,t) = (Ax + \varphi(t)\mathrm{v}^s, \psi(t))$$

where $\varphi : S^1 \to \mathbb{R}$ will be defined later. We get that

$$f^{-1}(x,t) = (A^{-1}x - \varphi(\psi^{-1}(t))\mathrm{v}^s, \psi(t))$$

We denote v^c to be a vector tangent to $\{x\} \times S^1$. So, at each point we have a basis for the tangent space given by the vectors $\mathrm{v}^s, \mathrm{v}^c, \mathrm{v}^u$, we denote as $\langle B \rangle$ to the vector space spanned by a subset B in a vector space. For such an f we have the following properties:

(P1) There exists a normally attracting torus $T^{cu} = \mathbb{T}^2 \times \{0\}$. Moreover, $f|_{T^{cu}}$ is partially hyperbolic with splitting given by $E^s = \langle \mathrm{v}^c \rangle$, $E^c = \langle \mathrm{v}^s \rangle$ and $E^u = \langle \mathrm{v}^u \rangle$.

(P2) There exists a repelling (but not normally repelling) torus $T^{su} = \mathbb{T}^2 \times \{1\}$. Moreover, $f|_{T^{su}}$ is partially hyperbolic with splitting given by $E^s = \langle \mathrm{v}^s \rangle$, $E^c = \langle \mathrm{v}^c \rangle$ and $E^u = \langle \mathrm{v}^u \rangle$.

We must choose $\varphi : S^1 \to \mathbb{R}$ in such a way that f is (globally) partially hyperbolic and that if there is a foliation $\mathcal{W}^{cu}$ tangent to E^{cu} then it must contain T^{cu} as a leaf. We will use the cone criterium (Proposition 2.6) to construct φ. Let us first write the derivative of f:

$$Df_{(x,t)}(u\mathrm{v}^u + s\mathrm{v}^s + c\mathrm{v}^c) = (\lambda^{-1}u)\mathrm{v}^u + (\lambda s + \varphi'(t)c)\mathrm{v}^s + (\psi'(t)c)\mathrm{v}^c$$

This implies that the subspaces $\langle \mathrm{v}^u \rangle$ and $\langle \mathrm{v}^s, \mathrm{v}^c \rangle$ are not only invariant[10] but a small cone field around $\langle \mathrm{v}^s, \mathrm{v}^c \rangle$ is contracted by Df^{-1} while vectors outside this cone field are expanded by Df. This implies that we must only concentrate on constructing the cone field C^{cu} for a suitable choice of φ. From the invariance seen above, it suffices to work in the plane $\langle \mathrm{v}^s, \mathrm{v}^c \rangle$ and define the cones in that plane.

An easy calculation shows that:

$$Df^n_{(x,t)}(s\mathrm{v}^s + c\mathrm{v}^c) = \left(\lambda^n s + \sum_{j=1}^{n} \lambda^{j-1} \varphi'(\psi^{n-j}(t))(\psi^{n-j})'(t)c \right) \mathrm{v}^s + ((\psi^n)'(t)c)\mathrm{v}^c$$

We will demand φ to verify the following.

- φ' are different from zero and have constant sign in $(0,1)$ and $(-1,0)$.
- $\varphi'(0) = \varphi'(1) = 0$ and $\varphi''(0)$, $\varphi''(1)$ are non-zero.

To fix ideas, we can assume that φ' is positive in $(0,1)$ and negative in $(-1,0)$. We claim that this is enough to guarantee that f is partially hyperbolic. We will work only in $(0,1)$ the other side is symmetric.

We have to define C^{cu} in $\mathbb{T}^3$. We will just define $C^{cu} \cap \langle \mathrm{v}^s, \mathrm{v}^c \rangle$ which as we mentioned is enough. First we define it in a neighborhood of T^{su}. There, it must contain the center direction of $f|_{T^{su}}$ so we choose a very narrow cone field in a small neighborhood of T^{su} of vectors of the form $v = a\mathrm{v}^s + b\mathrm{v}^c$ with $|a| \leq \varepsilon|b|$. In a small neighborhood of T^{su}, since $\psi'(t)$ is larger than λ we know that the cone field is invariant.

Now, we will propagate this cone field which we have defined in $\mathbb{T}^2 \times [t_0, 1]$ (with t_0 very close to 1) by iterating[11] it by Df^n which defines a cone field in $\mathbb{T}^2 \times (0, 1]$. Since φ'

[10]Notice that $\langle v^s \rangle$ is also invariant. However, in general it will not have a cone field around it separating it from the rest of the bundles, this will be important in the construction. In fact, since the map is not transitive, there is no obstruction for a partially hyperbolic diffeomorphism to have a globally defined invariant subspace which is not one of the bundles of the partial hyperbolicity.

[11]One must consider its iterate and then thicken it a little in order to have that the closure of the cone gets mapped into the interior of its image.

is positive in all of $(0, 1)$ we can see that after some iterates, the cone field gets twisted into the quadrant of vectors $v = a\mathrm{v}^s + b\mathrm{v}^c$ with $ab > 0$ (or possibly $a = b = 0$). This is crucial for the construction of the cone-field, and it is where the need to add φ becomes clear (notice that if $\varphi = 0$ the diffeomorphism cannot be partially hyperbolic).

Once the points arrive to the region $(0, t_1)$ where ψ' is much smaller than λ one gets that this cone field starts getting thinner and closer to the subspace $\langle \mathrm{v}^s \rangle$. Since in $\mathbb{T}^2 \times [0, t_1)$ one can consider the cone field of vectors of the form $v = a\mathrm{v}^s + b\mathrm{v}^c$ with $|b| \leq \varepsilon|a|$ which is also Df-invariant one gets that one can glue both cone-fields in order to get a well defined global cone-field C^{cu} which is Df-invariant in the sense of Proposition 2.6. It is easy to check that vectors outside C^{cu} when iterated by Df^{-n} get expanded uniformly so that the cone-field we have constructed verifies the hypothesis of Proposition 2.6. The same argument can be done in $\mathbb{T}^2 \times [-1, 0]$ which implies partial hyperbolicity.

Notice that in $\mathbb{T}^2 \times [-1, 0]$ we obtain that the cone-field twists to the opposite quadrant, so that every curve tangent to the cone-field will approach the torus T^{cu} in the same side. This implies that if there is a foliation tangent to E^{cu} then it must have T^{cu} as a leaf (see figure 4).

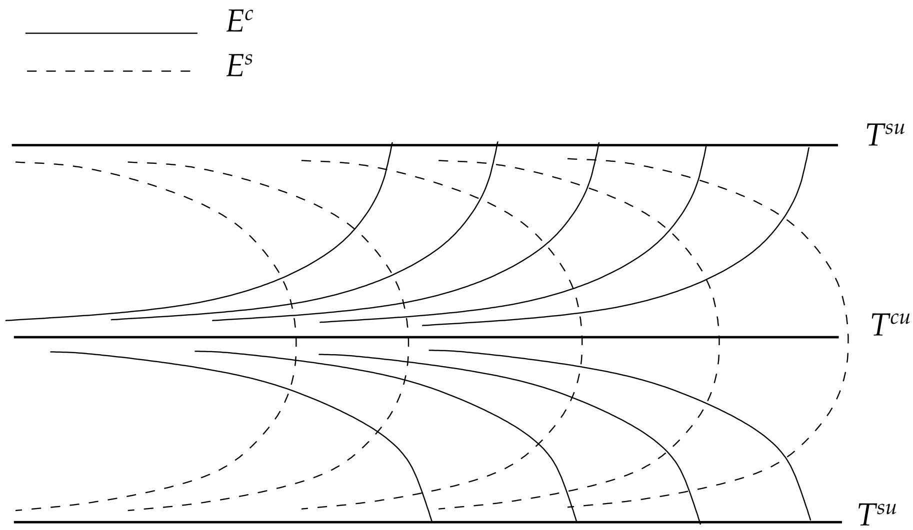

FIGURE 4. How the bundles twist from one torus to the other inside a center-stable leaf. This implies that if there is a foliation tangent to E^{cu} it must have T^{cu} as a leaf.

It is possible to write explicitly[12]. the subspaces E^c and E^s using the properties mentioned above (see [RHRHU$_4$]).

$\diamond$

[12] I believe that using the fact that $\varphi''(0) \neq 0$ it is possible to show that once one writes the bundles explicitly they are not integrable close to T^{cu}. I have not checked this much, see [RHRHU$_4$].

Digression. In dimension 2, it was recently proved (see [FG]) that the space of Anosov diffeomorphisms in a certain isotopy class is connected (in fact, it is homotopically equivalent to $\mathbb{T}^2$). In this case, the examples constructed in [RHRHU$_4$] show that the space of partially hyperbolic diffeomorphisms in $\mathbb{T}^3$ isotopic to $A \times Id_{S^1}$ with $A : \mathbb{T}^2 \to \mathbb{T}^2$ a linear Anosov automorphism is not connected (in fact, one can put a lot of center-unstable torus and show that there are infinitely many connected components). However, by looking into the construction, one can see that one can go from the partially hyperbolic diffeomorphism $A \times Id_{S^1}$ to the non-dynamically coherent examples constructed above by a path of diffeomorphisms which lie in the *closure* of the space of partially hyperbolic diffeomorphisms. In view of the previously mentioned result of [FG] it seems natural to ask wether the following question is true:

Question 5. *Is the closure of the space of partially hyperbolic diffeomorphisms in a given isotopy class of diffeomorphisms of* $\mathbb{T}^3$ *connected?*

$\diamondsuit$

We remark that if the Conjecture above on dynamical coherence is true, then, in terms of the classification of partially hyperbolic systems, we can assume dynamical coherence, since such a torus allows one to expect a quite strong understanding of the underlying dynamics (notice also that in [RHRHU$_3$] it is proved that such a torus implies that the manifold is either $\mathbb{T}^3$ or the mapping torus of a matrix commuting with an Anosov matrix, i.e. either $-id$ or a hyperbolic matrix). We refer the reader to sections 3.8 and 5.4 for more discussion on these examples and related topics.

Given two partially hyperbolic diffeomorphisms $f, g : M \to M$, even if they are isotopic to each other (even C^1-close), one can not expect that the dynamics will be conjugated: Indeed, a small perturbation along the central direction can affect the dynamics of the systems (for example, a small C^∞-perturbation of the product of an Anosov diffeomorphism with the identity on the circle can be made to be transitive [BD]), this is very similar to what happens for flows, were instead of conjugacy one considers orbit equivalence. If $f, g : M \to M$ are dynamically coherent, one can regard the center foliation as the orbits in the flow case, and there exists a natural equivalence that one can consider between those systems which was introduced in [HPS]:

Definition 2.11 (Leaf conjugacy)**.** We say that two dynamically coherent partially hyperbolic diffeomorphisms $f, g : M \to M$ with center foliations $\mathcal{W}^c_f$ and $\mathcal{W}^c_g$ are *leaf conjugate* if there exists a homeomorphism $h : M \to M$ which sends leaves of $\mathcal{W}^c_f$ into leaves of $\mathcal{F}^c_g$ and conjugates the dynamics of the leaves. More precisely,

$$h(\mathcal{W}^c_f(f(x))) = \mathcal{W}^c_g(g \circ h(x))$$

$\diamondsuit$

As in the case of Anosov diffeomorphisms, one cannot expect the conjugacy to be smother. In [HPS] it was proved that many examples of dynamically coherent partially hyperbolic diffeomorphisms (including those for which the center foliation is C^1) are *stable* in the sense that any C^1-perturbation remains dynamically coherent and leaf conjugate to the original one. It was not until [BoW] that some global results of this kind were obtained. In [H, H_2] Hammerlindl gave leaf conjugacy results for absolutely partially hyperbolic diffeomorphisms on $\mathbb{T}^3$ and nilmanifolds.

With these definitions, the informal conjecture mentioned above can be given a more precise form:

Conjecture (Pujals/Bonatti-Wilkinson [BoW])**.** *Let $f : M \to M$ be a transitive partially hyperbolic diffeomorphism. Then, f is dynamically coherent and (modulo finite cover and taking an iterate) leaf conjugate to one of the following:*

- *An Anosov diffeomorphism of $\mathbb{T}^3$.*
- *A skew-product over an Anosov diffeomorphism of $\mathbb{T}^2$ (and M is either $\mathbb{T}^3$ or a nil-manifold).*
- *The time one map of an Anosov flow.*

This conjecture is the main motivation for the work we present in this notes. We remark that the hypothesis of transitivity is necessary due to the examples we have presented above, however, one could ask a weaker question if one assumes that Hertz-Hertz-Ures conjecture is true. In fact, Hertz-Hertz-Ures have recently conjectured that the above conjecture is true also for non-transitive dynamically coherent partially hyperbolic diffeomorphisms.

2.5. **Torus bundles over the circle.** In this section we will introduce the 3-dimensional manifolds on which we will work. The reason we restrict to this class will be evident from the proofs when some properties of these manifolds enter. We refer the reader to [Hat] for a nice introduction to the basic tools in 3-manifold topology.

The first thing we will assume from our manifold M is that it is *prime*: This means that if the manifold can be decomposed as a connected sum $M = N_1 \sharp N_2$ then either N_1 or N_2 is the three sphere S^3. This will not be a restriction to our goals, indeed, it follows from the results in [BI] that a 3-dimensional manifold admitting a partially hyperbolic diffeomorphism must be prime (we will explain this result in section 3.3); in fact, it is true even that the manifold must be *irreducible* (i.e. every embedded 2-sphere bounds a ball) which after the proof of Poincare's conjecture is equivalent to knowing that $\pi_2(M) = 0$.

The class of manifolds we will be interested in is the ones whose fundamental group is *almost solvable*. This means that the group has a finite index normal[13] subgroup which is solvable. Let us recall briefly the definition of solvable and nilpotent for groups.

First, remember that if G is a group, then, its commutator subgroup is the subgroup $[G, G]$ of G generated by all the commutators, i.e. elements of the form $ghg^{-1}h^{-1}$ with $g, h \in G$. If H_1, H_2 are subgroups of G, we define $[H_1, H_2]$ the subgroup generated by elements of the form $h_1h_2h_1^{-1}h_2^{-1}$ with $h_i \in H_i$ $(i = 1, 2)$. Notice that a group G is *abelian* if and only if $[G, G] = \{e\}$.

We define $G_0 = G^0 = G$ and define recursively

$$G_n = [G, G_{n-1}] \qquad ; \qquad G^n = [G^{n-1}, G^{n-1}]$$

We say that a group G is *solvable* (resp. *nilpotent*) if there exists $n > 0$ such that $G^n = \{e\}$ (resp. $G_n = \{e\}$). Clearly, a nilpotent group is also solvable, the same holds for almost (virtually) nilpotent groups which are trivially almost (virtually) solvable. We will give examples later of groups which are solvable but not nilpotent (as well as nilpotent which are not abelian).

Exercise. Give an example of a virtually nilpotent group which is not almost solvable (Hint: Finite groups are always virtually whatever you want).

♢

We will be interested in (irreducible) 3-dimensional manifolds whose fundamental group is almost solvable and infinite. In fact, the class of manifolds we are really interested in is torus bundles over the circle, but due to the following result of Evans-Moser, this is essentially the same.

Theorem 2.12 (Evans-Moser [EM]). *Let M be a 3-dimensional manifold such that $\pi_1(M)$ is infinite and almost solvable. Moreover, assume that $\pi_2(M) = \{0\}$. Then, M is finitely covered by one of the following manifolds:*

- *A torus bundle over the circle.*
- *A Klein-bottle bundle over the circle.*
- *The union of two twisted I bundles over the Klein bottle sewn together along their boundaries.*

The proof of this theorem is not incredibly hard, but it is without a doubt out of the scope of this set of notes. In fact, it is also possible in our setting to discard the last

[13]This is the difference with the word *virtually* which does not demand the finite index subgroup to be normal.

two types of manifolds of the list (we will assume in general that the manifold admits a foliation without torus leaves, and as in the case of one-dimensional foliations of the Klein-bottle one can show that foliations of such manifolds must have compact leaves).

We now state what we mean by a *torus bundle over the circle*. We say that M is a torus bundle over the circle if there exists a differentiable map $p : M \to S^1$ such that $p^{-1}(t) = \mathbb{T}^2$ for every $t \in S^1$ and there is a local trivialization property: for every $t \in S^1$ there exists an interval I containing t in its interior such that $p^{-1}(I)$ is homeomorphic to $\mathbb{T}^2 \times I$ via a homeomorphism $\varphi_t : \mathbb{T}^2 \times I \to p^{-1}(I)$ which satisfies that $p \circ \varphi_t(x, s) = s$.

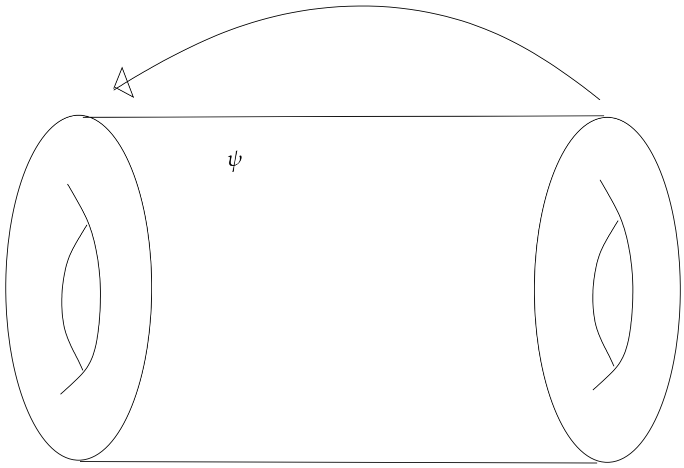

FIGURE 5. A torus bundle over the circle with gluing map ψ.

It is not hard to show that if M is a torus bundle over S^1 then $M \cong \mathbb{T}^2 \times [0,1]/_\sim$ where we identify points $(x, 1) \sim (\psi(x), 0)$ where $\psi : \mathbb{T}^2 \to \mathbb{T}^2$ is some diffeomorphism. We call M_ψ to the torus bundle over the circle obtained by this procedure with identification ψ.

Exercise. Show that if ψ is isotopic to ψ' then M_ψ and $M_{\psi'}$ are diffeomorphic.

$\diamondsuit$

We will concentrate then in manifolds of this form where ψ are matrices in $SL(2, \mathbb{Z})$ and infinite order. If $\psi = Id$, then $M_\psi = \mathbb{T}^3$.

In the case where ψ is of the form

$$\psi_k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$$

the resulting manifold is what is called a *nilmanifold*. This manifold can be thought of also as a circle bundle over the torus (exercise). We will denote these manifolds as

N_k when the gluing map is ψ_k. We will see later that the fundamental group of this manifold is nilpotent.

Finally, when $\psi = A$ is a hyperbolic matrix (with no eigenvalues of modulus one) we obtain what is called a *solvmanifold*. We will denote such manifold S_A. We will later prove that the fundamental group of such a manifold is solvable.

2.6. **Statement of the main results.** The goal of this notes is to present a recent result obtained in collaboration with Andy Hammerlindl which gives a complete solution to the conjectures above for a certain class of 3-dimensional manifolds. Our results extend the work that has been done in the absolute partially hyperbolic case when the manifold has a fundamental group with polynomial growth ([BBI, BBI$_2$, Par, H, H$_2$]) but more importantly, it contains a class of manifolds whose fundamental groups have exponential growth. We will review all of these previous results in Section 3. A departure point for extending the results to the poinwise case are the results obtained in [BI] which we shall also present here.

Main Theorem (joint with A. Hammerlindl [Pot, HP, HP$_2$])**.** *Let $f : M \to M$ be a partially hyperbolic diffeomorphism. Assume moreover that M has almost solvable fundamental group and that there does not exist a f-periodic two dimensional torus tangent to either E^{cs} or E^{cu}. Then, there exist unique f-invariant foliations $\mathcal{W}^{cs}$, $\mathcal{W}^{cu}$ and $\mathcal{W}^c$ tangent respectively to E^{cs}, E^{cu} and E^c (in particular, f is dynamically coherent). Moreover, (modulo taking finite lifts) f belongs to one of the following classes:*

(i) *$M = \mathbb{T}^3$ and f is leaf conjugate to its linear part.*
(ii) *M is a non-toral nilmanifold and f is leaf conjugate to a skew-product.*
(iii) *M is a 3-dimensional solvmanifold and f has an iterate which is leaf conjugate to the time one map of the suspension of a linear Anosov automorphism of $\mathbb{T}^2$.*

We divide the results here into several separate results whose proofs are quite different although they share some common features. The first step is the following:

Theorem 2.13 ([BI, Par, H$_2$])**.** *Let $f : M \to M$ be a partially hyperbolic diffeomorphism of a manifold M with almost solvable fundamental group, then M is finitely covered by a torus bundle over the circle. Moreover, if M has fundamental group with polynomial growth of volume then f is isotopic to a partially hyperbolic diffeomorphism which belong to the classes introduced in section 2.2.*

In the absolutely partially hyperbolic setting, there are conditions that allow to use a similar argument as the one in the proof of Theorem 2.7 in order to show dynamical coherence (see section 3), so, a main difference already appears when trying to show dynamical coherence. In the pointwise setting, a main difference is that the reason why these systems are dynamically coherent is different depending on the isotopy class.

Theorem 2.14 ([Pot]). *Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a partially hyperbolic diffeomorphism isotopic to an Anosov diffeomorphism. Then, f is dynamically coherent, and the center-stable and center-unstable foliations are "close" to the correct ones.*

In the case where the isotopy class is that of a skew-product we obtain dynamical coherence with a different argument.

Theorem 2.15 ([Pot, H_2, HP]). *Let $f : M \to M$ be a partially hyperbolic diffeomorphism such that $\pi_1(M)$ has polynomial growth and admits no f-periodic torus T tangent to either E^{cs} nor E^{cu}. Then, f is dynamically coherent and the center-stable and center-unstable foliations are "close" to the correct ones.*

In the isotopy class of Anosov, the fact that foliations are close to the correct ones follows a posteriori by using dynamical coherence, on the other hand, the fact that dynamical coherence holds in the isotopy class of skew-products uses a priori that certain objects remain close to where they should be. This is a key difference.

Then, using the results and ideas from [H, H_2] we are able to deduce that:

Theorem 2.16 ([HP]). *If $f; M \to M$ is a dynamically coherent partially hyperbolic diffeomorphism such that $\pi_1(M)$ has polynomial growth, then f is finitely covered by a partially hyperbolic diffeomorphism which is leaf conjugate to either a skew-product or an Anosov diffeomorphism.*

For the case where the manifold has solvable fundamental group, there had been no previous results, so we are led to introduce some new ideas. The main difficulty is that there is no a priori candidate to be leaf conjugate to. We are able however to prove:

Theorem 2.17 ([HP_2]). *Let $f : M \to M$ be a partially hyperbolic diffeomorphism of a manifold M with almost solvable fundamental group whose growth is not polynomial and such that there is no f-periodic torus T tangent to either E^{cs} nor E^{cu}. Then, f is dynamically coherent and moreover, there is an iterate of a finite lift which is leaf conjugate to the time one map of the suspension of a linear Anosov automorphism of $\mathbb{T}^2$.*

2.7. **Beyond leaf conjugacy.** We want to point out that leaf conjugacy is far from being the end of the story. In fact, what leaf conjugacy provides is a topological classification of partially hyperbolic diffeomorphisms, however, the dynamical consequences of this classification yet to be understood. In my opinion, this represents an important problem and we refer the reader to [BG] in order to see leaf conjugate examples having very different dynamical properties. We also mention a recent surprsing example by

Y.Shi ([Shi]) in the same spirit as [BG] which shows how the picture can be different in the conservative and non-conservative world. We will not enter in this aspect, but we mention that there exist some dynamical consequences of this classification (see [HP]).

3. Brin-Burago-Ivanov's results and further developments

In this section we present the results obtained by Brin-Burago-Ivanov in their sequence of papers [BBI, BI, BBI_2] as well as some extensions proven afterwards in [Par, H_2]. Also, we introduce some preliminaries that will be important in the rest of the notes and we show that the example of Hertz-Hertz-Ures ([$RHRHU_4$]) presented in Section 2 is not dynamically coherent. The results in [BBI, BI] provide the first (and sole for the moment) topological obstructions for admitting partially hyperbolic diffeomorphisms. The result itself is very complete: it combines a beautiful idea with a difficult (and very useful) technical result which is the starting point of any attempt to classify partially hyperbolic diffeomorphisms.

We mention that in this section some knowledge about foliations will be assumed. The reader not familiar with this subject can go first to section 4.1 where foliations are introduced in a more systematic way (or better, go to some of the following references [CC, CaLN, Cal]).

3.1. **Reebless foliations.**

We recall that if $\mathcal{F}$ is a foliation of a 3-dimensional manifold, a *Reeb component* is a diffeomorphic image of the solid torus $\mathbb{D}^2 \times S^1$ (here, we recall that we see $S^1 = \mathbb{R}/_{\mathbb{Z}}$ and $\mathbb{D}^2 = \{z \in \mathbb{C} \ : \ |z| \leq 1\}$) such that the boundary $\partial\mathbb{D}^2 \times S^1 \cong \mathbb{T}^2$ is a leaf of the foliation and the interior is foliated by the graphs of the functions from $\text{int}(\mathbb{D}^2) \to \mathbb{R}$ of the form:

$$x \mapsto a + \frac{1}{1 - \|x\|^2}$$

Since these graphs are invariant under translation in the $\mathbb{R}$-coordinate, we get that this foliation descends to $\text{int}(\mathbb{D}^2) \times S^1$. It is not hard to see that it defines a foliation even when one adds the boundary as a leaf.

It was proved in the 60's that every 3-dimensional manifold admits a codimension one foliation (see [CC]). The proof of this result involves doing surgery along certain solid tori in S^3 and regluing differently and putting Reeb-components inside the torus one introduced in order to in a certain sense ignore the effect of the surgery. Thus, codimension one foliations do not yield much information on the topology of the 3-dimensional manifold. On the other hand, due to a landmark result of Novikov, the existence of a foliation without Reeb components has deep implications on the

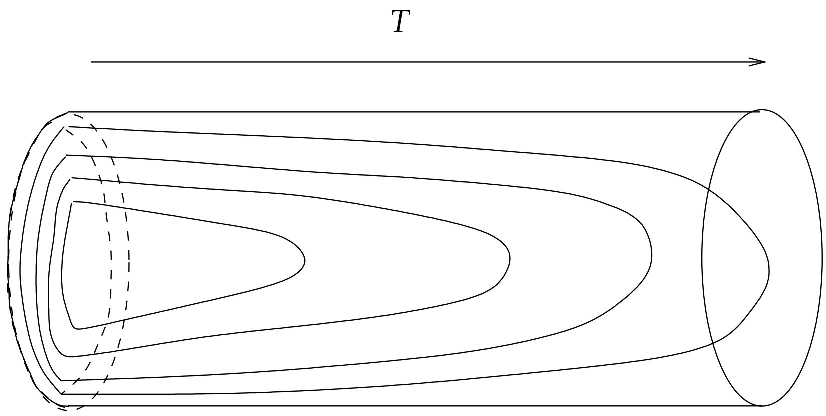

Figure 6. A Reeb component is obtained after quotienting the above foliation of the cylinder by the translation T.

topology of the manifold admitting it. We say that a foliation is *Reebless* if it has no Reeb components.

The following simple remark is one of the most important tools we will use:

Proposition 3.1. *If $\mathcal{L}$ is a one-dimensional foliation which is transverse to a Reeb component then $\mathcal{L}$ has a closed leaf. In particular, if $f : M \to M$ is a partially hyperbolic diffeomorphism then every foliation of M which is transverse to E^u or E^s is Reebless.*

Proof. The fact that a foliation transverse to a Reeb component has a compact leaf follows as a direct aplication of Brower's fixed point theorem. One can find a disc which moved by *holonomy* of the one-dimensional foliation returns to itself thus giving a fixed point which corresponds to a compact leaf. We leave this argument as an exercise.

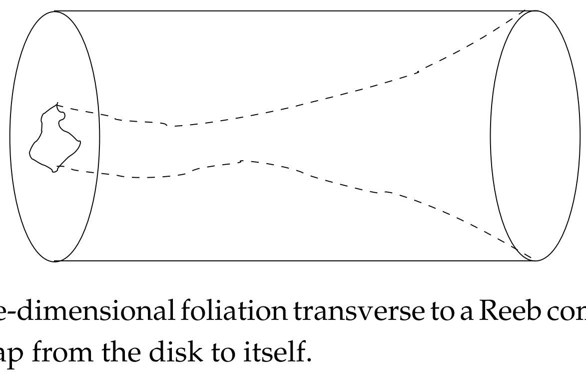

Figure 7. A one-dimensional foliation transverse to a Reeb component induces a continuous map from the disk to itself.

The second statement follows from the fact that a partially hyperbolic diffeomorphism cannot have a compact strong stable or unstable leaf (we also leave this easy fact as an exercise).

□

In view of the previous proposition, and the fact that Reebless foliations provide information on the topology of the manifolds admitting them. The following result is of capital importance to understand the topology of manifolds which admit partially hyperbolic diffeomorphisms:

Theorem 3.2 (Burago-Ivanov [BI] Key Lemma)**.** *Let $f : M \to M$ be a partially hyperbolic diffeomorphism such that the bundles E^s, E^c and E^u are orientable, then, M admits foliations $\mathcal{S}$ and $\mathcal{U}$ transverse respectively to E^u and E^s. In particular, these foliations are Reebless.*

We get the following as an immediate consequence of the previous Theorem and Novikov's theorem (see Theorem 3.4 bellow) asserting that every foliation of S^3 has a Reeb-component.

Corollary 3.3 (Burago-Ivanov)**.** *The sphere S^3 does not admit partially hyperbolic diffeomorphisms.*

Proof. Since S^3 is simply connected, the bundles E^s, E^c and E^u must be orientable so that Theorem 3.2 applies. Novikov's theorem combined with Proposition 3.1 concludes.

□

Notice that in general, to apply Theorem 3.2 we need the distributions E^σ ($\sigma = s, c, u$) to be orientable, however, it is not hard to see that every partially hyperbolic diffeomorphism has a finite lift which has this property.

Exercise.
(i) Show that if $f : M \to M$ verifies that Df preserves a continuous decomposition of the form $TM = E \oplus F$ then it is possible to find a finite lift $\hat{f} : \hat{M} \to \hat{M}$ such that the lift of the bundles is orientable.
(ii) Give an example of a diffeomorphism $f : M \to M$ (for some M) and a finite lift $p : \hat{M} \to M$ such that f does not lift to a diffeomorphism of $\hat{M}$.
(iii) Show[14] that given a finite lift $p : \hat{M} \to M$ and $f : M \to M$ a diffeomorphism, there exists an iterate of f which lifts to a diffeomorphism of $\hat{M}$.

3.2. **Novikov's Theorem and manifolds with solvable fundamental group.** In this section we collect some results on 3-manifolds that will be used to obtain topological obstructions for the existence of partially hyperbolic systems. As we mentioned above, to the moment, the sole obstruction we know is related with Theorem 3.2.

The fact that this gives topological obstructions is related with the following classical result on foliation theory proved by Novikov for C^2 foliations and then extended to

[14]This is quite hard as an exercise, it relies on a group theoretical property which at least to me it is not evident. I thank J. Brum for explaining me a proof.

the C^0 case by Solodov ([So]). We will not provide the proof nor the idea of the proof of this result and we refer the reader to [CaLN] for a nice sketch (see also [CC, Cal]).

Theorem 3.4 (Novikov [CC, So]). *Let $\mathcal{F}$ be a foliation of a 3-dimensional manifold M. If one of the following conditions holds:*

- *There exists a closed curve γ transverse to $\mathcal{F}$ such that γ is homotopically trivial. Equivalently, for $\tilde{\mathcal{F}}$ the lift of $\mathcal{F}$ to the universal cover, there exists a closed curve γ transverse to $\tilde{\mathcal{F}}$.*
- *There exists a continuous map $\varphi : S^2 \to M$ such that φ is not homotopically trivial (i.e. $\pi_2(M) \neq \{0\}$).*
- *There exists a leaf L of $\mathcal{F}$ and a closed curve $\gamma \in L$ which is homotopically trivial in M but not in L (i.e. the inclusion of L in M does not induce an injection in the fundamental groups).*

Then $\mathcal{F}$ has a Reeb component.

Exercise. Show that if in the universal cover there is a curve transverse to the lift $\tilde{\mathcal{F}}$ which intersects one leaf of $\tilde{\mathcal{F}}$ twice then there is a Reeb component for the foliation (see figure 8).

$\diamond$

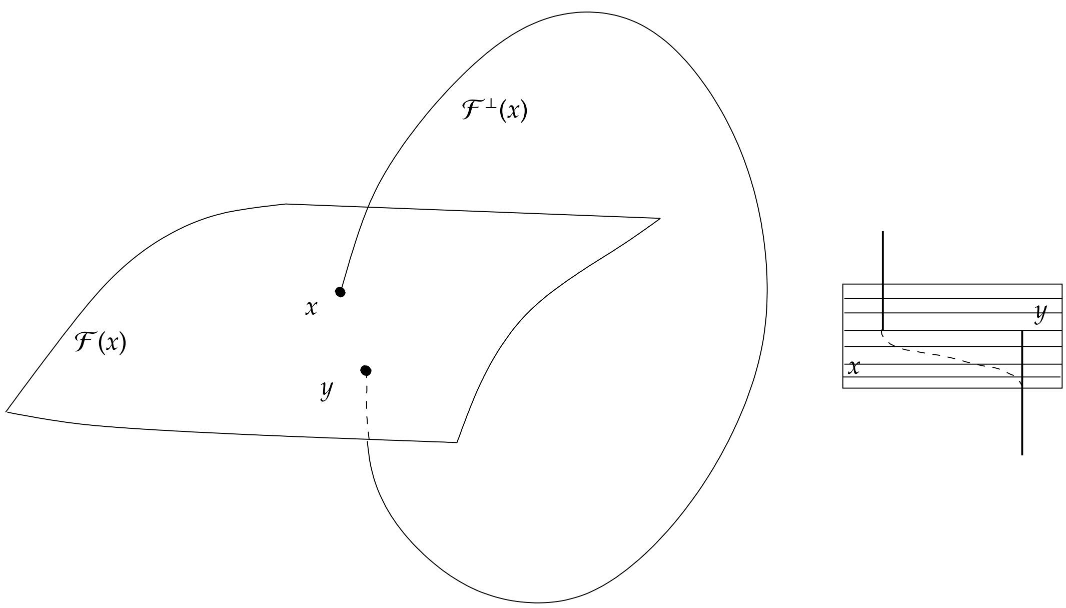

FIGURE 8. At the left, a transverse curve in the universal cover intersecting a leaf twice, at the right, an image on how to change it to create a closed transversal in the universal cover.

We will make more emphasis on the regularity of foliations in later sections, but let us mention here the reason why the C^2-hypothesis was used by Novikov. For this, there are essentially two reasons (see the introduction to chapter 9 of [CC]):

- It allows to construct easily curves transverse to the foliation. For this C^1-regularity (and in fact, what we will call $C^{1,0+}$-foliations are enough for this as we will see). In the C^0-case this is possible but difficult.
- The C^2-regularity allows one to use general position arguments. Say one has a C^∞-immersion of a disk D in M such that the boundary of D is transverse to $\mathcal{F}$ (this can be obtained for example when one has a homotopically trivial curve transverse to $\mathcal{F}$). The pull back of the foliation defines a partition in D given by preimages of each leaf. If the foliation is of class C^2, the function locally defines a C^2-function from D to $\mathbb{R}$ given by the restriction to D of a local trivialization chart of the foliation. By a classical result in Morse-theory one can easily perturb the immersion in the C^2 topology in order that this pull back becomes a singular foliation of D such that the singularities are either saddles or focus points and that there are no saddle connections. However, Solodov ([So]) remarked that the C^0-case need not be harder, even if putting in general position needs second derivatives, all we want is something topological, and perturbing in the C^0 topology is sometimes even simpler. We recomend the reader to see section 2 of [So] where this argument is done in a clean way.

We now study in more detail some aspects related with the volume growth of the manifolds we are interested in. In particular, we will show the following:

Proposition 3.5. *The growth of volume of a ball of radius R in the universal cover of $\mathbb{T}^3$ and N_k is polynomial in R.*

This must be understood as considering the universal cover of such manifolds and lifting the metric to this universal cover, once this is done, the ball of radius R as well as its volume has perfect sense (we do not care about which is the specific metric). In fact, this is well known to be equivalent to the fact that the growth of the fundamental group is polynomial. See [Pl, H_2] (and references therein) for more precise results.

Proof. The fundamental group of $\mathbb{T}^3$ is $\mathbb{Z}^3$ which has polynomial growth of order 3. To fix ideas, consider the flat metric, and we have that the volume of the ball of radius R is exactly $\frac{4\pi R^3}{3}$. For any other metric, a compactness argument shows that the growth is still polynomial of degree 3 (with possibly different constants).

In the case of N_k we will work only with the fundamental group, showing that it is nilpotent (and it is well known that this implies polynomial growth of volume). It is an interesting exercise to show that if you consider a metric invariant under deck transformations then the volume of the ball of radius R growths polynomially (in fact, one can prove that this is with degree 4). See also section 8 for arguments in this line.

A way to understand the fundamental group of N_k is to consider the group of deck transformations of the universal cover. We consider the universal cover $\tilde{N}_k$ of N_k which we identify with $\mathbb{R}^2 \times \mathbb{R}$. It is not hard to see that the following diffeomorphisms are a generator of the group G of deck transformations:

$$\gamma_1(x, y, t) = (x + 1, y, t)$$
$$\gamma_2(x, y, t) = (x, y + 1, t)$$
$$\gamma_3(x, y, t) = (x + ky, y, t + 1)$$

It is not hard to check that $[G, G]$ is the group generated by γ_1. Since γ_1 commutes with every deck transformation, one deduces that G is nilpotent as desired (in fact, one can see that the group generated by γ_1 is the *center* of G and is the group that allows one to see N_k also as a circle bundle over the torus).

□

Exercise. Fill in the details of the previous proof.

3.3. **Topological obstructions for admitting partially hyperbolic systems.** In this section we present a sketch of the proof of a result which shows that if the manifold is not so "big" (in terms of its fundamental group) then the induced action of a partially hyperbolic diffeomorphism in homology cannot be the identity. The argument in the proof is one of the main techniques available in the problem of classification of partially hyperbolic diffeomorphisms and variants of it will appear many times in this notes.

Theorem 3.6 (Brin-Burago-Ivanov [BBI, BI, Par]). *If $f : M \to M$ is a partially hyperbolic diffeomorphism and $\pi_1(M)$ has polynomial growth, then, the induced map $f_* : H_1(M, \mathbb{R}) \cong \mathbb{R}^k \to H_1(M, \mathbb{R})$ is partially hyperbolic. This means, it is represented by an invertible matrix $A \in GL(k, \mathbb{Z})$ which has an eigenvalue of modulus larger than 1 and determinant of modulus 1 (in particular, it also has an eigenvalue of modulus smaller than 1).*

Sketch We prove the result when $\pi_1(M)$ is abelian, so that it coincides with $H_1(M, \mathbb{Z})$. When the fundamental group is nilpotent, this follows from the fact that the 3-manifolds with this fundamental group are well known (they are circle bundles over the torus) so that one can make other kind of arguments with the same spirit (see [Par] Theorem 1.12): The main point is that in such manifold it is possible to show that if the action in homology has all its eigenvalues of modulus smaller or equal to one then f is isotopic to a representative f_* (its *algebraic part*) so that $\tilde{f}^n$ is at linear distance in n of a diffeomorphism f_*^n which one knows explicitly and distorts the diameter of a fundamental domain in the universal cover polynomially in n (in other manifolds, for example in surfaces of higher genus this is false: There are pseudo-anosov maps which act trivially in homology).

Let us work for simplicity in $\mathbb{T}^3$ which shows the main ideas. Assume that every eigenvalue of $f_* : \mathbb{R}^3 \to \mathbb{R}^3$ (f_* is a matrix in $SL(3, \mathbb{Z})$) is smaller or equal to 1.

For such f_* we know that it is a linear diffeomorphism so that we know that its derivative is everywhere with eigenvalues smaller or equal to one, this implies that the diameter of a fundamental domain D in $\mathbb{R}^3$ grows at most polynomially in n.

Given $R > 0$ the number of fundamental domains needed to cover a ball of radius R in $\mathbb{R}^3$ is polynomial in R (of degree 3).

Consider $\mathcal{S}$ the foliation given by Theorem 3.2 and $\tilde{\mathcal{S}}$ its lift to the universal cover.

Let I be an unstable arc contained in a fundamental domain D. We obtain that the diameter of $\tilde{f}^n(I)$ is smaller than or equal to $p(n)$ with p a polynomial.

On the other hand, using the uniform expansion along E^u we have that the length of $\tilde{f}^n(I)$ is larger than or equal to $C\lambda^n$ for some $\lambda > 1$ and $C > 0$.

We obtain that given ε we find points of a leaf of $\tilde{\mathcal{W}}^u$ which are not in the same local unstable manifold but are at distance smaller than ε, this implies the existence of a close transversal to $\tilde{\mathcal{S}}$ which via Theorem 3.4 implies the existence of a Reeb component for $\mathcal{S}$ and contradicts Proposition 3.1.

□

Notice that if the growth of the fundamental group is exponential, one can make partially hyperbolic diffeomorphisms which are isotopic to the identity (for example, the time-one map of an Anosov flow). This is because in such a manifold, a sequence K_n of sets with exponentially (in n) many points but polynomial (in n) diameter may not have accumulation points.

3.4. **Semiconjugacies in certain isotopy classes.** When a diffeomorphism of a manifold M is isotopic to a diffeomorphism which has some hyperbolic properties it is possible, via a shadowing argument to obtain semiconjugacies with certain models. This was studied by Franks ([Fr]) and we reproduce here some properties we will use (see also [Pot$_2$]).

We have the following results:

Theorem 3.7. *Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a diffeomorphism such that is isotopic to a linear Anosov automorphism with matrix $A \in SL(3, \mathbb{Z})$ then, there exists $h : \mathbb{T}^3 \to \mathbb{T}^3$ a continuous surjective map homotopic to the identity such that $h \circ f = A \circ h$.*

Theorem 3.8. *Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a diffeomorphism such that $f_* : H_1(\mathbb{T}^3, \mathbb{R}) \to H_1(\mathbb{T}^3, \mathbb{R})$ has one eigenvalue larger than one and one equal to one. Then, there exists $h : \mathbb{T}^3 \to \mathbb{T}^2$ continuous and surjective such that $h \circ f = A \circ h$ where A is given by the matrix induced in the eigenplane corresponding to the eigenvalues different from* 1*. The same holds for diffeomorphisms of* $\mathbb{T}^2 \times [0, 1]$ *which act hyperbolically in homology.*

Theorem 3.9. *Let $f : N_k \to N_k$ be a diffeomorphism such that $f_* : H_1(N_k, \mathbb{R}) \to H_1(N_k, \mathbb{R})$ is partially hyperbolic, then, there exists $h : N_k \to \mathbb{T}^2$ continuous and surjective such that $h \circ f = A \circ h$ where A is given by the matrix induced by f_* in $H_1(N_k, \mathbb{R}) \cong \mathbb{R}^2$.*

The proofs of these propositions are quite classical (see [Fr]). The idea is to lift the the universal cover and compare the dynamics with a suitable model in the isotopy class which by the hyperbolicity properties asked in the action in homology has some infinite expansiveness in certain directions. The classical shadowing argument and the fact that diffeomorphisms are at bounded distance from each other allows to find for each point in the universal cover an orbit of the model that will shadow the orbit. See also [Pot_2] section 2.3 for a more detailed account.

3.5. **Dynamical coherence in the absolute case-Brin's argument.** We review in this section a simple criterium given by Brin in [Br] which guaranties dynamical coherence for absolutely dominated partially hyperbolic diffeomorphisms. It involves the concept of quasi-isometry which we will use afterwards.

Definition 3.10 (Quasi-Isometric Foliation)**.** Consider a Riemannian manifold M (not necessarily compact) and a foliation $\mathcal{F}$ in M. We say that the foliation $\mathcal{F}$ is *quasi-isometric* if distances inside leaves can be compared with distances in the manifold. More precisely, for $x, y \in \mathcal{F}(x)$ we denote as $d_{\mathcal{F}}(x, y)$ as the infimum of the lengths of curves contained in $\mathcal{F}(x)$ joining x to y. We say that $\mathcal{F}$ is *quasi-isometric* if there exists $a, b \in \mathbb{R}$ such that for every x, y in the same leaf of $\mathcal{F}$ one has that:

$$d_{\mathcal{F}}(x, y) \leq a d(x, y) + b$$

$\diamond$

A similar argument as the one given in the proof of Theorem 2.7 can be pushed to obtain the following result for absolute partial hyperbolicity.

Proposition 3.11 ([Br])**.** *Let $f : M \to M$ be an absolutely partially hyperbolic diffeomorphism with splitting such that the foliation $\tilde{\mathcal{W}}^u$ is quasi-isometric in $\tilde{M}$ the universal cover of M. Then, f is dynamically coherent.*

It shows that in fact, the foliation is unique in (almost) the strongest sense, which is that every C^1-embedding of a ball of dimension $\dim E^{cs}$ which is everywhere tangent to E^{cs} is in fact contained in a leaf of the foliation $\mathcal{W}^{cs}$.

We give a sketch of the proof in order to show how the hypothesis are essential to pursue the argument. See [Br] for a clear exposition of the complete argument.

Sketch We will work in $\tilde{M}$. Assume that there are two embedded balls B_1 and B_2 through a point x which are everywhere tangent to E^{cs} and whose intersection is not relatively open in (at least) one of them.

It is possible to construct a curve η which has non-zero length, is contained in a leaf of $\tilde{\mathcal{W}}^u$ and joins these two embedded balls.

Let γ_1 and γ_2 two curves contained in B_1 and B_2 respectively joining x to the extremes of η.

Since η is an unstable curve, its length growths exponentially, and by quasi-isometry, we know that the extremal points of the curve are at a distance which grows exponentially with the same rate as the rate the vectors in E^u expand.

On the other hand, the curves γ_1 and γ_2 are forced to grow with at most an exponential rate which is smaller than the one in E^u (by using absolute domination) and so we violate the triangle inequality.

$\square$

3.6. **Branching foliations.** In this section we introduce the concept of branching foliations which is a key tool introduced in [BI] for the study of integrability and classification of partially hyperbolic diffeomorphisms.

Then, we introduce the notions of "almost aligned", "almost parallel" for branching and non-branching foliations which will be useful for classification results of foliations in certain 3-manifolds.

A (*complete*) *surface* in a 3-manifold M is a C^1-immersion $\imath : U \to M$ of a connected smooth 2-dimensional manifold without boundary U which is complete with the metric induced by the metric on M pulled back by $\imath$.

A *branching foliation* on M is a collection of complete surfaces tangent to a given continuous 2-dimensional distribution on M such that:

- Every point is in the image of at least one surface.
- There are no topological crossings between any two surfaces of the collection.
- It is complete in the following sense: if $x_k \to x$ and $\imath_k$ are surfaces of the partition having x_k in its image we have that $\imath_k$ converges in the C^1-topology to a surface of the collection with x in its image (see [BI] Lemma 7.1).

The image $\imath(U)$ of each surface in a branching foliation is called a *leaf*.

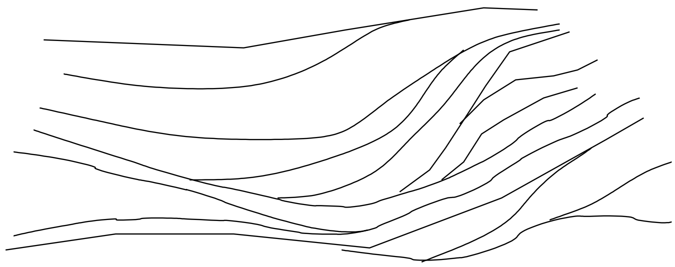

FIGURE 9. A one-dimensional branching foliation.

Theorem 3.12 (Burago-Ivanov [BI], Theorem 4.1)**.** *If f is a partially hyperbolic diffeomorphism on a 3-manifold M, such that the bundles E^s, E^c and E^u are orientable and Df preserves their orientation, then, there is a (not necessarily unique) f-invariant branching foliation $\mathcal{F}^{cs}_{bran}$ tangent to E^{cs}. Further, any curve tangent to E^s lies in a single leaf of $\mathcal{F}^{cs}_{bran}$.*

A similar foliation $\mathcal{F}^{cu}_{bran}$ is defined tangent to E^{cu} under the same hypothesis.

For a branching foliation $\mathcal{F}$ on M, there is a unique branching foliation $\tilde{\mathcal{F}}$ on the universal cover $\tilde{M}$ such that if L is a leaf of $\mathcal{F}$ then its pre-image in $\tilde{M}$ is a disjoint union of leaves in $\tilde{\mathcal{F}}$. Call $\tilde{\mathcal{F}}$ the lift of $\mathcal{F}$ to $\tilde{M}$.

A branching foliation $\mathcal{F}_1$ is *almost aligned* with a branching foliation $\mathcal{F}_2$ if there is $R > 0$ such that each leaf of $\tilde{\mathcal{F}}_1$ lies in the R-neighborhood of a leaf of $\tilde{\mathcal{F}}_2$.

Two branching foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ are *almost parallel* if there exists $R > 0$ such that:

- For every leaf $L_1 \in \tilde{\mathcal{F}}_1$ there exists a leaf $L_2 \in \tilde{\mathcal{F}}_2$ such that $L_1 \subset B_R(L_2)$ and $L_2 \subset B_R(L_1)$ (i.e. The Hausdorff distance between L_1 and L_2 is smaller than R).
- For every[15] leaf $L_2 \in \tilde{\mathcal{F}}_2$ there exists a leaf $L_1 \in \tilde{\mathcal{F}}_1$ such that $L_1 \subset B_R(L_2)$ and $L_2 \subset B_R(L_1)$.

The following properties are verified:

Proposition 3.13. *The following properties are verified:*

(i) *Being almost parallel is an equivalence relation among branching foliations. Almost aligned is a transitive relation.*
(ii) *If $\mathcal{W}$ is almost aligned with $\mathcal{W}'$ a foliation in M and φ a diffeomorphism of M isotopic to the identity, then $\varphi(\mathcal{W})$ is almost parallel to $\mathcal{W}$ and almost aligned to $\mathcal{W}'$.*
(iii) *If $\mathcal{F}_{bran}$ is a branching foliation and $\mathcal{W}$ a foliation such that there exists a continuous map $h : M \to M$ at small distance from the identity such that it maps leaves of $\mathcal{W}$ diffeomorphically onto leaves of $\mathcal{F}_{bran}$ then $\mathcal{F}_{bran}$ is almost parallel to $\mathcal{W}$.*

[15] I do not know if this second condition is necessary, but it makes it easier to show that this relation is an equivalence relation. Notice that this definition is stronger than asking that $\mathcal{F}_1$ is almost aligned with $\mathcal{F}_2$ and $\mathcal{F}_2$ almost aligned with $\mathcal{F}_1$. See Figure 10.

Proof. Property (i) follows from the triangle inequality. Properties (ii) and (iii) follow from the fact that the maps in the universal cover are at bounded distance from the identity.

□

It then follows that (see Theorem 7.2 of [BI] for a stronger version):

Theorem 3.14 (Burago-Ivanov [BI])**.** *In the hypothesis of the previous theorem, there is a (non-branching) $C^{1,0+}$ Reebless foliation $\mathcal{S}$ almost parallel to $\mathcal{F}^{cs}_{bran}$ and tangent to an arbitrarily small cone field around E^{cs}. Similarly, for $\mathcal{F}^{cu}_{bran}$.*

The fact that the foliation is Reebless is a consequence of Proposition 3.1.

3.7. **Ideas on the proofs of Burago-Ivanov's results.**

3.7.1. *In two dimensions:* We shall first explain a similar problem in dimension 2. Say you have a continuous non-singular vector field X in $\mathbb{T}^2$. We are interested in finding a branching foliation tangent to X which will be invariant under *any* diffeomorphism which preserves the distribution generated by X (i.e. $f : \mathbb{T}^2 \to \mathbb{T}^2$ such that $Df(X(x)) = \alpha(x)X(f(x))$ with $\alpha(x) > 0$).

The problem here is not integrability, since by Peano's existence theorem, we know that there exist integral curves through each point of $\mathbb{T}^2$, but as it is well known, what may fail is *unique integrability*. If the vector field X is uniquely integrable, then the problem of finding a foliation which is invariant under any diffeomorphism preserving X is trivial by the flow box theorem (which says that the solutions of a uniquely integrable vector field form a foliation). The failure of unique integrability can be quite radical, in fact, it is possible for a (Holder) continuous vector field of $\mathbb{T}^2$ to be tangent to uncountably many different foliations or many other pathological properties[16] (see [BF]).

We have chosen to start with a vector field and not a distribution on purpose, since this gives us an orientation for the distribution as well as (using the fact that $\mathbb{T}^2$ is orientable) a transverse orientation.

We can choose a finite covering of $\mathbb{T}^2$ by charts (φ_i, U_i) which are coherent with a chosen orientation and verify the following:

- $\varphi_i : U_i \to \mathbb{R}^2$ is a C^1 map.
- The pushforward $(\varphi_i)_* X$ by φ_i of the vector field X in $\mathbb{R}^2$ is of the form $(v_1(x, y), v_2(x, y))$ with $\frac{v_2}{v_1} \leq 1$ for every point.

[16]Nicolas Gourmelon showed me how to construct a vector field such that there exists a one parameter family of pairwise (everywhere) topologically transverse foliations tangent to the vector field.

We now can choose the *lowest* solutions of the differential equation $\gamma'(x) = \frac{v_2(x,\gamma(x))}{v_1(x,\gamma(x))} = F(x,\gamma(x))$, this means, we consider for each point (x_0, y_0) the function $\gamma_{x_0,y_0} : \mathbb{R} \to \mathbb{R}$ defined by:

- $\gamma_{x_0,y_0}|_{[x_0,+\infty)}$ is the supremum of the C^1-functions $\eta : [x_0, +\infty) \to \mathbb{R}$ such that $\eta(x_0) = y_0$ and $g'(x) < F(x, \eta(x))$ for every x.
- $\gamma_{x_0,y_0}|_{(-\infty,x_0]}$ is the infimum of the C^1-functions $\eta : (-\infty, x_0] \to \mathbb{R}$ such that $\eta(x_0) = y_0$ and $\eta'(x) < F(x, \eta(x))$ for every x.

With this constructions, one obtains what one was looking for:

Exercise. (i) Show that the curves f_{x_0,y_0} are integral curves of $(\varphi_i)_* X$.
(ii) There are no topological crossings between the curves curves f_{x_0,y_0}.
(iii) One can add some curves in order to obtain a branching foliation.
(iv) Show that this glues well in $\mathbb{T}^2$.
(v) Show that if $f : \mathbb{T}^2 \to \mathbb{T}^2$ is a diffeomorphism preserving the vector field X and the orientation, then f leaves the branching foliation obtained above invariant.

Notice that the orientability hypothesis is crucial in order to pass from the local to the global problem of integrating the vector field: It allows to define a branching foliation by *lowest* curves tangent to the vector field. In fact, this gives essentially two different branching foliations which are in some way canonical. See section 5 of [BI] for more details.

3.7.2. *Extending the ideas to higher dimensions:* In higher dimensions the strategy developed in dimension two stops right away: In dimension 3 there is no way to define the *lowest* or *highest* integral curve for the vector field. The idea used in [BI] is to saturate the integral curves of the center direction by strong stable manifolds (using the fact that the latter are uniquely integrable, cf. Theorem 2.7) to have a well defined notion of being up or down when the bundles are orientable. The idea is successful but not without pain, a large number of technical difficulties appear which make the result not only beautiful but difficult. We will try to explain here which are these difficulties and give an idea on how to solve them (and then refer the reader to [BI] for more details).

The two main difficulties that appear are the following:

- It is not clear that after saturating the center curves by strong stable manifolds one obtains a C^1-manifold. Even if we knew that, it is also not clear in principle that this manifold should be tangent to E^{cs}.
- When saturating by strong stable manifolds, the resulting manifold could not be complete with its intrinsic metric. Completeness of the leaves is an important

requirement in the definition of branching foliations (more in view that we want afterwards to obtain a true foliation close to the branching foliation).

The first difficulty is mainly technical since it is possible to prove:

Proposition 3.15 (Proposition 3.1 of [BI]). *Let $\gamma : [0,1] \to M$ a curve parametrized by arc length such that $\gamma'(t) \in E^{cs}(\gamma(t)) \setminus E^s(\gamma(t))$ for every $t \in [0,1]$. Let Φ^s denote the flow generated by integrating a unitary vector field contained in E^s. Then, the map $S : I \times \mathbb{R} \to M$ defined by $S(t,s) = \Phi^s(\gamma(t))$ is a C^0-parametrization of a C^1-immersed 2-dimensional manifold tangent to E^{cs}.*

Proof. If the bundle E^{cs} were differentiable of class C^1, it is possible to show by dynamical reasons that the Frobenius conditions is satisfied so that it is uniquely integrable (see [BuW$_2$], notice also that for this the one-dimensionality of the center and certain symmetry conditions are very important).

If the bundles are not differentiable one can make a Frobenius argument "by hand".

Since S is trivially continuous and injective we have only to prove that around each (t^*, s^*) there is a neighborhood V such that $S(V)$ is an embedded surface tangent to E^{cs}. Since this is local, we can always assume that V is small enough so that we can work in local coordinates and look at the tangent spaces as subsets of M. By this, we mean that to prove that $S(V)$ is a C^1-surface tangent to E^{cs} it is enough to prove that for every $p = S(t_0, s_0)$ in $S(V)$ we have that for every $\varepsilon > 0$ there exists δ such that if $d(p,q) < \delta$ with $q \in S(V)$ then we have that $d(q, E^{cs}(p)) < \varepsilon d(p,q)$ (here $E^{cs}(p)$ denotes the subspace through p which in local coordinates can be thought of as a subset of M).

Assume this does not hold, so, there exists $\varepsilon > 0$ and a sequence $q_n \to p$ such that $d(q_n, E^{cs}(p)) \geq \varepsilon d(p, q_n)$.

We can write $q_n = S(t_n, s_n)$ and since we are in a compact part of $S(I \times \mathbb{R})$ we can assume that the strong stable holonomy is bounded so that there exists $C_H > 0$ such that:

$$C_H^{-1}(|t_n - t_0| + |s_n - s_0|) \leq d(p, q_n) \leq C_H(|t_n - t_0| + |s_n - s_0|)$$

Since the domination is pointwise, we cannot compare it at large scales, but as we are looking to prove something local we can assume that all points and the iterates we will use are at distance smaller than δ_0 where we know that if v^{cs} is a unit vector in E^{cs} and v^u in E^u we have that $\lambda \|Dfv^{cs}\| < \|Dfv^u\|$ with $\lambda > 1$. Also, using the continuity of the bundles, we can further assume that in a ball of radius δ_0 the three bundles are almost orthogonal and almost constant (say, up to a factor of $10^{-100}\varepsilon$). In particular, if two points x, y are in the curve tangent to E^{cs} at scale δ_0, the distance between their local strong stable manifolds in the direction of E^u is smaller than $10^{-10}\varepsilon d(x,y)$.

Since p, q_n are in strong stable manifolds of γ, we know that after iterating k_0 times points are at distance smaller than $\frac{\delta_0}{10}$ from $f^n(\gamma)$ which is almost tangent to E^c.

Now, let C_0 be the supremum of $\|Df\|$ in M and consider $k > k_0$ such that for n large enough we have:

$$\varepsilon C_H^{-1} \lambda^k \gg 10 \qquad , \qquad C_0^k |t_n - t_0| \ll \delta_0$$

Now, we can see that if $d(q_n, E^{cs}(p)) \geq \varepsilon d(p, q_n)$ for such a large n then there exist an unstable curve γ_n^u which joins q_n with $E^{cs}(p)$. Iterating k times we get that $f^n(p)$ and $f^n(q)$ are at distance smaller than δ_0 but the length of $f^n(\gamma_n^u)$ is larger than δ_0 a contradiction.

See [BI] Proposition 3.1 for more details.

□

The second difficulty is a real difficulty, it may happen that after saturating by strong stable manifolds the resulting surface is not complete under the metric induced by the manifold from the immersion, indeed, this happens in the example by [RHRHU$_4$] presented above.

To solve this problem, in [BI] after saturating by stables they make a completion of the surface (which involves adding certain strong stable manifolds) and then extending again the surface by an inductive process. By compactness and uniform transversality, at each step one extends the surface by a definite amount, so, this allows to obtain complete surfaces. However, this is not so simple, since to perform this inductive process one has to take care on how to define globally which are the *lowest* and *highest* surfaces and this represents a big difficulty which they are able to deal by using an abstract procedure of extending pre-foliations with a method which is not far from the one used in dimension two but which has to take care of more details (this is done in sections 4,5 and 6 of [BI], we recommend reading at least section 4 where the difficulty is reduced to a technical proposition and the strategy is explained).

3.7.3. *Blowing up into a true foliation:* In this section we will try to explain how to obtain Theorem 3.14 which states that it is possible to construct a true foliation which is almost parallel to the branching foliation obtained in Theorem 3.12 and which is tangent to small cones around the bundles. We will explain this construction in dimension 2 (see [BI] Section 7 for more details).

We remark that in dimension two, it is very easy to construct a true foliation which is tangent to a distribution which lies in small cones of the original distribution: Indeed, it suffices to consider a C^1-vector field close to the original one and by uniqueness of solutions this gives the desired foliation. In dimension 3 this is more delicate, but in fact, this is not the real problem, since already in dimension 2, even if this argument gives a true foliation almost tangent to the original distribution, the difficult part is to obtain that the resulting foliation is almost parallel to the original one.

To obtain this, the proof of [BI] provides a way to suitably blow up the branching points in order to obtain a true foliation while remaining close to the original one. Let us briefly go through the main ideas.

In a local product chart, it is not hard to separate leaves in order to get a foliation. To fulfill the other requirements it is important to keep track how the leaves are separated in order to make coordinate changes compatible with this separation (and remain close to the leaves globally as desired). To do this, we remark that in each local product structure box it is possible to define an ordering between local leaves of the branching foliation: Each leaf separates the chart in two components and this allows to define a notion of above and below which gives the desired ordering. Since the branching foliation is complete, this ordering defines a topology in the space of leaves in a chart which makes it homeomorphic to an interval (we choose the chart to be bounded by local leaves).

This parametrization of leaves is what will allow us to keep track of the separation. We consider a finite covering of M with such charts and separate in each in a way that coordinate changes send leaves into leaves.

3.8. **More on branching.** A key remark is the following (see Proposition 1.10 and Remark 1.16 of [BoW]), we can state it as "a branching foliation without branching is a foliation":

Proposition 3.16. *If $\mathcal{F}$ is a branching foliation such that each point belongs to a unique leaf, then $\mathcal{F}$ is a true foliation.*

It is important to know that branching is indeed possible, at least in the case where f is not transitive. As we mentioned, the example of [RHRHU$_4$] is not dynamically coherent, so, clearly it has non-trivial branching. Let us expand more on this by giving the main ideas on the proof of Theorem 2.10.

The departure point is a very nice result of [RHRHU$_3$] which state that not many 3-manifolds can admit a partially hyperbolic diffeomorphism which has a periodic two-dimensional torus (their result is much stronger, in fact, under certain assumptions one can even drop the fact that f is partially hyperbolic). Of course, this result is interesting in its own right so we will briefly comment on some of the ideas in its proof.

Theorem 3.17. *Let $f : M \to M$ a partially hyperbolic diffeomorphism such that it admits a periodic two-dimensional C^1-embedded torus T. Then, M is either $\mathbb{T}^3$, the manifold M_{-Id} or S_A for some hyperbolic A.*

Sketch The idea is the following. Assume without loss of generality that T is fixed. Since it is C^1 and periodic it must be tangent to one of the invariant distributions (i.e. E^{cs}, E^{cu} or $E^s \oplus E^u$). This implies that $f|_T$ is isotopic to Anosov by a growth argument very similar to the one in Theorem 3.6 using Poincare-Bendixon's theorem instead of Novikov's theorem.

This already shows that T must be incompressible so that we can cut M by T to obtain a connected manifold N with boundary two copies of T. Since M is already irreducible (because it admits a partially hyperbolic diffeomorphism, see above) and by some arguments on 3-dimensional topology one deduces that N must be of the form $\mathbb{T}^2 \times [0,1]$ (see [Hat, RHRHU$_3$]).

Since the gluing map must commute with the action in T one deduces that the options mentioned above are the only possible ones.

□

We are now ready to sketch the proof of Theorem 2.10 which states that if f is partially hyperbolic diffeomorphism with and f-invariant foliation $\mathcal{W}^{cu}$ tangent to E^{cu} then, $\mathcal{W}^{cu}$ cannot contain a torus leaf.

The proof is by contradiction, so we assume there exists an f-invariant foliation $\mathcal{W}^{cu}$ having a torus leaf. The first point is that we can assume that there exists a f-periodic torus T tangent to E^{cu} (see the proof of the Proposition 3.18 bellow) and after considering an iterate assume that T is in fact fixed.

After cutting along T we obtain that we can work in $T \times [0,1]$ from the result above. We know moreover that f in $T \times [0,1]$ must be isotopic to an Anosov times the identity. Using Theorem 3.8 we get a semiconjugacy $h : T \times [0,1] \to \mathbb{T}^2$ with an Anosov diffeomorphism A of $\mathbb{T}^2$.

Let us now work in the universal cover $\mathbb{R}^2 \times [0,1]$ and denote as $\tilde{f}$ to the lift of f and H to the lift of h, as usual, we denote also as A to the action of the matrix A in $\mathbb{R}^2$ as in $\mathbb{T}^2$. We denote as $\mathcal{W}^{\sigma}_A$ $(\sigma = s, u)$ to the strong foliations of A. We have the following characterization of H, we know that if two points verify that $d(\tilde{f}^k((x,t)), \tilde{f}^k((y,s)))$ is bounded for $k > 0$ then $H((x,t)) \in \mathcal{W}^s_A(H(y,s))$. Similarly, $d(\tilde{f}^k((x,t)), \tilde{f}^k((y,s)))$ bounded for $k < 0$ implies $H((x,t)) \in \mathcal{W}^u_A(H(y,s))$ and if $d(\tilde{f}^k((x,t)), \tilde{f}^k((y,s)))$ is bounded for $k \in \mathbb{Z}$ then $H((x,t)) = H(y,s)$ (this follows from the proof of Theorem 3.8).

The previous remark together with Theorem 3.4 and Proposition 3.1 implies that H is injective in strong-stable and strong-unstable manifolds since otherwise one can find a transverse loop in $\mathbb{R}^2 \times [0,1]$ to the foliations given by Theorem 3.2. Moreover, we know that $H(\tilde{\mathcal{W}}^{\sigma}(x)) = \mathcal{W}^{\sigma}_A(H(x))$ $(\sigma = s, u)$.

Since H is surjective when restricted to $\mathbb{R}^2 \times \{0\}$ we get that there exists a sequence $(x_n, t_n) \to (x, 0)$ such that $H(x_n, t_n) = H(x, 0)$, we can further assume that the points

(x_n, t_n) pairwise do not belong to the same leaf of $\tilde{\mathcal{W}}^{cu}$. Consider $K > 0$ such that $\tilde{f}^k(H^{-1}(H(x,0)))$ has always diameter smaller than K and a finite subset of points $\{(x_i, t_i)\}_{i\in F}$ such that for every $k > 0$ there are at least two of them that are at distance smaller than $\varepsilon > 0$ (the size of the local product structure between $\tilde{\mathcal{W}}^{cu}$ and $\tilde{\mathcal{W}}^s$).

Since $\mathbb{R}^2\times[0,1]$ is simply connected, we can give an orientation to E^{cu} and E^s and when two points are sufficiently close (at less than ε as chosen above) it has sense to say that one is above or bellow the other. Taking an iterate, we can assume that $D\tilde{f}$ preserves these orientations. We assume that all the (x_n, t_n) belong to an ε-neighborhood of $(x, 0)$ so that they can all be compared to each other.

Exercise. Use Theorem 3.4 and Proposition 3.1 to show that if x is above a point y then we have that if points $\mathsf{z} \in \tilde{\mathcal{W}}^{cu}(\mathsf{x})$ and $\mathsf{w} \in \tilde{\mathcal{W}}^{cu}(\mathsf{y})$ verify that $d(\tilde{f}^k(\mathsf{z}), \tilde{f}^k(\mathsf{w}))$ are at distance smaller than ε then z must be above w.

♢

For every $i \neq j \in F$ we can join the points (x_i, t_i) and (x_j, t_j) by a small (non-trivial) strong stable curve $I_{i,j}$ plus a small curve $\gamma_{i,j}$ contained in $\mathcal{W}^{cu}$. Consider $N > 0$ large enough so that $\tilde{f}^{-N}(I_{i,j})$ has length much larger than 100ε for every $i \neq j \in F$. From the remark above, one can choose $i \neq j$ so that $d(\tilde{f}^{-N}(x_i, t_i), \tilde{f}^{-N}(x_j, t_j)) < \varepsilon$. Theorem 3.4 and Proposition 3.1 imply that $I_{i,j}$ intersect the leaves $\tilde{\mathcal{W}}^{cu}((x_i, t_i))$ and $\tilde{\mathcal{W}}^{cu}((x_j, t_j))$ only at their extremes.

We can assume that (x_i, t_i) is above (x_j, t_j) so that using the exercise above we know that $\tilde{f}^{-N}(x_i, t_i)$ is above $\tilde{f}^{-N}(x_j, t_j)$. This implies that $\tilde{f}^{-N}(I_{i,j})$ intersects $\tilde{\mathcal{W}}^{cu}(\tilde{f}^{-N}(x_i, t_i))$ in its interior, a contradiction. This finishes the sketch of the proof of Theorem 2.10.

□

We end this section by showing a result which we will use later.

Proposition 3.18. *Let $f : M \to M$ be a partially hyperbolic diffeomorphism with orientable bundles such that Df preserves their orientation. Assume that there is no f-periodic torus T tangent to either E^{cs} or E^{cu}. Then, the foliation given by Theorem 3.14 has no torus leaves.*

Proof. The proof uses a stronger version of Theorem 3.14 which says that the foliation almost parallel to the branching foliation comes with a continuous and surjective map which sends leaves of the foliation onto leaves of the branching foliation. Using this, if the foliation (say) $\mathcal{U}$ has a torus leaf, then so does $\mathcal{F}^{cu}_{bran}$. From the properties of these foliations (being Reebless) it is clear that the torus leaf is incompressible.

There are finitely many disjoint incompressible torus modulo isotopy, so, we can assume that iteration of the torus fixes their isotopy class. In particular, in the lift to the universal cover they are all fixed by the same deck transformations. An argument of Haefliger allows one to see that the space of such tori is compact (see [CC] or Proposition 5.1.11 of [Pot_2]) so that iterating forward there will be a recurrent one.

Using that it is transversally contracting we obtain that there is a periodic normally attracting torus.

□

4. Classifying Reebless foliations in some 3-manifolds

4.1. **Some preliminaries on foliations.** We will give a partial overview of foliations influenced by the results we use here. The main sources will be [Cal, CaLN, CC].

Definition 4.1 (Foliation)**.** A *foliation* $\mathcal{F}$ of dimension d ($d = 1, 2$) on a 3-manifold M is a partition of M on injectively immersed connected C^1-submanifolds tangent to a continuous d-dimensional subbundle E of TM satisfying:

- For every $x \in M$ there exists a neighborhood U and a continuous homeomorphism $\varphi : U \to \mathbb{R}^d \times \mathbb{R}^{3-d}$ such that for every $y \in \mathbb{R}^{3-d}$:
$$L_y = \varphi^{-1}(\mathbb{R}^d \times \{y\})$$
is a connected component of $L \cap U$ where L is an element of the partition $\mathcal{F}$.

◊

In most of the texts about foliations, this notion refers to a C^0-foliation with C^1-leaves (or foliations of class $C^{1,0+}$ in [CC]). For reasons that appear in the extensions of some of these results to higher dimensions, we will sometimes call 2-dimensional foliations of 3-manifolds *codimension 1 foliations*.

In dynamical systems, particularly in the theory of Anosov diffeomorphisms, flows and or partially hyperbolic systems, this notion is the best suited since it is the one guarantied by these dynamical properties.

Notation. We will denote $\mathcal{F}(x)$ to the *leaf* (i.e. element of the partition) of the foliation $\mathcal{F}$ containing x. Given a foliation $\mathcal{F}$ of a manifold M, we will always denote as $\tilde{\mathcal{F}}$ to the lift of the foliation $\mathcal{F}$ to the universal cover $\tilde{M}$ of M.

◊

We will say that a foliation is *orientable* if there exists a continuous choice of orientation for the subbundle $E \subset TM$ which is tangent to $\mathcal{F}$. Similarly, we say that the foliation is *transversally orientable* if there exists a continuous choice of orientation for the subbundle $E^\perp \subset TM$ consisting of the orthogonal bundle to E. Notice that if M is orientable, then the fact that E is orientable implies that $E^\perp$ is also orientable.

Given a foliation $\mathcal{F}$ of a manifold M, one can always consider a finite covering of M and $\mathcal{F}$ in order to get that the lifted foliation is both orientable and transversally orientable.

We remark that sometimes, the definition of a foliation is given in terms of atlases on the manifold, we state the following consequence of our definition:

Proposition 4.2. *Let M be a 3-dimensional manifold and $\mathcal{F}$ a d-dimensional foliation of M. Then, there exists a C^0-atlas $\{(\varphi_i, U_i)\}$ of M such that:*

- *$\varphi_i : U_i \to \mathbb{R}^d \times \mathbb{R}^{3-d}$ is a homeomorphism.*
- *If $U_i \cap U_j \neq \emptyset$ one has that $\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \mathbb{R}^k \times \mathbb{R}^{d-k}$ is of the form $\varphi_i \circ \varphi_j^{-1}(x, y) = (\varphi_{ij}^1(x, y), \varphi_{ij}^2(y))$. Moreover, the maps φ_{ij}^1 are C^1.*
- *The preimage by φ_i of a set of the form $\mathbb{R}^d \times \{y\}$ is contained in a leaf of $\mathcal{F}$.*

Being an equivalence relation, we can always make a quotient from the foliation and obtain a topological space (which is typically non-Hausdorff) called the *leaf space* endowed with the quotient topology. For a foliation $\mathcal{F}$ on a manifold M we denote the leaf space as $M/_{\mathcal{F}}$.

Proposition 4.3. *Given a codimension 1 foliation $\mathcal{F}$ of a compact manifold M there exists a one-dimensional foliation $\mathcal{F}^\perp$ transverse to $\mathcal{F}$. Moreover, the foliations $\mathcal{F}$ and $\mathcal{F}^\perp$ admit a* local product structure, *this means that for every $\varepsilon > 0$ there exists $\delta > 0$ such that:*

- *Given $x, y \in M$ such that $d(x, y) < \delta$ one has that $\mathcal{F}_\varepsilon(x) \cap \mathcal{F}_\varepsilon^\perp(y)$ consists of a unique point. Here, $\mathcal{F}_\varepsilon(x)$ and $\mathcal{F}_\varepsilon^\perp(y)$ denote the local leaves[17] of the foliations in $B_\varepsilon(x)$ and $B_\varepsilon(y)$.*

Proof. Assume first that $\mathcal{F}$ is transversally orientable. To prove the existence of a one dimensional foliation transverse to $\mathcal{F}$ consider E the continuous subbundle of TM tangent to $\mathcal{F}$. Now, there exists an arbitrarily narrow cone $\mathcal{E}^\perp$ transverse to E around the one dimensional subbundle $E^\perp$ (the orthogonal subbundle to E).

In $\mathcal{E}^\perp$ there exists a C^1 subbundle F. Since $E^\perp$ is orientable, so is F so we can choose a C^1-vector field without singularities inside F which integrates to a C^1 foliation which will be of course transverse to $\mathcal{F}$.

If $\mathcal{F}$ is not transversally orientable, one can choose a C^1-line field inside the cone field and taking the double cover construct a C^1-vector field invariant under deck transformations. This gives rise to an orientable one dimensional foliation transverse to the lift of $\mathcal{F}$ which projects to a non-orientable one transverse to $\mathcal{F}$.

By compactness of M one checks that the local product structure holds.

[17] More precisely, $\mathcal{F}_\varepsilon(x) = cc_x(\mathcal{F}(x) \cap B_\varepsilon(x))$ and $\mathcal{F}_\varepsilon^\perp(y) = cc_y(\mathcal{F}^\perp(y) \cap B_\varepsilon(y))$. Here $cc_x(A)$ denotes the connected component of A containing x.

□

In codimension 1 the behavior of the transversal foliation may detect non-simply connected leafs, this is the content of this well known result of Haefliger which can be thought of a precursor of the celebrated Novikov's theorem:

Proposition 4.4 (Haefliger Argument). *Consider a codimension one foliation $\mathcal{F}$ of a compact manifold M. Let $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}^{\perp}$ be the lifts to the universal cover of both $\mathcal{F}$ and the transverse foliation given by Proposition 4.3. Assume that there exists a leaf of $\tilde{\mathcal{F}}^{\perp}$ that intersects a leaf of $\tilde{\mathcal{F}}$ in more than one point, then, $\tilde{\mathcal{F}}$ has a non-simply connected leaf.*

This can be restated in the initial manifold by saying that if there exists a closed curve in M transverse to $\mathcal{F}$ which is nullhomotopic, then there exists a leaf of $\mathcal{F}$ such that its fundamental group does not inject in the fundamental group of M.

This result was first proven by Haefliger for C^2 foliations and then extended to general C^0-foliations by Solodov (see [So]). The idea is to consider a disk bounding a transverse curve to the foliation and making general position arguments (the reason for which Haefliger considered the C^2-case first) in order to have one dimensional foliation with Morse singularities on the disk, classical Poincare-Bendixon type of arguments then give the existence of a leaf of $\mathcal{F}$ with non-trivial holonomy.

Other reason for considering codimension one foliations is that leaves with finite fundamental group do not only give a condition on the local behaviour of the foliation but on the global one (see for example Theorem 6.1.5 of [CC]):

Theorem 4.5 (Reeb's global stability theorem). *Let $\mathcal{F}$ be a codimension one foliation on a compact manifold M and assume that there is a compact leaf L of $\mathcal{F}$ with finite fundamental group. Then, M is finitely covered by a manifold $\hat{M}$ admitting a fibration $p : \hat{M} \to S^1$ whose fibers are homeomorphic to $\hat{L}$ which finitely covers L and by lifting the foliation $\mathcal{F}$ to $\hat{M}$ we obtain the foliation given by the fibers of the fibration p.*

Corollary 4.6. *Let $\mathcal{F}$ be a codimension one foliation of a 3-dimensional manifold M having a leaf with finite fundamental group. Then, M is finitely covered by $S^2 \times S^1$ and the foliation lifts to a foliation of $S^2 \times S^1$ by spheres.*

For a codimension one foliation $\mathcal{F}$ of a manifold M, such that the leafs in the universal cover are properly embedded, there is a quite nice description of the leaf space $\tilde{M}/_{\tilde{\mathcal{F}}}$ as a (possibly non-Hausdorff) one-dimensional manifold. When the leaf space is homeomorphic to $\mathbb{R}$ we say that the foliation is $\mathbb{R}$-*covered* (see [Cal]).

Consider the foliation of the band $[-1,1] \times \mathbb{R}$ given by the horizontal lines together with the graphs of the functions $x \mapsto \exp\left(\frac{1}{1-x^2}\right) + b$ with $b \in \mathbb{R}$.

Clearly, this foliation is invariant by the translation $(x,t) \mapsto (x, t+1)$ so that it defines a foliation on the annulus $[-1,1] \times S^1$ which we call *Reeb annulus*.

In a similar way, we can define a two-dimensional foliation on $\mathbb{D}^2 \times \mathbb{R}$ given by the cylinder $\partial\mathbb{D}^2 \times \mathbb{R}$ and the graphs of the maps $(x, y) \mapsto \exp\left(\dfrac{1}{1 - x^2 - y^2}\right) + b$.

Definition 4.7 (Reeb component)**.** Any foliation of $\mathbb{D}^2 \times S^1$ homeomorphic to the foliation obtained by quotienting the foliation defined above by translation by 1 is called a *Reeb component*.

♢

Another important component of 3-dimensional foliations are dead-end components. They consist of foliations of $\mathbb{T}^2 \times [-1, 1]$ such that any transversal which enters the boundary cannot leave the manifold again. An example would be the product of a Reeb annulus with the circle.

Definition 4.8 (Dead-end component)**.** A foliation of $\mathbb{T}^2 \times [-1, 1]$ such that no transversal can intersect both boundary components is called a *dead-end component*.

♢

Exercise. Show that a Reeb annulus times the circle gives rise to a dead end component.

As in the previous section, we will say that a (transversally oriented) codimension one foliation of a 3-dimensional manifold is *Reebless* if it does not contain Reeb components. Similarly, we say that a Reebless foliation is *taut* if it has no dead-end components.

As a consequence of Novikov's theorem (Theorem 3.4) we obtain the following corollary on Reebless foliations on 3-manifolds which we state without proof. We say that a surface S embedded in a 3-manifold M is *incompressible* if the inclusion $\imath : S \to M$ induces an injective morphism of fundamental groups.

Corollary 4.9. *Let $\mathcal{F}$ be a Reebless foliation on an orientable 3-manifold M and $\mathcal{F}^\perp$ a transversal one-dimensional foliation. Then,*

(i) *For every $x \in \tilde{M}$ we have that $\tilde{\mathcal{F}}(x) \cap \tilde{\mathcal{F}}^\perp(x) = \{x\}$.*
(ii) *The leafs of $\tilde{\mathcal{F}}$ are properly embedded surfaces in $\tilde{M}$. In fact there exists $\delta > 0$ such that every euclidean ball U of radius δ can be covered by a continuous coordinate chart such that the intersection of every leaf S of $\tilde{\mathcal{F}}$ with U is either empty of represented as the graph of a function $h_S : \mathbb{R}^2 \to \mathbb{R}$ in those coordinates.*
(iii) *Every leaf of $\mathcal{F}$ is incompressible. In particular, $\tilde{M}$ is either $S^2 \times \mathbb{R}$ and every leaf is homeomorphic to S^2 or $\tilde{M} = \mathbb{R}^3$.*
(iv) *For every $\delta > 0$, there exists a constant C_δ such that if J is a segment of $\tilde{\mathcal{F}}^\perp$ then* $\mathrm{Vol}(B_\delta(J)) > C_\delta \, \mathrm{length}(J)$.

Notice that item (iii) implies that every leaf of $\tilde{\mathcal{F}}$ is simply connected, thus, if the manifold M is not finitely covered by $S^2 \times S^1$ then every leaf is homeomorphic to $\mathbb{R}^2$. Also, if M is $\mathbb{T}^3$ one can see that every closed leaf of $\mathcal{F}$ must be a two-dimensional torus (since for every other surface S, the fundamental group $\pi_1(S)$ does not inject in $\mathbb{Z}^3$).

The last statement of (iii) follows by the fact that the leaves of $\mathcal{F}$ being incompressible they lift to $\tilde{M}$ as simply connected leaves. Applying Reeb's stability Theorem 4.5 we see that if one leaf is a sphere, then the first situation occurs, and if there are no leaves homeomorphic to S^2 then all leaves of $\tilde{\mathcal{F}}$ must be planes and by a result of Palmeira ([Cal]) we obtain that $\tilde{M}$ is homeomorphic to $\mathbb{R}^3$.

Exercise. Prove Corollary 4.9.

4.2. **Foliations of $\mathbb{T}^2 \times [0,1]$.** In view of this result and in order to classify foliations in torus bundles over the circle it is natural to look at foliations of $\mathbb{T}^2 \times [0,1]$. By considering a gluing of the boundaries by the identity map, we get a foliation of $\mathbb{T}^3$. These foliations (in the C^0-case) were classified in [Pot] by applying the ideas developed in [BBI_2] (which stopped not far from obtaining the result bellow), and the result can be restated in the terms used here as follows:

Theorem 4.10 (Theorem 5.4 and Proposition 5.7 of [Pot])**.** *Let $\mathcal{W}$ be a Reebless foliation of $\mathbb{T}^3$, then, $\mathcal{W}$ is almost aligned with a linear foliation of $\mathbb{T}^3$. Moreover, if the linear foliation is not a foliation by tori, then $\mathcal{W}$ is almost parallel to the linear foliation and if it is a foliation by tori then there is at least one torus leaf.*

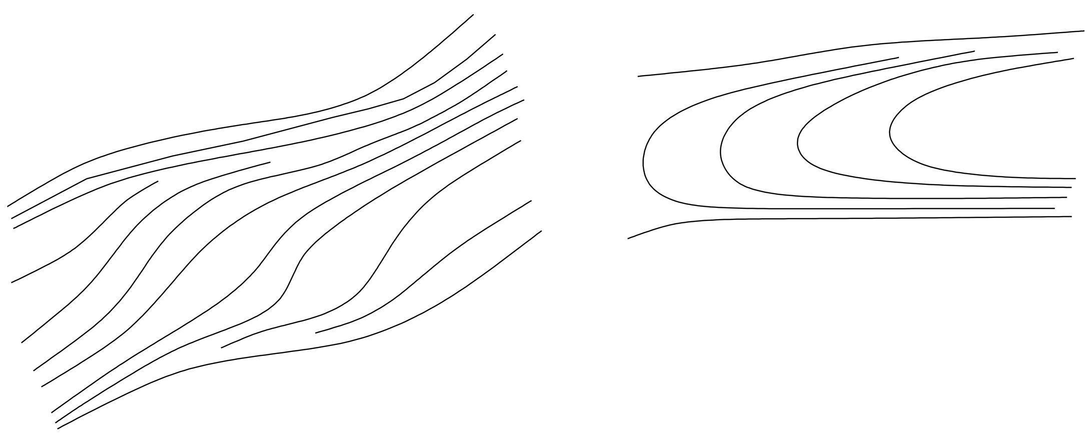

FIGURE 10. Possibilities for foliations.

A *linear foliation* of $\mathbb{T}^3$ is the projection by the natural projection $p : \mathbb{R}^3 \to \mathbb{R}^3/\mathbb{Z}^3 \cong \mathbb{T}^3$ of a linear foliation of $\mathbb{R}^3$. It is a foliation by tori if the linear foliation is given by a plane generated by two vectors in $\mathbb{Z}^3$. The same classification can be done for one-dimensional foliations of $\mathbb{T}^2$ for which the proof is easier (see for example section 4.A of [Pot$_2$]). This allows us to classify foliations of $\mathbb{T}^2 \times [0,1]$ transverse to the boundary:

Proposition 4.11. *Let $\mathcal{W}$ be a foliation of $\mathbb{T}^2 \times [0,1]$ which is transverse to the boundary and has no torus leaves. Then, the foliation $\mathcal{W}$ is almost aligned to a foliation of the form $\mathcal{L} \times [0,1]$ where $\mathcal{L}$ is a linear foliation of $\mathbb{T}^2$. If $\mathcal{L}$ is not a foliation by circles, then $\mathcal{W}$ is almost parallel to $\mathcal{L} \times [0,1]$.*

Proof. The proof can be done directly (see [Rou, Pl] for the C^2-case). We will use Theorem 4.10 instead. Consider the foliation $\mathcal{W}'$ of $\mathbb{T}^2 \times [0,2]$ obtained by gluing $\mathbb{T}^2 \times [0,1]$ with the foliation $\mathcal{W}$ with $\mathbb{T}^2 \times [1,2]$ with the foliation $\varphi(\mathcal{W})$ where $\varphi_1 : \mathbb{T}^2 \times [0,1] \to \mathbb{T}^2 \times [1,2]$ is given by $\varphi_1(x,t) = (x, 2-t)$. It is not hard to check that this gives rise to a well defined foliation $\mathcal{W}'$ of $\mathbb{T}^2 \times [0,2]$ (is like putting a mirror in the torus $\mathbb{T}^2 \times \{1\}$).

We can now construct a foliation of $\mathbb{T}^3$ as follows: we glue $\mathbb{T}^2 \times \{0\}$ with $\mathbb{T}^2 \times \{2\}$ by the diffeomorphism $\varphi_2 : \mathbb{T}^2 \times \{0\} \to \mathbb{T}^2 \times \{2\}$ given by $\varphi_2(x,0) = (x,2)$. Again, it is easy to show that the foliation can be defined in $\mathbb{T}^3 = \mathbb{T}^2 \times [0,2]/_{\varphi_2}$.

By the previous Theorem, we know that the resulting foliation is almost aligned to a linear foliation of $\mathbb{T}^3$. Since we have assumed that there is no torus leaves of $\mathcal{W}$ we know that this linear foliation cannot be the one given by the planes $\mathbb{R}^2 \times \{t\}$ so, it must be transverse to the boundaries of $\mathbb{T}^2 \times [0,1]$. This concludes.

□

Remark 4.12. As a consequence of the previous result we get the following: The foliations $\mathcal{W} \cap (\mathbb{T}^2 \times \{0\})$ and $\mathcal{W} \cap (\mathbb{T}^2 \times \{1\})$ are almost aligned to each other. In particular, one can prove that if in one of the boundary components is almost parallel to a linear foliation, then the whole foliation $\mathcal{W}$ is almost parallel to a linear foliation of $\mathbb{T}^2$ times $[0,1]$.

4.3. **Transverse tori.** For C^2-foliations, Plante (see [Pl]) gave a classification of foliations without torus leaves in 3-dimensional manifolds with almost solvable fundamental group. His proof relies on the application of a result from [Rou] which uses the C^2-hypothesis in an important way (other results which used the C^2-hypothesis such as Novikov's Theorem are now well known to work for C^0-foliations thanks to [So]). We shall use a recent result of Gabai [Ga] which plays the role of Roussarie's result and allows the argument of Plante to be recovered.

We state now a consequence of Theorem 2.7 of [Ga] which will serve our purposes[18]:

[18]Notice that a foliation of a 3-dimensional manifold without closed leaves is *taut*, see [Cal] Chapter 4 for definitions and these results.

Theorem 4.13. *Let $\mathcal{F}$ be a foliation of a 3-dimensional manifold M without closed leaves and let T be an embedded two-dimensional torus whose fundamental group injects in the one of M, then, T is isotopic to a torus which is transverse to $\mathcal{F}$.*

On the one hand, Gabai proves that a closed incompressible surface is homotopic to a surface which is either a leaf of $\mathcal{F}$ or intersects $\mathcal{F}$ only in isolated saddle tangencies. Since the torus has zero Euler characteristic, this implies that it must be transverse to $\mathcal{F}$. We remark that Gabai's result is stated by the existence of a homotopy, and this must be so since Gabai starts with an *immersed* surface, however, it can be seen that Lemma 2.6 of [Ga] can be done by isotopies if the initial surface is embedded. The rest of the proof uses only isotopies. See also [Cal] Lemma 5.11 and the Remark after Corollary 5.13.

Just to give a taste on the ideas of the proof of this result, let us first explain an heuristic proof in the C^2-case (which is not the one that Roussarie gave). When a foliation of a 3-dimensional manifold is without torus leaves, it can be seen to be what is called *taut*. This has several equivalences, but in the C^2-case, Sullivan showed that for such a foliation there exists a Riemannian metric g which makes every leaf of $\mathcal{F}$ a minimal surface (this result is I believe the reason for the name taut, see [Cal] Theorem 4.31). Once this is done, one can consider the embedded torus as a surface in M with this metric, and a result of Shoen-Yau ([Cal] Theorem 3.29) states that the torus is isotopic to a minimal surface. By the maximum principle this implies that the torus can only intersect the foliation in singularities with negative index, but since the torus has zero Euler characteristic one deduces that it is transverse to the foliation.

In the C^0 case one the previous proof makes no sense as it is, since the minimal surfaces arguments require differentiability of at least class C^2. However, in [Ga] showed that it is possible to make a macroscopic version of Sullivan's result (see [Cal] Example 4.32) which allows to reduce the problem to a local problem (essentially the global problems are solved by going to a minimal position with respect to this macroscopic metric) which becomes a problem of general position and Roussarie arguments can be performed in the same way as Novikov's result works in the C^0-case thanks to the general position arguments of [So].

4.4. **Classification of foliations in 3-manifolds which are torus bundles over the circle.** Consider the manifold M_ψ obtained by $\mathbb{T}^2 \times [0,1]$ by identifying $\mathbb{T}^2 \times \{0\}$ with $\mathbb{T}^2 \times \{1\}$ by a diffeomorphism ψ. Let $p : M_\psi \to S_1 = [0,1]/_\sim$ given by the projection in the second coordinate.

The construction of M_ψ determines a incompressible torus in M_ψ which we will assume remains fixed. Under this choice of incompressible torus we can consider a family of foliations of M_ψ transverse to such torus.

We are now able to classify foliations in torus bundles over S^1 depending on the isotopy class of ψ.

In the case that $\psi : \mathbb{T}^2 \cong S^1 \times S^1 \to \mathbb{T}^2$ is a Dehn-twist of the form:

$\psi(t,s) = (t, s+kt)(\mod \mathbb{Z}^2)$

M_ψ is homeomorphic to a nilmanifold N_k. We define the foliations $\mathcal{F}_\theta$ on N_k given by starting with the linear foliation $\mathcal{L}$ of $\mathbb{T}^2$ by circles of the form $\{t\} \times S^1$ and we consider the foliation $\mathcal{L} \times [0,1]$ of $\mathbb{T}^2 \times [0,1]$. The foliation $\mathcal{F}_\theta$ will be the foliation obtained by gluing $\mathbb{T}^2 \times \{0\}$ with $\mathbb{T}^2 \times \{1\}$ by the diffeomorphism

$$\psi_\theta : \mathbb{T}^2 \times \{0\} \to \mathbb{T}^2 \times \{1\} \qquad \psi_\theta(t,s,0) = (t+\theta, s+kt, 1)$$

Remark 4.14. Notice that if $\mathcal{W}$ is a foliation of N_k which is transverse to T the torus obtained by projection of $\mathbb{T}^2 \times \{0\} \sim \mathbb{T}^2 \times \{1\}$ we know that it must be invariant by a map of T which is isotopic to ψ.

The foliation $\mathcal{F}_\infty$ is the foliation by the fibers of the torus bundle. It is not hard to prove that the foliations $\mathcal{F}_\theta$ are pairwise not almost parallel.

Theorem 4.15. *Let $\mathcal{W}$ be a codimension one Reebless foliation of N_k. Then, $\mathcal{W}$ is almost aligned to $\mathcal{F}_\theta$ for some $\theta \in \mathbb{R} \cup \{\infty\}$. Moreover, if θ is irrational then $\mathcal{W}$ is almost parallel to $\mathcal{F}_\theta$.*

Proof. If $\mathcal{W}$ has a torus leaf, this torus must be incompressible by Novikov's Theorem ([So, CC]). We can cut the foliation along this torus. By doing the same doubling proceedure as in Proposition 4.11 we obtain a foliation of $\mathbb{T}^3$ and using Theorem 4.10 we deduce that $\mathcal{W}$ is almost aligned to a foliation of the form $\mathcal{F}_\theta$ with θ being irrational.

If $\mathcal{W}$ has no torus leaves, we can consider the torus $\mathbb{T}^2 \times \{0\} \subset M_\psi$ which is incompressible. Using Theorem 4.13 we can make an isotopy and assume that the foliation $\mathcal{W}$ is transverse to this torus (recall from Proposition 3.13 that the isotopy does not affect the equivalence class of the foliation under the relation of being almost parallel). Here we are using the fact that the isotopy of the torus can be extended to a global isotopy of M (see for example Theorem 8.1.3 of [Hi]).

We can cut M_ψ by this torus and apply Proposition 4.11 to obtain that $\mathcal{W}$ in $\mathbb{T}^2 \times [0,1]$ is almost aligned to a linear foliation of $\mathbb{T}^2$ times $[0,1]$. In fact, if the foliation is not almost parallel to the linear foliation we deduce that the foliation in $\mathbb{T}^2 \times \{0\}$ must have Reeb annuli (see section 4.A of [Pot_2]). Since the foliation in $\mathbb{T}^2 \times \{0\}$ must be glued by ψ with the foliation in $\mathbb{T}^2 \times \{1\}$ we deduce that it must permute these Reeb annuli which are finitely many. So, we get that there is a periodic circle of the foliation of

$\mathbb{T}^2 \times \{0\}$ by ψ which implies the existence of a torus leaf for $\mathcal{W}$. We deduce that $\mathcal{W}$ in $\mathbb{T}^2 \times [0,1]$ is almost parallel to a linear foliation of $\mathbb{T}^2$ times $[0,1]$.

Now, we must show that this linear foliation corresponds to the linear foliation $\mathcal{L}$ by circles of the form $\{t\} \times S^1$ but this follows from the fact that the foliation is invariant by ψ.

Now we must see that after gluing the foliation is almost parallel to some $\mathcal{F}_\theta$. This follows from the following fact, since in the boundary it is almost parallel to the foliation $\mathcal{L}$, we know that it has at least one circle leaf L. Now we obtain the value of θ by regarding the relative order of the images of $\psi^n(L)$ and performing a classical rotation number argument as in the circle.

□

When ψ is isotopic to Anosov, the classification gives only three possibilities.

We consider then A a hyperbolic matrix in $SL(2,\mathbb{Z})$ and in S_A we consider the following linear foliations: $\mathcal{F}^{cs}_A$ is given by the linear foliation which is the projection of $\mathcal{L}^s \times [0,1]$ where $\mathcal{L}^s$ is the linear foliation corresponding to the strong stable foliation of A and similarly we obtain $\mathcal{F}^{cu}_A$ as the projection of $\mathcal{L}^u \times [0,1]$ where $\mathcal{L}^u$ is the linear foliation which corresponds to the strong unstable foliation.

Finally, we consider the foliation $\mathcal{F}_T$ which is the projection of foliation by tori $\mathbb{T}^2 \times \{t\}$ to M_A. Clearly, these 3 foliations are pairwise not almost parallel to each other.

Theorem 4.16. *Let $\mathcal{W}$ be a Reebless foliation of S_A, then, $\mathcal{W}$ is almost aligned to one of the foliations $\mathcal{F}^{cs}_A, \mathcal{F}^{cu}_A$ or $\mathcal{F}_T$. Moreover, if $\mathcal{W}$ has no torus leaves, then $\mathcal{W}$ is almost parallel to either $\mathcal{F}^{cs}_A, \mathcal{F}^{cu}_A$ and if it is almost aligned with $\mathcal{F}_T$ it has a torus leaf.*

Proof. The first part of the proof is as in the previous Theorem: If $\mathcal{W}$ has a torus leaf, it must be isotopic to T the projection of $\mathbb{T}^2 \times \{0\}$ since it is the only incompressible embedded torus in M_A and we get that we get that $\mathcal{W}$ is almost parallel to $\mathcal{F}_T$.

Otherwise, we can assume that $\mathcal{W}$ is transverse to T and we obtain a foliation of $\mathbb{T}^2 \times [0,1]$ which is almost aligned with a foliation of the form $\mathcal{L} \times [0,1]$ and which in T is invariant under a diffeomorphism f isotopic to A.

This implies that the linear foliation $\mathcal{L}$ is either the strong stable or the strong unstable foliation for A, and in particular, since it has no circle leaves, we get that $\mathcal{W}$ in $\mathbb{T}^2 \times [0,1]$ is almost parallel to $\mathcal{L} \times [0,1]$.

Now, since the gluing map f is isotopic to A, we know it is semiconjugated to it, so, we get that after gluing, the foliations remain almost parallel.

□

We will denote as $\hat{\varphi}^s_A : \hat{S}_A \to \hat{S}_A$ to the flow in the coordinates $\mathbb{T}^2 \times \mathbb{R}$ given by

$$\hat{\varphi}_A^s(v,t) = (v, t+s)$$

It is the lift of a flow $\varphi_A^s : S_A \to S_A$ which we call the suspension of A. It is not hard to check that it is an Anosov flow. We will denote as $\varphi_A^1 : S_A \to S_A$ to the time one map of the suspension of the Anosov flow which can be seen as

$$\varphi_A^1(v,t) = (v, t+1) = (Av, t)$$

The suspension of A^{-1} is exactly $\varphi_{A^{-1}}^s = \varphi_A^{-s}$. It interchanges the stable and unstable foliations of these Anosov flows.

The key fact we will use about the flow φ_A^s and its time one map φ_A^1 is the structure of its invariant foliations which we will denote as $\mathcal{F}_A^{cs}$ and $\mathcal{F}_A^{cu}$ whose properties can be summarized in the following proposition and are easy to verify:

Proposition 4.17. *Let $\mathcal{F}_A^{cu}$ be the center-unstable foliation of φ_A^1 and $\tilde{\mathcal{F}}_A^{cu}$ its lift to the universal cover. Then, if L_1 and L_2 are two leaves of $\tilde{\mathcal{F}}_A^{cu}$ we have the following properties:*

(i) *The distance between leaves factorizes through p_1. This means that for $x, y \in L_1$ with $p_1(x) = p_1(y)$ we have that $d(x, L_2) = d(y, L_2)$.*
(ii) *For every $\varepsilon > 0$ there exists $T > 0$ such that if $p_1(z) > T$ and $z \in L_1$ then $d(z, L_2) < \varepsilon$.*
(iii) *For every $C > 0$ there exists $T < 0$ such that if $p_1(z) < T$ and $z \in L_1$ then $d(z, L_2) > C$.*

The same exact property holds for $\tilde{\mathcal{F}}_A^{cs}$ exchanging $p_1(z)$ by $-p_1(z)$.

Here $p_1 : \tilde{S}_A \to \mathbb{R}$ denotes the lift of $p : S_A \to S^1$ (the bundle projection) to the universal cover (which is the one which generates the homology, see Section 8 for more details).

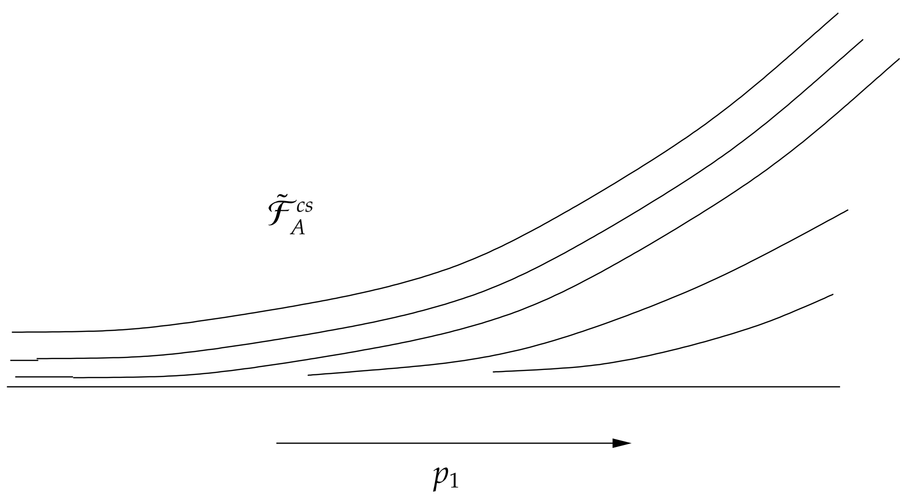

FIGURE 11. The foliation $\mathcal{F}_A^{cs}$.

Notice that $\mathcal{F}^{cs}_A$ is obtained by considering in $\hat{S}_A$ the product of the stable foliation of A in $\mathbb{T}^2$ with $\mathbb{R}$ projecting by the quotient map $\hat{\pi} : \hat{S}_A \to S_A$. The symmetric property holds for $\mathcal{F}^{cu}_A$.

4.5. **Foliations without torus leaves.** We state here a result which can be ignored if the reader only wants to understand the classification result for torus bundles over the circle. Indeed this works as (another) black box in our proof and it is not relevant in terms of the ideas of partial hyperbolicity.

This result was proved by Plante in [Pl] based on Evans-Moser's Theorem (Theorem 2.12) for C^1-foliations. However, the proof holds for C^0-foliations since the only place where the C^1-hypothesis is used is to obtain general position results which were solved in the C^0-case by Solodov in [So].

Theorem 4.18 (Theorem 3.1 of [Pl])**.** *Let M be a closed 3-manifold with almost solvable fundamental group. Suppose M admits a transversely oriented codimension one foliation which does not have any compact leaves. Then M is a torus bundle over the circle. Furthermore, any attaching map for M either has 1 as an eigenvalue or is hyperbolic (i.e. M is either $\mathbb{T}^3$, N_k or S_A).*

We will not prove this Theorem, but we leave the following exercise to the reader which we think may show the idea behind this result (see also Theorem 2.12):

Exercise. Show that a foliation of the Klein bottle has a circle leaf.

$\diamondsuit$

4.6. **Global product structure.** An important tool in the proof of dynamical coherence for partially hyperbolic diffeomorphisms of $\mathbb{T}^3$ or nilmanifolds is the use of global product structure of foliations.

Definition 4.19 (Global product structure)**.** We say that a two dimensional foliation $\mathcal{F}$ and a transverse one dimensional foliation $\mathcal{F}^{\perp}$ have *global product structure* if when lifted to the universal cover, we get that for every $x \neq y$ we have that:

$$\tilde{\mathcal{F}}(x) \cap \tilde{\mathcal{F}}^{\perp}(y) \neq \emptyset$$

and consists of no more than one point.

$\diamondsuit$

There exists a criterium for obtaining global product structure. It has to do with the fact that when a foliation has no *holonomy* (or at least not much) then in the universal cover, if one considers a one dimensional foliation then it must traverse a lot of leaves. In [Pot] a quantitative version of it was obtained and this was useful to study the case of partially hyperbolic diffeomorphisms with higher dimensional center bundle, we refer the reader to [Pot, Pot$_2$] for more information on such result.

Theorem 4.20 (Theorem VIII.2.2.1 of [HeHi]). *Consider a codimension one foliation $\mathcal{F}$ of a compact manifold M such that all the leaves of $\mathcal{F}$ are simply connected. Then, for every $\mathcal{F}^{\perp}$ foliation transverse to $\mathcal{F}$ we have that $\mathcal{F}$ and $\mathcal{F}^{\perp}$ have global product structure.*

Since this Theorem is fundamental for the classification result in the case where f is isotopic to Anosov in $\mathbb{T}^3$ we sketch the proof in the lineas of [Pot] (which works for $\mathbb{T}^3$ but not for general manifolds). We refer the reader to [Pot, Pot$_2$] for a more detailed proof in a more general setting.

Sketch We assume $M = \mathbb{T}^3$ and every leaf of $\mathcal{F}$ is simply connected. Consider $\tilde{\mathcal{F}}$ the lift of $\mathcal{F}$ to $\mathbb{R}^3$.

The proof is organized as follows:

- Consider $\mathcal{O} \neq \mathbb{T}^3$ an open set which is $\mathcal{F}$ saturated. Then, there cannot be a closed transversal to $\mathcal{F}$ contained in $\mathcal{O}$. This closed transversal would correspond to a deck transformation and imply the existence of a non-simply connected leaf in the boundary of $\mathcal{O}$.
- Let $\mathcal{F}^{\perp}$ be a transverse foliation, using Novikov's Theorem and the previous remark one deduces that there exists $\ell > 0$ such that every transversal of length ℓ intersects every leaf of $\mathcal{F}$.
- Using the fact that the fundamental group of $\mathbb{T}^3$ is abelian (and thus deck transformations correspond to free homotopy classes of curves) one deduces that in the universal cover, a curve of length ℓ in $\tilde{\mathcal{F}}^{\perp}$ intersects a translate of the initial leaf in $\tilde{\mathcal{F}}$ by a uniform translation.

This allows one to deduce global product structure. See [Pot$_2$] section 4.3 for more details.

□

When one obtains global product structure, some consequences follow:

Proposition 4.21. *Let $\mathcal{F}$ be a codimension one foliation of $\mathbb{T}^3$ and $\mathcal{F}^{\perp}$ a transverse foliation. Assume the foliations $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}^{\perp}$ lifted to the universal cover have global product structure. Then, the foliation $\tilde{\mathcal{F}}^{\perp}$ is quasi-isometric. Moreover, if $\mathcal{F}$ is almost parallel to a foliation by planes parallel to P then, there exists a cone $\mathcal{E}$ transverse to P in $\mathbb{R}^3$ and $K > 0$ such that for every $x \in \mathbb{R}^3$ and $y \in \tilde{\mathcal{F}}^{\perp}(x)$ at distance larger than K from x we have that $y - x$ is contained in the cone $\mathcal{E}$.*

Proof. Notice that the global product structure implies that $\mathcal{F}$ is Reebless. Moreover, having global product structure implies that $\mathcal{F}$ is almost parallel to a foliation by planes parallel to a certain subspace P.

Consider v a unit vector perpendicular to P in $\mathbb{R}^3$.

Global product structure implies that for every $N > 0$ there exists L such that every segments of $\tilde{\mathcal{F}}^\perp$ of length L starting at a point x intersect $P + x + Nv$. Indeed, if this was not the case, we could find arbitrarily large segments of leaves of $\tilde{\mathcal{F}}^\perp$ not satisfying this property, by taking a subsequence and translations such that the initial point is in a bounded region, we obtain a leaf of $\tilde{\mathcal{F}}^\perp$ which does not intersect every leaf of $\tilde{\mathcal{F}}$.

This implies quasi-isometry since having length larger than kL implies that the endpoints are at distance at least kN.

Moreover, assuming that the last claim of the proposition does not hold, we get a sequence of points x_n, y_n such that the distance is larger than n and such that the angle between $y_n - x_n$ with P is smaller than $1/\|x_n - y_n\|$.

In the limit (by translating x_n we can assume that it has a convergent subsequence), we get a leaf of $\tilde{\mathcal{F}}^\perp$ which cannot intersect every leaf of $\tilde{\mathcal{F}}$ contradicting the global product structure.

□

5. General strategy for the classification result

In this section we try to present an overview of the proof of our Main Theorem joint with A. Hammerlindl. We will try to present the main ideas an fundamentally try to explain the main difficulties to give an heuristic idea on how to solve them. In this notes we will emphasize mainly in the solvmanifold case since there are already available preprints on the other results. Hopefully the solvmanifold one will be available soon, but one never knows.

5.1. **Dynamical Coherence.** Probably the hardest part of our result is to establish dynamical coherence for partially hyperbolic diffeomorphisms in the manifolds we are studying. The starting point will be to use the existence of f-invariant branching foliations and trying to prove that those do not branch. In the absolute partially hyperbolic case, this is done for $\mathbb{T}^3$ and nilmanifolds by showing quasi-isometry of the strong foliations which due to Brin's result is enough to obtain unique integrability of the bundles (see Proposition 3.11). In the pointwise setting this fails so we need to work on different arguments. In the next section we explain briefly how the proof in the absolute case works to show the differences between the proofs. We remark here that in the solvmanifold case, to my knowledge at least, there is no easier proof of dynamical coherence even by assuming absolute (or stronger forms of) partial hyperbolicity.

The reason we say that proving dynamical coherence is the hardest part is for two reasons: One is that in our proof of dynamical coherence we include some parts which are already preparation for leaf conjugacy (such as proving that the corresponding foliations are almost parallel to the corresponding ones). The second one is that obtaining leaf conjugacy for the absolute partially hyperbolic case is not so different than for the pointwise case and this had already been done by Hammerlindl ([H, H_2]) at least in the nilmanifold and torus case.

In my opinion, a key ingredient of the proof of dynamical coherence is the correct separation in cases. The reason coherence holds does not really depend on the manifold in question but on the action in homology. We can thus separate in cases.

- When $M = \mathbb{T}^3$ and f is isotopic to Anosov. The first point is to prove that the branching foliations are almost parallel to linear foliations which are invariant by f_*, the action in homology of f. This implies that all leaves are planes, in particular simply connected. This allows us to show that there is a global product structure between the strong foliations and the branching ones which is enough to conclude no branching.
- When $M = \mathbb{T}^3$ or N_k and f is isotopic to a skew product. In this case, we first show that the branching foliations are almost parallel to the leaves of the skew product. Here, the main point is that since there is only one eigenvalue larger than one, the unstable direction must escape to infinity in that direction since otherwise, taking an unstable arc J we get that $f^n(J)$ would have diameter which cannot grow more than polynomially in n and the volume of a neighborhood of $f^n(J)$ should grow exponentially which is a contradiction with the polynomial volume growth of M. This exact argument can be also applied for the strong unstable foliation in the Anosov case by assuming that there are two eigenvalues of modulus smaller than one (but in the Anosov case it will not be symmetric and the other foliation would pose problems).
- When $M = S_A$ there is more work to be done. First, we show that the topology of S_A allows us to assume from the start that f is isotopic to the identity. This is important since there are very few possible foliations in S_A and from their structure in the universal cover and the fact that f is isotopic to the identity it is possible to show that leaves of the branching foliation are fixed by a suitable lift to the universal cover. Then, the idea is to adapt an argument of [BoW] which implies that there are no fixed strong stable leaves and show that the existence of branching would imply the existence of a fixed strong stable leaf.

We hope that this very brief outline works as a motivation for separating the proof of the main theorem into three separate sections. We mention here that is is well plausible that many of the arguments in the solvmanifold case work in the case where f is isotopic to the identity in other manifolds. I believe that one of the most relevant questions remaining in the classification problem of partially hyperbolic diffeomorphisms in 3-manifolds is the following:

Question 6. *Let $f : M \to M$ be a partially hyperbolic diffeomorphism such that M is not finitely covered by $\mathbb{T}^3$ or N_k. Is it true that f has an iterate isotopic to the identity?*

Notice that by Mostow rigidity the previous question has a positive answer in hyperbolic 3-manifolds, so they should be a possible place to continue the classification (however, this is not trivial, since foliations of such manifolds are not as simple as in torus bundles over the circle).

5.2. **The absolute case.** In this section we shall review the proof of dynamical coherence in some particular cases. The reason to do this is to show how some cases are more difficult than others and show what kind of difficulties appear. As we mentioned, we will restrict to the case with polynomial growth of volume since we do not know if there is some easier proof in the solvmanifold case even by restricting further the hypothesis on partial hyperbolicity.

First, we will prove dynamical coherence in a special case under some strong form of absolute partial hyperbolicity (this is based on an argument from [H_2]):

Proposition 5.1. *Assume that $f : M \to M$ is a partially hyperbolic diffeomorphism of a 3-manifold M with polynomial growth of volume of degree d. Assume that we have that there exist constants $\lambda_1 < \lambda_2^d < \lambda_2 < 1 < \mu_1 < \mu_1^d < \mu_2$ such that we have $N > 0$ verifying:*

$$\|Df^N|_{E^s}\| < \lambda_1 < \lambda_2 < \|Df^N|_{E^c}\| < \mu_1 < \mu_2 < \|Df^N|_{E^u}\|$$

Then, f is dynamically coherent and the bundles are uniquely integrable.

Proof. Assume that E^{cs} it is not uniquely integrable (the argument for E^{cu} is symmetric). Then, as we have already done we can find a non trivial unstable arc J and a point $x \in \tilde{M}$ and not in J such that for every point $z \in J$ there is a curve γ from x to z which is tangent to E^{cs} (see also the proof of Proposition 3.11). Moreover, these curves can be all chosen to have length smaller than a certain universal constant K by compactness.

Iterating forward the arc J we deduce that its length growths exponentially with rate larger than μ_2. Using Corollary 4.9 (iv) we obtain that for some δ we have that:

$$\mathrm{Vol}(B_\delta(f^n(J))) \geq C_\delta \mu_2^n$$

Using the polynomial growth of volume we deduce that

$$\operatorname{diam}(f^n(J)) \geq C_0 \mu_2^{n/d}$$

On the other hand, we get that there exists C_1 such that all the curves γ joining x with J as chosen above have length smaller or equal to

$$\operatorname{length}(f^n(\gamma)) \leq C_1 \mu_1^n$$

Which gives a contradiction with the triangle inequality since this implies that the diameter of $f^n(J)$ is bounded by $2C_1\mu_1^n$.

□

In the skew product case, this argument above can be adapted in such a way to obtain dynamical coherence for all absolute partially hyperbolic diffeomorphisms (see [H_2] Theorem 4.9), or to prove unique integrability of the center-stable bundle in the case isotopic to Anosov with two eigenvalues of modulus smaller than one. The main point is that in that case it is possible to compare the constants of partial hyperbolicity with those of the algebraic model (and since there is only one eigenvalue larger than one this allows to obtain nice properties, moreover, one can project in certain directions and this allows not to use the polynomial growth exactly but to project in this direction and have linear growth, see [H_2]). This is by no means trivial, but we wanted to emphasize that this line of argument already poses problems in the Anosov case.

The argument in [BBI_2] uses absolute partial hyperbolicity to show that even when one cannot compare exactly the constants of partial hyperbolicity one can get that the strong foliations are quasi-isometric so that Brin's result applies (Proposition 3.11). Their argument also depends on comparing the absolute constants of partial hyperbolicity with the ones of the action in homology.

5.3. **Leaf conjugacy.** To prove leaf conjugacy we will also separate into the same three classes. We explain briefly the main ideas in each:

- If $M = \mathbb{T}^3$ and f is isotopic to Anosov we will use the semiconjugacy given by Theorem 3.7. This involves first showing that the leaves of the obtained foliations are close to the correct ones (here the proof uses an ad-hoc argument involving *accessibility*, it could be nice to have a more direct aproach). Using the same kind of arguments as above it is not hard to show that the semiconjugacy will be injective on strong foliations which allows to prove that the preimages of points by the semiconjugacy will be segments inside the center leaves. This is not far from leaf conjugacy[19]. To get leaf conjugacy one can adapt an argument of [H] which gives a way to construct a homeomorphism once we have decided which leaf to send to which leaf.

- If $M = \mathbb{T}^3$ or N_k and f is isotopic to a skew-product we must show that every center leaf is a circle. To show this we use a kind of shadowing argument. We must show that if two points remain close in the universal cover then their center leaves coincide: This follows from the fact that the strong foliations are more or less close to linear (closeness here means that the distance to a linear foliation is subexponential) so that we obtain the desired property. Since in the isotopy class the action in the fundamental group of translation in the direction of the skew product commutes with the lift of f to the universal cover, we get that the translate by this deck transformation of any point has all its iterates at bounded distance. This implies that the center foliation is invariant under that deck transformation which is what we wanted. Now leaf conjugacy follows quite easily since all leaves being compact and of bounded length we get that the quotient is a torus and the dynamics expansive so that we can conjugate to a linear Anosov to obtain the leaf conjugacy with the skew-product as desired.
- In the case where $M = S_A$ when proving coherence we showed that center leaves must be fixed. After that, one must show that every point "turns" in the same direction. This allows one to construct a torus which is transverse to the leaves of the center foliation and using this we get that the return map is also expansive and so conjugate to a linear Anosov diffeomorphism. This allows to construct the leaf conjugacy.

5.4. **When there are periodic torus.** In what follows, a characterization of partially hyperbolic diffeomorphisms admitting periodic torus T tangent to E^{cs} will be given. Informally, the idea is to show that the existence of such a tori forces the diffeomorphism to be more or less the example presented in [RHRHU$_4$].

Along this section $f : M \to M$ will denote a partially hyperbolic diffeomorphism which admits a periodic torus T_1 tangent to E^{cs}. Now, let $T_1, \dots, T_k$ be the finite family of f-periodic tori tangent to either E^{cs} or E^{cu}. This family is clearly finite since each tori must be either normally attracting or normally repelling and thus a (uniformly) bounded distance apart from the rest.

After considering an iterate, it is possible to assume that every tori T_i is fixed by f. It is enough to study the following situation (otherwise, consider g^{-1}):

[19]In fact, in a certain sense it is more, since it provides not only complete information on the topology of the leaves but also on the dynamics *inside* center leaves which is not provided by leaf conjugacy.

> $g : \mathbb{T}^2 \times [0,1] \to \mathbb{T}^2 \times [0,1]$ is a partially hyperbolic diffeomorphism such that $T_0 = \mathbb{T}^2 \times \{0\}$ is fixed and tangent to E^{cs} and $T_1 = \mathbb{T}^2 \times \{1\}$ is tangent to either E^{cs} or E^{cu}. Moreover, there is no tori T in $\mathbb{T}^2 \times (0,1)$ tangent to E^{cs} or E^{cu}.

The fact that this is enough follows from [RHRHU$_3$] which shows that after cutting along a tori, the resulting manifold is $\mathbb{T}^2 \times [0,1]$.

For such a g there exists a linear hyperbolic automorphism A of $\mathbb{T}^2$ and a continuous and surjective map $h : \mathbb{T}^2 \times [0,1] \to \mathbb{T}^2$ such that $h \circ g = A \circ h$. Moreover, in the universal cover, there is a lift $H : \mathbb{R}^2 \times [0,1] \to \mathbb{R}^2 \times [0,1]$ at bounded distance from the identity.

Let us call U^0 the basin of repulsion of $T_0 = \mathbb{T}^2 \times \{0\}$. Define U^1 accordingly depending on the case. One can also define $K^i = (U^i)^c$ and $K = K^1 \cap K^2$.

The dynamics on the basin is quite simple:

Lemma 5.2. *The basins U^0 and U^1 are homeomorphic to $\mathbb{T}^2 \times [0,1)$ and admit a g-invariant foliation by tori. Moreover, if $U^0 \cap U^1 = \mathbb{T}^2 \times (0,1)$, the foliation can be chosen to cover the whole $\mathbb{T}^2 \times [0,1]$.*

Proof. We work with U^0, the case of U^1 is symmetric. Consider a small neighborhood N of T_0 homeomorphic to $T_0 \times (0,1)$ which is contained in the basin of repulsion of T. There exist C^1-tori T' such that $g(T') \cap T' = \emptyset$. Moreover, the region bounded by T' and $g(T')$ is diffeomorphic to $\mathbb{T}^2 \times [0,1]$. Call this region D.

It is possible to fill up D by a C^1-foliation by tori. The foliation can be extended to $\hat{D} = \bigcup_{n \in \mathbb{Z}} g^n(R)$.

Normal hyperbolicity of T_0 implies that $U^0 = T_0 \cup \hat{D}$ admits a foliation[20] by tori which is trivially g-invariant. If $U^0 \cap U^1 = \mathbb{T}^2 \times (0,1)$ one has that the foliation can be further extended to T_1. $\square$

The study divides in two different cases which are treated in the next two propositions.

Proposition 5.3. *If T_1 is tangent to E^{cs}, then there exists a unique cu-foliation tangent to E^{cu} and moreover, the set K is a u-saturated set such that $h(K) = \mathbb{T}^2$ and such that the preimage of every point by h intersected with K consist of a (possibly trivial) compact interval tangent to E^c.*

The fact that in this case there exists a unique *cu*-foliation tangent to E^{cu} follows directly from Theorem B of [Pot]. In fact, such a g can be glued to itself to give a

[20] As A. Hammerlindl pointed out to me, it is possible to construct examples such that this foliation cannot be made to be C^1 at the boundary torus.

partially hyperbolic diffeomorphism of $\mathbb{T}^3$ not admitting *cu*-tori. It is the other part of the statement that is relevant about this proposition.

Proposition 5.4. *If T_1 is tangent to E^{cu} one has that $h(K^i) = \mathbb{T}^2$ for $i = 0, 1$ and there are two possibilities:*

(a) *If $U^0 \cap U^1 = \emptyset$ then $h(K) = \mathbb{T}^2$ the set the preimage of every point by h intersected with K consists of a (possibly trivial) compact interval tangent to E^c a bounded distance apart from the boundaries.*
(b) *If $U^0 \cap U^1 \neq \emptyset$ then the preimage of any point by h intersected with $K^i \cap \mathbb{T}^2 \times (0, 1)$ is a (possibly empty or trivial) interval tangent to E^c.*

First some results which hold in both cases are presented. As in the proof of Lemma 5.2 we will work with U^0, of course, the same results hold for U^1 with the caveat that in the hypothesis of Proposition 5.4 one has to exchange g by g^{-1}.

Lemma 5.5. *The sets U^0, K^0 and the boundary of U^0 are saturated by $\mathcal{W}^u$.*

Proof. The set U^0 is f-invariant and contains segments of unstables of uniform lenght in a neighborhood of T_0. This implies that U^0 is $\mathcal{W}^u$-saturated, then, by the continuous variation of $\mathcal{W}^u$-leaves so is $\overline{U}_0$. This implies that both K^0 and the boundary of U^0 being the complement of U^0 in $\mathcal{W}^u$-saturated sets are $\mathcal{W}^u$-saturated too. □

As a consequence, we deduce that, since K^0 is compact and non-empty, its image by h is the whole $\mathbb{T}^2$. This is because h cannot collapse an unstable leaf into a point (since that would contradict Corollary 4.9 item (iv) in the universal cover) so $h(K^0)$ is a closed set which contains at least a half unstable leaf of A, thus the whole torus.

From now on, we shall consider the branching foliations $\mathcal{F}^{cs}_{bran}$ and $\mathcal{F}^{cu}_{bran}$ given by Theorem 3.12, in particular from now on we will start to work in the universal cover. Let $\tilde{g}$ be a lift of g to the universal cover, the lift of A will still be called A and the relation $H \circ \tilde{g} = A \circ H$ holds. The lifts of U^i, K^i, T_i will keep the same notation.

Lemma 5.6. *Given a neighborhood U of the boundary tori tangent to E^{cs} there exists R such that the branching foliation $\mathcal{F}^{cs}$ intersected with the complement of U is almost parallel to the foliation by translates of $E^s_A \times [0, 1]$ with constant R in the definition of almost parallel. A symmetric argument works for $\mathcal{F}^{cu}$ outside a neighborhood*

Proof. By a straightforward adaptation of the arguments in [BBI$_2$, Pot] or [HP, Appendix A] using the fact that there are no *cs*-tori outside those neighborhoods one gets that $\mathcal{F}^{cs}$ is almost parallel to a foliation of the form $(E + v) \times [0, 1]$ where E is a subspace of $\mathbb{R}^2$. Using a growth argument one obtains that E is indeed E^s_A. □

Lemma 5.7. *Let S be a leaf of $\mathcal{F}^{cs}$ and T be a leaf of $\mathcal{F}^{cu}$ such that both intersect K^0. Then, for every γ connected component of $S \cap T \cap (U^0 \setminus T_0)$ one has that the closure of γ is a compact arc with one end point in T_0 and the other one in the boundary of U^0.*

Proof. Let γ be a connected component of $S \cap T \cap (U^0 \setminus T_0)$. We first show that $H(\gamma)$ is a unique point.

Since T_0 is a cs-tori, the intersection $T \cap T_0$ is a center curve in T_0 which must be mapped by the semiconjugacy to $E^u_A + H(z)$ for any $z \in T \cap T_0$ (see [Pot$_2$, Section 4.A]). Now, since U^0 is the union of strong unstable manifolds through points in T_0 one gets that every point in γ is in the strong unstable leaf of some point in $T \cap T_0$, so, we obtain that $H(\gamma) \subset E^u_A + H(x)$ for some $x \in \gamma$.

Now, assume that there is a point $y \in \gamma$ such that $H(x) \neq H(y)$ one can consider the segment I of γ between x and y. Their forward iterates by $\tilde{g}$ converge uniformly to the boundary of U_0 and by the semiconjugacy property and the fact that $H(y) - H(x) \in E^u_A$ one obtains that the images of I grow exponentially in the E^u_A direction. In particular, for some large n one gets that $\tilde{g}^n(S)$ cannot be at distance R from a translate $E^s_A \times [0,1]$ in a neighborhood of the boundary of U^0, a contradiction with Lemma 5.6 (notice that the boundary of U^0 cannot be close to cs-tori).

Since center leafs are uniformly properly embedded (as a consequence of Corollary 4.9) one gets that the length of $\tilde{g}^n(\gamma)$ remains bounded. In particular, the closure of γ consists of a closed interval whose extremal points are in the boundary of $U^0 \setminus T_0$. We must show that one point is in T_0 and the other in the boundary of U^0.

First assume that both points are contained in T_0. This implies that the boundary points of γ are mapped by H into the same point and so there is an arc γ' in T_0 tangent to E^c which is collapsed into that point by H. On the other hand, T is transverse to T_0 so one gets that an unstable leaf intersects both γ and γ'. One obtains that an interval of a $\mathcal{W}^u$-leaf must be mapped by H to $H(\gamma)$ which is a point, this contradicts Corollary 4.9.

Now, assume that both end points are contained in the boundary of U^0. Since the length of $\tilde{g}^n(\gamma)$ remains bounded for $n \in \mathbb{Z}$, there exists a subsequence $n_k \to -\infty$ and a sequence of translates $r_k \in \mathbb{Z}^2$ such that $\tilde{g}^{n_k}(\overline{\gamma}) + r_k$ converges uniformly to a center interval I. Since the interior of γ is contained in U^0 one gets that this interval must intersect T_0. Since the boundaries of γ are contained in the boundary of U^0 which is closed and $\tilde{g}$-invariant, the same happens to I. Now, consider the saturation of I by strong stable manifolds, this surface S_I is uniformly approached in compact sets by $\tilde{g}^{n_k}(S) + r_k$. Call S^+_k to be the connected component of $\mathbb{R}^2 \times [0,1] \setminus (\tilde{g}^{n_k}(S) + r_k)$ disjoint from S_I. As in the proof of [Pot, Theorem 5.4] (see also [HP, Appendix A]) one has that the translates of S^+_k by elements in $\mathbb{Z}^2$ cover the whole $\mathbb{R}^2 \times (0,1)$. On the other hand, by Theorem 3.12, no translate can intersect S_I since it is uniformly approached by surfaces of $\mathcal{F}^{cs}$, thus we get a contradiction.

□

Once this has been proved, the proof of Propositions 5.3 and 5.4 follow quite directly.

From the previous Lemma one obtains that for every pair of leafs $T \in \mathcal{F}^{cu}$ and $S \in \mathcal{F}^{cs}$, the set $T \cap S$ consists of a (unique) curve γ going from T_0 to T_1. Moreover, once it intersects K^0 it does not come back to U_0 again. These intersections consist exactly of the preimages by H of points in $\mathbb{R}^2$.

The case of Proposition 5.3 and the case where $U^0 \cap U^1 = \emptyset$ in Proposition 5.4 are identical and follow directly from the previous considerations.

When $U^0 \cap U^1 \neq \emptyset$ in Proposition 5.4 the result follows from Lemma 5.2.

Exercise. Make a modification of the example of [RHRHU$_4$] such that it has an invariant region homeomorphic to $\mathbb{T}^2 \times [0,1]$. If acquainted with [BD], use blenders to show that one can make the dynamics transitive in this region. Construct other examples showing that in a certain sense the "classification" given above is optimal.

To end this section we will show how with our results we can recover [BBI$_2$]'s result. What they prove in the absolute case is in principle stronger since they do not ask for the hypothesis of not having a f-periodic torus T tangent to either E^{cs} or E^{cu}. We can recover this by showing:

Proposition 5.8. *Let $f : M \to M$ be a partially hyperbolic diffeomorphism having an f-periodic torus T tangent to E^{cu} (or E^{cs}). Then, f is not absolutely partially hyperbolic.*

Sketch Let us assume that T^{cu} is a fixed torus invariant under f and tangent to E^{cu}. Then, the dynamics in T^{cu} must be semiconjugated to a certain linear Anosov diffeomorphism A of $\mathbb{T}^2$. We obtain that the entropy of $f|_{T^{cu}}$ is at least as big as the entropy of A. Using the variational principle and Ruelle's inequality for f^{-1} (see [M$_3$]) we deduce that for every $\varepsilon > 0$ there is an ergodic measure μ such that the center Lyapunov exponent of μ is $\geq h_{top}(A) - \varepsilon$.

On the other hand, by iterating a small neighborhood of T^{cu} backwards we have seen that we obtain a C^0-torus T which is saturated by strong stable leaves and such that the dynamics of $f|_T$ is conjugated to that of A. We claim that this implies that there exists at least one measure μ supported on T such that the strong-stable Lyapunov exponent is $\leq h_{top}(A)$. In fact (at least if f is $C^{1+\alpha}$ which can be obtained by approximation) assuming otherwise one obtains that for a Gibbs s-state μ of T (see [BDV] Chapter 11) the entropy is equal to the strong-stable exponent (via Ledrappier-Young's Theorem see also [BDV] Chapter 12.6) and again by the variational principle one deduces that the entropy of μ should be smaller or equal to $h_{top}(A)$.

This concludes since for absolutely partially hyperbolic diffeomorphisms there must be a gap between the spectrum of the center Lyapunov exponents and the strong-stable Lyapunov exponents.

□

6. The isotopy class of Anosov in $\mathbb{T}^3$

In this section $f : \mathbb{T}^3 \to \mathbb{T}^3$ will denote a partially hyperbolic diffeomorphism isotopic to Anosov. We will assume that Df preserves an orientation on E^s, E^c, E^u.

6.1. **Global product structure implies coherence.** One of the key facts in $\mathbb{T}^3$ is that foliations and branching foliations verify that their leaves do not separate much: Let $\mathcal{F}$ be a branching foliation of $\mathbb{T}^3$ which is almost parallel to a plane P, then, there exists $K > 0$ such that if $x, y \in \mathbb{R}^3$ are two different points then the Hausdorff distance between any leaf through x and any leaf through y is bounded by $K + d(x, y)$.

The idea then will be to show that if there are two leaves through the same point, iterating an unstable arc which joins them one can make them separate as much as you want getting a contradiction. For this, the concept of global product structure is essential.

Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a partially hyperbolic diffeomorphism. Let $\mathcal{F}^{cs}_{bran}$ be the f-invariant branching foliation given by Theorem 3.12 and let $\mathcal{S}$ the foliation which is almost parallel to $\mathcal{F}^{cs}_{bran}$ given by Theorem 3.14. We can prove the following criteria for dynamical coherence:

Proposition 6.1. *Assume that there is a global product structure between the lift of $\mathcal{S}$ and the lift of $\mathcal{W}^u$ to the universal cover and that $\mathcal{S}$ is almost parallel to a foliation by planes P^{cs}. Then there exists a f-invariant foliation $\mathcal{W}^{cs}$ everywhere tangent to $E^s \oplus E^u$.*

Proof. We will show that the branching foliation $\tilde{\mathcal{F}}^{cs}_{bran}$ must be a true foliation. This is reduced to showing that each point in $\mathbb{R}^3$ belongs to a unique leaf of $\tilde{\mathcal{F}}^{cs}_{bran}$ (see Proposition 3.16).

Assume otherwise, i.e. there exists $x \in \mathbb{R}^3$ such that $\tilde{\mathcal{F}}^{cs}_{bran}(x)$ has more than one complete surface. We call L_1 and L_2 different leaves in $\tilde{\mathcal{F}}^{cs}_{bran}(x)$. There exists y such that $y \in L_1 \setminus L_2$. Using global product structure and the fact that $\mathcal{F}^{cs}_{bran}$ is almost parallel to $\mathcal{S}$ we get $z \in L_2$ such that:

- $y \in \tilde{\mathcal{W}}^u(z)$.

Consider γ the arc in $\tilde{\mathcal{W}}^u(z)$ whose endpoints are y and z. Let R be the value given by the fact that $\mathcal{F}^{cs}_{bran}$ is almost parallel to a plane P^{cs} and $\ell > 0$ given by Proposition 4.21. We consider N large enough so that $\tilde{f}^N(\gamma)$ has length larger than $n\ell$ with $n \gg R$.

By Proposition 4.21 we get that the distance between $P^{cs} + \tilde{f}^N(z)$ and $\tilde{f}^N(y)$ is much larger than R. However, we have that, by $\tilde{f}$-invariance of $\tilde{\mathcal{F}}^{cs}_{bran}$ there is a leaf of $\tilde{\mathcal{F}}^{cs}_{bran}$ containing both $\tilde{f}^N(z)$ and $\tilde{f}^N(x)$ and another one containing both $\tilde{f}^N(y)$ and $\tilde{f}^N(x)$. This contradicts the fact that $\mathcal{F}^{cs}_{bran}$ is almost parallel to P^{cs} showing that $\tilde{\mathcal{F}}^{cs}_{bran}$ must be a true foliation.

□

A similar statement holds for $\mathcal{F}^{cu}_{bran}$.

An important fact, that we will use again in the next section is the following:

Lemma 6.2. *If P^{cs} is the plane almost parallel to $\mathcal{F}^{cs}_{bran}$ then P^{cs} is f_*-invariant.*

The proof is left as an exercise.

6.2. **Dynamical coherence.** We are now ready to prove dynamical coherence for a partially hyperbolic diffeomorphism f such that $f_* : H_1(\mathbb{T}^3, \mathbb{R}) \to H_1(\mathbb{T}^3, \mathbb{R})$ is hyperbolic.

Proof of dynamical coherence. Notice that if f_* is hyperbolic, then, every invariant plane must be totally irrational, so that it projects into a plane in $\mathbb{T}^3$.

Let $\mathcal{F}^{cs}_{bran}$ be the branched foliation tangent to E^{cs} given by Theorem 3.12. Using Theorems 3.14 and 4.10 we get a f_*-invariant plane P^{cs} in $\mathbb{R}^3$ such that $\mathcal{F}^{cs}_{bran}$ is almost aligned with the linear foliation given by P^{cs}. We know that P^{cs} cannot project into a two-dimensional torus since f_* has no invariant planes projecting into a torus. This implies by Theorem 4.10 that $\mathcal{F}^{cs}_{bran}$ is almost parallel to P^{cs}. For the foliation $\mathcal{S}$ given by Theorem 3.14 we know that all leaves must be simply connected since otherwise there would be a deck transformation fixing a leaf in the universal cover: such deck transformation would also fix P^{cs} since the foliations are almost parallel. We can apply Theorem 4.20 and we obtain that there is a global product structure between $\tilde{\mathcal{S}}$ and $\tilde{\mathcal{W}}^u$.

Dynamical coherence follows from applying Proposition 6.1.

□

Uniqueness of the foliation tangent to E^{cs} is a little more delicate but not difficult. With what we have done it is easy to see that there cannot be another foliation which is almost parallel to the same plane. To prove that there cannot be a foliation almost parallel to another plane demands a little more work (see [Pot] section 7).

6.3. **Leaf conjugacy.** Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a partially hyperbolic diffeomorphism isotopic to a linear Anosov automorphism. The main obstacle to show leaf conjugacy is to show that the foliations we have found are close to the correct ones. After this is done, at least one can use the semiconjugacy to obtain something more or less equivalent to leaf conjugacy (in fact, it is in some sense stronger as we have mentioned, because it gives that center leaves are homeomorphic, they are mapped correctly and it gives some information on their dynamics, however, it does not imply directly leaf conjugacy).

Notice first that the eigenvalues of f_* verify that they are all different. We shall name them $\lambda_1, \lambda_2, \lambda_3$ and we assume (without loss of generality) that they verify:

$$|\lambda_1| < |\lambda_2| < |\lambda_3| \quad ; \quad |\lambda_1| < 1 \ , \ |\lambda_2| \neq 1 \ , \ |\lambda_3| > 1$$

we shall denote as E^i_* to the eigenline of f_* corresponding to λ_i.

Proposition 6.3. *The foliation $\tilde{\mathcal{W}}^{cs}$ is almost parallel to the foliation by planes parallel to the eigenplane corresponding to the eigenvalues of smaller modulus (i.e. the eigenspace $E^1_* \oplus E^2_*$ corresponding to λ_1 and λ_2). Moreover, there is a global product structure between $\tilde{\mathcal{W}}^{cs}$ and $\tilde{\mathcal{W}}^u$. A symmetric statement holds for $\tilde{\mathcal{W}}^{cu}$ and $\tilde{\mathcal{W}}^s$.*

Proof. This proposition follows from the existence of a semiconjugacy H between $\tilde{f}$ and its linear part f_* which is at bounded distance from the identity (Theorem 3.7).

Denote as P^{cs} to the plane which is almost parallel to $\tilde{\mathcal{W}}^{cs}$.

The existence of a global product structure was proven above. Assume first that $|\lambda_2| < 1$, in this case, we know that $\tilde{\mathcal{W}}^u$ is sent by the semiconjugacy into lines parallel to the eigenspace of λ_3 for f_*. This readily implies that P^{cs} must coincide with the eigenspace of f_* corresponding to λ_1 and λ_2 otherwise we would contradict the global product structure.

The case were $|\lambda_2| > 1$ is more difficult. First, it is not hard to show that the eigenspace corresponding to λ_1 must be contained in P^{cs} (see Proposition 7.3 and the discussion afterwards which proves this in a very similar context).

Assume by contradiction that P^{cs} is the eigenspace corresponding to λ_1 and λ_3.

First, notice that by the basic properties of the semiconjugacy H, for every $x \in \mathbb{R}^3$ we have that $\tilde{\mathcal{W}}^u(x)$ is sent by H into $E^u_* + H(x)$ (where $E^u_* = E^2_* \oplus E^3_*$ is the eigenspace corresponding to λ_2 and λ_3 of f_*).

We claim that this implies that in fact $H(\tilde{\mathcal{W}}^u(x)) = E^2_* + H(x)$ for every $x \in \mathbb{R}^3$. In fact, we know from Proposition 4.21 that points of $H(\tilde{\mathcal{W}}^{cs}(x))$ which are sufficiently far apart are contained in a cone of $(E^2_* \oplus E^3_*) + H(x)$ bounded by two lines L_1 and L_2 which are transverse to P^{cs}. If P^{cs} contains E^3_* this implies that if one considers points in the same unstable leaf which are sufficiently far apart, then their image by H makes an angle with E^3_* which is uniformly bounded from below. If there is a point $y \in \tilde{\mathcal{F}}^u(x)$ such that $H(y)$ not contained in E^2_* then we have that $d(\tilde{f}^n(y), \tilde{f}^n(x))$ goes to ∞ with

n while the angle of $H(y) - H(x)$ with E^3_* converges to 0 exponentially contradicting Proposition 4.21.

Consider now a point $x \in \mathbb{R}^3$ and let y be a point which can be joined to x by a finite set of segments $\gamma_1, \dots, \gamma_k$ tangent either to E^s or to E^u (an *su*-path, see [DW]). We know that each γ_i verifies that $H(\gamma_i)$ is contained either in a translate of E^1_* (when γ_i is tangent to E^s, i.e. it is an arc of the strong stable foliation $\tilde{\mathcal{F}}^s$) or in a translate of E^2_* (when γ_i is tangent to E^u from what we have shown in the previous paragraph). This implies that the *accesibility* class of x (see [DW] for a definition and properties) verifies that its image by H is contained in $(E^1_* \oplus E^2_*) + H(x)$. The projection of $E^1_* \oplus E^2_*$ to the torus is not the whole $\mathbb{T}^3$ so in particular, we get that f cannot be accesible.

Since small perturbations cannot change the plane P^{cs} (see [Pot] Corollary 7.14) this situation should be robust under C^1-perturbations since those perturbations cannot change the direction of P^{cs}.

On the other hand, in [DW] it is proved that by an arbitrarily small (C^1) perturbation of f one can make it accessible. This gives a contradiction and concludes the proof.

□

Once this is obtained one can use the semiconjugacy and show that it is injective on strong stable and unstable leafs (this follows for example from Corollary 4.9 (iv) and the fact that the preimage of points by the semiconjugacy in $\mathbb{R}^3$ has uniformly bounded diameter). This, together with the fact that the center foliation must be contained in a cylinder around a translate of the eigenspace of the central eigenvalue of f_* implies with some work that the preimage of points by the semiconjugacy is a (possibly trivial) arc contained in a center leaf (see also Proposition 7.4 bellow). As a consequence, we get that all leaves are homeomorphic to lines and that the dynamics of the leafs is as in the Anosov diffeomorphism. Moreover, we obtain that the dynamics inside each leaf is semiconjugated to the one of the Anosov in its center foliation.

However, this does not imply directly leaf conjugacy. To obtain this, in [H] the following is done. First we lift to the universal cover, and (for example) with the semiconjugacy we can make a homeomorphism between the space of center leaves of $\tilde{f}$ and f_*. By choosing an appropriate section (what is called *us*-pseudoleaf in [H]) it is possible to construct a global homeomorphism of $\mathbb{R}^3$ which maps center leafs into center leafs and conjugates the dynamics modulo this. Finally, an averaging procedure allows to construct the desired leaf conjugacy (see [H] for details).

7. Skew-products

In this section $f : M \to M$ will denote a partially hyperbolic diffeomorphism of $\mathbb{T}^3$ or N_k such that Df preserves an orientation in E^s, E^c, E^u (we will not enter in the problem of lifting or taking iterates, etc, see [HP]). If $M = \mathbb{T}^3$, then f will be isotopic to a skew-product of the form $A \times id_{S^1}$ with $A \in SL(2, \mathbb{Z})$ hyperbolic.

7.1. **Global Product Structure.** In $\mathbb{T}^3$ we have already shown that global product structure is enough to guarantee dynamical coherence. To do this for nilmanifolds we must show that a similar property holds for foliations of N_k. Since the property holds for foliations almost parallel to one having the property we prove it only for the foliations $\mathcal{F}_\theta$ in view of Theorem 4.15.

Proposition 7.1. *The foliations $\mathcal{F}_\theta$ of N_k verify the following, there exists $K > 0$ such that given $x, y \in \tilde{N}_k$ we have that if $d(z, \tilde{\mathcal{F}}_\theta(y)) < K + d(x, \tilde{\mathcal{F}}_\theta(y))$ for every $z \in \tilde{\mathcal{F}}_\theta(x)$.*

In fact, the constant K can be chosen to be as small as one wants if the distance is well chosen. We leave this fact as an exercise as well as filling the details in the following proof.

Proof. We look N_k as $\mathbb{R}^3$ quotiented by the deck transformations:

$$\gamma_1(x, y, t) = (x + 1, y, t)$$
$$\gamma_2(x, y, t) = (x, y + 1, t)$$
$$\gamma_3(x, y, t) = (x + ky, y, t + 1)$$

as in section 3.2. The foliation $\mathcal{F}_\theta$ lifts to $\tilde{\mathcal{F}}_\theta$ which with this coordinates can be written as the foliation by planes parallel to the plane $y = \theta t$ (notice that this foliation is invariant under deck transformations).

Now, take two points $x, y \in \tilde{N}_k$. We can assume without loss of generality that $x \in [0, 1]^3$. Consider now the (Euclidean) ball B of radius $d(x, \tilde{\mathcal{F}}_\theta(y))$ around $[0, 1]^3$. Given $z \in \tilde{\mathcal{F}}_\theta(x)$ we can make a translation by a deck transformation so that $z \in [0, 1]^3$. From the form of the deck transformations (which do not alter the Euclidean distance between leaves of the foliation $\mathcal{F}_\theta$) we see that the transformation of $\mathcal{F}_\theta(y)$ will still intersect B.

□

Exercise. Show that the argument above fails for the leaves of the foliation $\mathcal{F}_A^{cs}$ in S_A.

With the previous proposition we recover the same result as in the torus case:

Proposition 7.2. *Let $f : N_k \to N_k$ be a partially hyperbolic diffeomorphism and $\mathcal{S}$ the foliation given by Theorem 3.14 which is almost parallel to $\mathcal{F}^{cs}_{bran}$ given by Theorem 3.12 and assume that $\mathcal{S}$ is almost parallel to some $\mathcal{F}_\theta$. Assume that $\tilde{\mathcal{S}}$ and $\tilde{\mathcal{W}}^u$ have a global product structure. Then $\mathcal{F}^{cs}_{bran}$ is a foliation.*

The proof is the same as Proposition 6.1.

7.2. **Dynamical coherence.** We show here how to establish global product structure which thanks to Propositions 6.1 and 7.2 gives dynamical coherence.

We consider first the case of $\mathbb{T}^3$ and then explain the differences. Let $\mathcal{F}^{cs}_{bran}$ be the branching foliation given by Theorem 3.12. Using Theorems 3.14 and 4.10 we deduce that $\mathcal{F}^{cs}_{bran}$ is almost aligned with a foliation by planes parallel to a certain plane P^{cs} in $\mathbb{R}^3$. Since $\mathcal{F}^{cs}_{bran}$ is f-invariant and any lift $\tilde{f}$ of f is at bounded distance from f_* we deduce that the plane P^{cs} is f_*-invariant.

We know that P^{cs} cannot be the plane generated by the eigenspaces of f_* associated to the eigenvalues different from one. This is because such a plane projects into a torus in $\mathbb{T}^3$ and by Theorem 4.10 and Proposition 3.18 we would get an f-periodic torus tangent to E^{cs} which we have assume there is not.

To show that P^{cs} corresponds to the eigenspace associated with the eigenvalues smaller or equal to 1 of f_* we must then prove the following. For notational purposes we work with $\mathcal{W}^u$ but the same proof holds for $\mathcal{W}^s$ by iterating backwards.

Proposition 7.3. *Let E^- be the eigenspace of f_* associated with the eigenvalues smaller or equal to one. Then, for every $R > 0$ we know that there exists $L > 0$ such that an arc J of $\tilde{\mathcal{W}}^u$ of length larger than L cannot be contained in the R-neighborhood of a translate of E^-.*

Proof. Let C be a connected set contained in an R-neighborhood of a translate of E^-, we will estimate the diameter of $\tilde{f}(C)$ in terms of the diameter of C.

Claim. *There exists K_R which depends only on $\tilde{f}$, f_* and R such that:*

$$\operatorname{diam}(\tilde{f}(C)) \leq \operatorname{diam}(C) + K_R$$

Proof. Let K_0 be the C^0-distance between $\tilde{f}$ and f_* and consider $x, y \in C$ we get that:

$$d(\tilde{f}(x), \tilde{f}(y)) \leq d(f_*(x), f_*(y)) + d(f_*(x), \tilde{f}(x)) + d(f_*(y), \tilde{f}(y)) \leq$$
$$\leq d(f_*(x), f_*(y)) + 2K_0$$

We have that the difference between x and y in the unstable direction of f_* is bounded by $2R$ given by the distance to the plane E^- which is transverse to E^u_*, the eigenspace associated to the eigenvalue larger than 1.

Since the eigenvalues of f_* along E^- are smaller or equal to 1 we have that f_* does not increase distances in this direction: we thus have that $d(f_*(x), f_*(y)) \leq d(x, y) + 2|\lambda^u|R$ where λ^u is the eigenvalue of modulus larger than 1. We have obtained:

$$d(\tilde{f}(x), \tilde{f}(y)) \leq d(x, y) + 2K_0 + 2|\lambda^u|R = d(x, y) + K_R$$

which concludes the proof of the claim.

◊

Now, this implies that if we consider an arc γ of $\tilde{\mathcal{W}}^u$ of length 1 and assume that its future iterates remain in a slice parallel to E^- of width $2R$ we have that

$$\operatorname{diam}(\tilde{f}^n(\gamma)) < \operatorname{diam}(\gamma) + nK_R \leq 1 + nK_R$$

So that the diameter grows linearly with n.

The volume of balls in the universal cover of $\mathbb{T}^3$ grows polynomially with the radius so that we have that $B_\delta(\tilde{f}^{-n}(\gamma))$ has volume which is polynomial $P(n)$ in n.

On the other hand, we know from the partial hyperbolicity that there exists $C > 0$ and $\lambda > 1$ such that the length of $\tilde{f}^n(\gamma)$ is larger than $C\lambda^n$.

Using Corollary 4.9 (iv), we obtain that there exists n_0 uniform such that every arc of length 1 verifies that $\tilde{f}^{n_0}(\gamma)$ is not contained in the R-neighborhood of a translate of $E^s_* \oplus E^c_*$. This implies that no unstable leaf can be contained in the R-neighborhood of a translate of $E^s_* \oplus E^c_*$ concluding the proof of the Proposition.

□

Notice that this implies in particular that $\tilde{\mathcal{F}}^{cs}_{bran}$ cannot be close to E^+ (the eigenspace corresponding to the eigenvalues larger than or equal to 1 of f_*) since $\tilde{\mathcal{F}}^{cs}_{bran}$ is saturated by leaves of $\tilde{\mathcal{W}}^s$. This implies that $\tilde{\mathcal{F}}^{cs}_{bran}$ is almost parallel to E^-.

The proposition also allows to show global product structure since we know at the same time that the strong unstable leaves cannot remain close to $\tilde{\mathcal{F}}^{cs}_{bran}$ which is close to E^-.

This concludes the proof of dynamical coherence in the case of $\mathbb{T}^3$. Let us now show how to adapt this in the case of N_k.

The idea is very similar. First, f is isotopic to a skew-product g which we can assume fixes two of the foliations $\mathcal{F}_{\theta_1}$ and $\mathcal{F}_{\theta_2}$ corresponding respectively to the center-stable and center-unstable foliations for g.

Using Proposition 7.1 we can apply the same argument as in Proposition 7.3 to show that the strong stable foliation $\tilde{\mathcal{W}}^s$ cannot remain close to $\tilde{\mathcal{F}}_{\theta_2}$ and the strong unstable foliation $\tilde{\mathcal{W}}^u$ cannot remain close to $\tilde{\mathcal{F}}_{\theta_1}$. This allows one to show that $\mathcal{F}^{cs}_{bran}$ is almost parallel to $\mathcal{F}_{\theta_1}$ and $\tilde{\mathcal{W}}^u$ has a global product structure with the foliation $\mathcal{S}$ almost parallel to $\mathcal{F}^{cs}_{bran}$. This gives coherence in the nilmanifold case.

In this case, showing unique integrability is much simpler since we have shown that the foliations are almost parallel to the correct foliations at the very start. We leave this as an exercise.

7.3. **Leaf conjugacy.** We have proved not only that f is dynamically coherent but also that the foliations remain close to the correct ones. We will restrict to the case of $M = \mathbb{T}^3$ to fix ideas and leave as an exercise to do the (minor) changes necessary in the case where $M = N_k$.

The main point is the following (based on [H_2]):

Proposition 7.4 (Central Shadowing)**.** *Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ a dynamically coherent partially hyperbolic diffeomorphism. Then, if $x, y \in \mathbb{R}^3$ verify that $d(\tilde{f}^n(x), \tilde{f}^n(y))$ is bounded with $n \in \mathbb{Z}$ then $y \in \tilde{\mathcal{F}}^c(x)$.*

Notice that we have not assumed that f_* is a skew-product.

Proof. We know that $\tilde{\mathcal{F}}^c(x)$ is contained in a R-neighborhood of a translate L of the center eigenline of f_*. Moreover, we know that $\tilde{f}^n(\tilde{\mathcal{F}}^c(x))$ is also contained in an R-neighborhood of $f_*^n(L)$ (since f_* is linear, this is also a line).

Assume that $y \notin \tilde{\mathcal{F}}^c(x)$. Then, we can join y with $\tilde{\mathcal{F}}^c(x)$ by two arcs (possibly trivial): one in a strong stable leaf and the other in a strong unstable one. Assume that one of them (say the unstable one) is non trivial.

Using Proposition 7.3 we deduce that after iteration by $\tilde{f}^n$ we have that the distance between y and $f_*^n(L)$ goes to infinity. This contradicts the fact that $d(\tilde{f}^n(x), \tilde{f}^n(y))$ so we deduce that $y \in \tilde{\mathcal{F}}^c(x)$.

□

Using this result we will show that every leaf of $\tilde{\mathcal{W}}^c$ is a circle of uniform length. After this is done, showing leaf conjugacy is not so hard (for example using Theorem 3.8 or the arguments at the end of section 2 of [BoW]).

To do this, notice that if $\gamma \in \mathbb{Z}^3$ is the deck transformation such that $f_*(\gamma) = \gamma$ we have that iterating by $\tilde{f}$ the points x and $x + \gamma$ remain at bounded distance for all iterates since $\tilde{f}^n(x+\gamma) = \tilde{f}^n(x) + \gamma$, so using the proposition above we deduce that they belong to the same center leaf. This concludes.

Proving this result in the nilmanifold case is now a matter of translating the above "proof" to the nilmanifold language.

8. Anosov flows

8.1. **Solvmanifolds.** Given a hyperbolic matrix $A \in GL(2, \mathbb{Z})$ we denote as

$$S_A = \mathbb{T}^2 \times \mathbb{R}/_\sim \qquad (v, t+k) \sim (A^k v, t) \;\; k \in \mathbb{Z}$$

It is a 3-manifold whose fundamental group is solvable yet with exponential growth (in particular, it is not nilpotent).

The manifold S_A admits two essentially different Anosov flows, the suspension of A and the suspension of A^{-1}. In particular, this shows that S_A is diffeomorphic to $S_{A^{-1}}$. Considering the time one maps of these flows one obtains examples of partially hyperbolic diffeomorphisms in S_A.

We will consider an intermediate covering space of S_A between S_A and its universal cover $\tilde{S}_A$. We will denote as

$$\hat{S}_A = \mathbb{T}^2 \times \mathbb{R}$$

to the covering space given by $\hat{\pi} : \hat{S}_A \to S_A$ given by the equivalence relation defined above.

We will give special coordinates to $\tilde{S}_A$ which will help to work there. Also, we will define a Riemannian metric in $\tilde{S}_A$ which will be invariant under deck transformations and so defines a metric also in S_A. It is with this metric which we will work. We identify $\tilde{S}_A = \mathbb{R}^2 \times \mathbb{R}$ and the deck transformations (isomorphic to $\pi_1(S_A)$) are the ones generated by the following three diffeomorphisms of $\tilde{S}_A$:

$$\gamma_1(v,t) = (v + (1,0), t)$$
$$\gamma_2(v,t) = (v + (0,1), t)$$
$$\gamma_3(v,t) = (A^{-1}v, t+1)$$

Let us call G to the subgroup of $\mathrm{Diff}^1(\tilde{S}_A)$ generated by γ_1,γ_2 and γ_3. Clearly, $G \cong \pi_1(S_A)$ since $S_A = \tilde{S}_A/_G$. Also, we have that $\hat{S}_A$ is the quotient of $\tilde{S}_A$ by the group generated by γ_1 and γ_2.

Notice that the subgroup generated by γ_1 and γ_2 is isomorphic to $\mathbb{Z}^2$ and coincides with the commutator subgroup $[G,G]$: It is then not hard to show that $\pi_1(S_A) = G$ is solvable (since the commutator subgroup is abelian) but not nilpotent since $[G,[G,G]] = [G,G] \neq \{0\}$.

Let $D_\mu \in GL(2,\mathbb{R})$ be the diagonal matrix with entries μ and μ^{-1}. We know that there exists a matrix P such that $A = PD_\lambda P^{-1}$, as usual, we denote $A^t = PD_{\lambda^t}P^{-1}$ which allows to extend the integer powers of A to all the reals. Consider then the following Riemannian metric in $\tilde{S}_A$ (identifying $T\tilde{S}_A \cong \tilde{S}_A \times (\mathbb{R}^2 \times \mathbb{R})$):

$$\langle (V_1,T_1),(V_2,T_2)\rangle_{(v,t)} = T_1T_2 + \langle A^tV_1, A^tV_2\rangle_{eucl}$$

Which gives rise to the following norm:

$$\|(V,T)\|^2_{(v,t)} = \|A^tV\|^2_{eucl} + |T|^2$$

It is not hard to check that g is invariant under all deck transformations, in fact, γ_1 and γ_2 leave trivially g invariant since it only depends on the coordinate t and by direct calculation one sees that $\|D_{(v,t)}\gamma_3(V,T)\|_{\gamma_3(v,t)} = \|(V,T)\|_{(v,t)}$. This metric is not at all consistent with our view of $\mathbb{R}^3$, of course, this should not be a surprise since S_A has fundamental group with exponential growth, so that $\tilde{S}_A$ cannot admit a flat metric invariant under such a group acting properly and discontinuously.

For a point $(v,t) \in \tilde{S}_A = \mathbb{T}^2 \times \mathbb{R}$ we will define $p_1(v,t) = t$ giving rise to a smooth function $p_1 : \tilde{S}_A \to \mathbb{R}$. Notice that $H_1(S_A, \mathbb{R}) = \mathbb{R}$ since the abelianization of $\pi_1(S_A)$ is $\mathbb{Z}$ (and a generator is the image of γ_3 by the quotient with the commutator subgroup). Thus, for some $g : S_A \to S_A$ which is isotopic to the identity and a lift $\tilde{g} : \tilde{S}_A \to \tilde{S}_A$ which lifts this homotopy we can define the *homological rotation set* of g as the set of limit points of:

$$\frac{1}{n}(p_1(\tilde{g}^n(x_n)) - p_1(x_n)) \qquad n \geq 0 \; , \; x_n \in \tilde{S}_A$$

Since $\hat{g}$ is at distance at most K_0 from the identity, we have that the set is contained in $[-K_0, K_0]$.

It is not hard to see that the function $p_1 : \tilde{S}_A \to \mathbb{R}$ (we will abuse notation and use the same notation for both functions) can be also defined for points in $\hat{S}_A$ in a similar way. This is because p_1 factors through the covering map $\tilde{\pi} : \tilde{S}_A \to \hat{S}_A$ which as we mentioned before consists on making the quotient respect to the commutator subgroup of $\pi_1(S_A)$.

In fact, the hypothesis of g being homotopic to the identity is almost redundant. We can prove the following (probably) well known result:

Proposition 8.1. *Let $g : S_A \to S_A$ be a diffeomorphism, then, there is a finite iterate of g which is isotopic to the identity.*

Proof. It is not hard to see that there is only one possible embedded incompressible two-dimensional torus in S_A modulo isotopy: Every $\mathbb{Z}^2$ subgroup of $\pi_1(S_A)$ is a subgroup of the inclusion of the fundamental group of $T = \hat{\pi}(\mathbb{T}^2 \times \{0\})$ in $S_A = \mathbb{T}^2 \times \mathbb{R}/_\sim$. Clearly, if the subgroup is proper, the torus cannot be embedded (it will have autointersections) so, any embedded torus[21] must be isotopic to T.

Now, we consider the image $g^2(T)$ of $T \subset S_A$ under g^2. We obtain that $g^2(T)$ is an injectively embedded incompressible two-dimensional torus and thus isotopic to T. We consider an isotopy from $g^2(T)$ to T which we can extend to a global isotopy by the Isotopy Extension Theorem (see [Hi] Theorem 8.1.3).

[21] Another way to see this is to consider an embedded torus $T_1 \subset S_A$: The lift to $\hat{S}_A$ must be a countable collection of two-dimensional torus which pairwise do not intersect because otherwise there would be a covering transformation of $\hat{S}_A$ fixing a connected component contradicting that the fundamental group is abelian. After this is done it is clear that T_1 is isotopic to T because each component has no autointersections.

We can then assume that g^2 fixes T and moreover that it fixes the orientation both in T as its transverse orientation.

Now, cut S_A along T in such a way to obtain a diffeomorphism $G : T\times[0,1] \to T\times[0,1]$ which verifies that (modulo isotopy) in the boundaries it commutes with A. This implies that in the boundaries it is isotopic to a power[22] of A and G can be isotoped to this map times the identity in $T \times [0,1]$. Now, one can make an isotopy from the identity to g^2 by moving forward along the suspension flow until one gets the desired power of A and then cutting and undoing the previous isotopy.

□

Remark 8.2. Let $g : S_A \to S_A$ be a diffeomorphism. We can assume (modulo considering an iterate) that there is a lift $\tilde{g} : \tilde{S}_A \to \tilde{S}_A$ such that for some $K_0 > 0$ we have that:

$$d(\tilde{g}(x), x) < K_0 \qquad \forall x \in \tilde{S}_A$$

This lift is obtained by lifting the isotopy from the identity to g. We will always consider such a lift and continue to denote it as $\tilde{g}$ (even if it is not an arbitrary lift).

♢

For a point $(v_0, t_0) \in \tilde{S}_A$ we can define the following set:

$$\hat{\mathcal{B}}_{a,b}((v_0, t_0)) = \{(v, t) \in \tilde{S}_A \ : \ \|v - v_0\|_{eucl} < a \ , \ |t - t_0| < b\}$$

Where $\|\cdot\|_{eucl}$ denotes the euclidean metric in $\mathbb{R}^2$. For a set $C \in \tilde{S}_A$ we define as usual $\hat{\mathcal{B}}_{a,b}(C) = \bigcup_{p\in C} \hat{\mathcal{B}}_{a,b}(p)$.

When points do not move in the t-direction (i.e. when the value of p_1 remains bounded) the behavior is much like in the euclidean space. An heuristic way to see this is that if you can find an embedded incompressible torus such that points do not cross, then you can cut along this torus and the dynamics would be very similar to something in $\mathbb{T}^2 \times [0,1]$ which has polynomial growth of volume. This can be formalized in the following way:

Lemma 8.3. *Let $g : S_A \to S_A$ be a diffeomorphism isotopic to the identity and $\tilde{g} : \tilde{S}_A \to \tilde{S}_A$ the lift at bounded distance from the identity. For every $K > 0$ there exists $D > 0$ such that if $C \subset \tilde{S}_A$ be a compact connected set such that for every $x \in C$ and every $n \in \mathbb{Z}$ one has that $|p_1(\tilde{g}^n(x))| < K$ then we have that $\tilde{g}^n(C)$ is contained in $\hat{\mathcal{B}}_{D|n|,K}(C)$.*

Proof. Fix $K > 0$ and let K_0 be the distance between $\tilde{g}$ and the identity.

[22] In fact, if A were an iterate of a hyperbolic matrix, then the power could be fractional. This implies that we should consider a larger iterate of g in order to do the argument, in any case, it is a finite iterate.

By compactness, there exists $D > 0$ such that if $(v_0, t_0) \in [0,1] \times [0,1] \times [-K, K] \subset \tilde{S}_A$ and $(v_1, t_1) = \tilde{g}((v_0, t_0))$ then we have that $\|v_1 - v_0\|_{eucl} \leq D$.

Call $H = [0,1] \times [0,1] \times [-K, K]$, if we denote as G_1 the subgroup of $G = \pi_1(S_A)$ generated by the deck transformations γ_1 and γ_2 defined above, we obtain that $\bigcup_{\gamma \in G_1} \gamma(H) = \mathbb{R}^2 \times [-K, K] = p_1^{-1}([-K, K])$.

Since all deck transformations preserve the euclidean distances in the first coordinate, we get that for every point (v_0, t_0) in $p_1^{-1}([-K, K])$ we have that $\|v_1 - v_0\|_{eucl} < D$.

Now, if a set has its whole (future) orbit contained in $p_1^{-1}([-K, K])$ we can prove inductively that its image by $\tilde{g}^n$ cannot move points more than nD as desired. The same argument works for negative iterates.

□

In $\tilde{S}_A$ with the Riemannian metric we used, the volumes can be measured exactly as in $\mathbb{R}^3$. An easy way to see this is to remark that all deck trasformations preserve the cannonical volume form of $\mathbb{R}^3$.

Proposition 8.4. *The volume in $\tilde{S}_A$ of $\hat{\mathcal{B}}_{R,K}(p)$ is equal to its euclidean volume, i.e.:*

$$\mathrm{Vol}(\hat{\mathcal{B}}_{R,K}(p)) = 2\pi R^2 K$$

Proof. There exists an euclidean orthonormal basis such that the coeficients of the metric of $\tilde{S}_A$ make a diagonal matrix with coeficients λ^t, λ^{-t} and 1. Thus, the determinant is equal to 1 and we obtain that the volume form is the same. The result follows from calculating the euclidean volume of $\hat{\mathcal{B}}_{R,K}(p)$ which is easy.

□

We remark however that the diameter of $\hat{\mathcal{B}}_{R,K}(p)$ is very far from being equal to the euclidean diameter. In fact, it is not hard to get an upper bound of the diameter of the form:

$$\mathrm{diam}(\hat{\mathcal{B}}_{R,K}(p)) \leq 2K + 2 + \frac{4}{\log \lambda} \log(\max\{1, 2R\})$$

which evidences the exponential growth of volume in $\tilde{S}_A$. We will not use this fact so we do not prove it, we leave it as an exercise but we mention that to prove this bound it suffices to choose appropriate curves joining points in the same t-coordiante (move forward if they are in different unstables or backwards if they are in different stables).

8.2. **Fixing leaves in the universal cover.** We now assume that $f : S_A \to S_A$ is a partially hyperbolic diffeomorphism with f-invariant branching foliations $\mathcal{F}^{cs}_{bran}$ and $\mathcal{F}^{cu}_{bran}$ given by Theorem 3.12. We assume moreover as a standing hypothesis that f has no periodic two-dimensional torus tangent to either $E^s \oplus E^c$ nor $E^c \oplus E^u$.

By considering an iterate, Proposition 8.1 allows us to assume that f is isotopic to the identity. Moreover, we can consider a lift $\tilde{f}$ of f which is at distance smaller than K_0 from the identity (see Remark 8.2).

We will show that the lift $\tilde{f}$ fixes some leaves of $\tilde{\mathcal{F}}^{cu}_{bran}$ which will be very important for the proof of our results. Ultimately, our goal in this section is to show that $\tilde{f}$ fixes every leaf of the branching foliation, but we will need to know that at least some are fixed to obtain that.

Recall that by Theorem 3.14 there exists a Reebless foliation which is almost parallel to $\mathcal{F}^{cu}_{bran}$ (respectively $\mathcal{F}^{cs}_{bran}$. This implies that after lifting to the universal cover each leaf of $\tilde{\mathcal{F}}^{cu}_{bran}$ is a properly embedded plane which thus separates $\tilde{S}_A$ into two connected components. Since $\tilde{S}_A$ is simply connected we can choose an orientation for E^s and this determines a relation between leaves of $\tilde{\mathcal{F}}^{cu}_{bran}$: We say that $L_1 \geq L_2$ if L_2 is not completely contained in the connected component of $\tilde{S}_A \setminus L_1$ with negative orientation of E^s (recall that the leaves of $\tilde{\mathcal{F}}^{cu}_{bran}$ have no topological crossings). In general, this may not be an order, but in S_A from the classification of foliations we will see that it is an order and in fact the set of leaves of $\tilde{\mathcal{F}}^{cu}_{bran}$ is totally ordered (see Section 5.1 of [Pot] for more details and explanations on this kind of ordering).

Lemma 8.5. *Given $K > 0$ and $x \in \tilde{S}_A$ there exist leaves $L_{min}(x)$ and $L_{max}(x)$ (which may coincide) in $\tilde{\mathcal{F}}^{cu}_{bran}$ such that every leaf of $\tilde{\mathcal{F}}^{cu}_{bran}$ which is larger than $L_{max}(x)$ or smaller than $L_{min}(x)$ verifies that has points which are at distance larger than K of $L_{min}(x)$ and $L_{max}(x)$. Moreover, every leaf between $L_{min}(x)$ and $L_{max}(x)$ is contained in a K-neighborhood of each of those leaves and there is at least one leaf between $L_{min}(x)$ and $L_{max}(x)$ which is contained in $\tilde{\mathcal{F}}^{cu}_{bran}(x)$.*

If for some point $L_{min}(x) \neq L_{max}(x)$ one can regard the situation as if the foliation is as the suspension of a Denjoy foliation in $\mathbb{T}^2$ (for example if one suspends the invariant foliation invariant by a DA-diffeomorphism of $\mathbb{T}^2$ isotopic to Anosov).

Proof. Consider the foliation $\mathcal{U}$ which is almost parallel to $\mathcal{F}^{cu}_{bran}$ given by Theorem 3.14. We know from Proposition 3.18 that $\mathcal{U}$ has no torus leaves so that Theorem 4.16 applies. We get that $\mathcal{F}^{cu}_{bran}$ is almost parallel to either $\mathcal{F}^{cs}_A$ or $\mathcal{F}^{cu}_A$.

Now, the proof of the Lemma follows directly from the properties of $\mathcal{F}^{cs}_A$ and $\mathcal{F}^{cu}_A$ given in Proposition 4.17. In particular, one obtains that the leaves of $\tilde{\mathcal{F}}^{cu}_{bran}$ are totally ordered.

□

Remark 8.6. We have in fact proved that the branching foliations of a partially hyperbolic diffeomorphisms of S_A with no periodic torus tangent to the center-stable or center-unstable distributions verifies that its branching foliations are almost parallel to one of the foliations $\mathcal{F}^{cs}_A$ or $\mathcal{F}^{cu}_A$. This implies in particular that for K large enough, if two leaves have a point at distance larger than K, then they have points at arbitrarily large distance.

$\diamondsuit$

Corollary 8.7. *For every $x \in \tilde{S}_A$ we have that $\tilde{f}$ fixes both $L_{min}(x)$ and $L_{max}(x)$ for $K > K_0$.*

Of course, the same results hold for $\tilde{\mathcal{F}}^{cs}_{bran}$.

Proof. Since $\tilde{f}$ is at distance smaller than K_0 from the identity, one can see that the image of a leaf L of $\tilde{\mathcal{F}}^{cu}_{bran}$ is contained in $B_{K_0}(L)$. So, this means that for K large enough we have that the image of these leaves must be contained in between the leaves $L_{min}(x)$ and $L_{max}(x)$. The same holds for $\tilde{f}^{-1}$ so by the order preservation we deduce that those boundary leaves must be fixed.

$\square$

Proposition 8.8. *Let $\tilde{\mathcal{F}}^{cu}_{bran}$ be the lift of $\mathcal{F}^{cu}_{bran}$ to the universal cover $\tilde{S}_A$. Then, for every two different leaves $L_1 \neq L_2$ of $\tilde{\mathcal{F}}^{cu}_{bran}$ and $K > 0$ we have that there is a point $x \in L_1$ such that $B_K(x) \cap L_2 = \emptyset$. In other words, for every $K > 0$ and $x \in \tilde{S}_A$ we have that $L_{min}(x) = L_{max}(x)$.*

We see this Proposition as giving that the branching foliation has no "Denjoy phenomena" since that would give a family of leaves which remain close in the universal covering space.

Proof. The heart of the proof lies in the following:

Claim. *Given $x \in \tilde{S}_A$ and $K > 0$, there exists $L > 0$ such that no unstable arc of length larger than L lies between $L_{min}(x)$ and $L_{max}(x)$.*

Proof. Consider an arc J of unstable of length 1 which we assume is contained in between $L_{min}(x)$ and $L_{max}(x)$. If no such arc exists we can consider $L = 1$ and we are done.

First, notice that since $L_{min}(x)$ and $L_{max}(x)$ lie close to each other in the universal cover, we deduce that intersected to the plane $\mathbb{R}^2 \times \{t\}$ in $\tilde{S}_A \cong \mathbb{R}^2 \times \mathbb{R}$ they are asymptotic to each other and moreover, we get that the area between the intersection of $L_{min}(x)$ and $L_{max}(x)$ in that plane is finite because deck transformations which fix some leaf of $\tilde{\mathcal{F}}^{cu}_{bran}$ must contain a translation in the coordinate given by the projection p_1. So, in the torus $\mathbb{T}^2 \times \{t\} \subset \hat{S}_A$ the area between the two leaves is finite.

We deduce that the volume between $L_{min}(x)$ and $L_{max}(x)$ inside $\mathbb{R}^2 \times [-N, N]$ is finite and moreover grows linearly with N (recall that the volume is calculated as in $\mathbb{R}^3$, see Proposition 8.4).

The iterates $\tilde f^n(J)$ of J must be contained in sets of the form $\mathbb{R}^2 \times [-p(n), p(n)]$ with $p : \mathbb{N} \to \mathbb{R}$ linear in n since $\tilde f$ is at bounded distance from the identity. On the other hand, the length of $\tilde f^n(J)$ grows exponentially so, using Corollary 4.9 (iv) we conclude the proof of the claim.

$\diamondsuit$

By Corollary 8.7 the boundary leaves $L_{min}(x)$ and $L_{max}(x)$ are fixed under $\tilde f$. Using the claim, we can consider a non-trivial unstable arc joining $L_{min}(x)$ and $L_{max}(x)$ whose interior is contained in between those leaves and has length smaller than L. Now, iterating backwards, we get on the one hand that the arc remains between those leaves but on the other hand the length grows exponentially contradicting the claim and concluding the proof of the proposition.

$\square$

As a consequence we obtain:

Corollary 8.9. *The lift $\tilde f$ of f fixes every leaf of the branching foliations $\tilde{\mathcal{F}}^{cs}_{bran}$ and $\tilde{\mathcal{F}}^{cu}_{bran}$.*

8.3. **Finding a model.** Again in this section we will consider $f : S_A \to S_A$ to be a partially hyperbolic diffeomorphism (isotopic to the identity) such that there is no f-periodic two-dimensional torus tangent to the center-stable nor the center-unstable distributions. So, by Remark 8.6 we know that each of the branching foliations are almost parallel to either $\mathcal{F}^{cs}_A$ or $\mathcal{F}^{cu}_A$.

Using the fact that the leaves of $\tilde{\mathcal{F}}^{cs}_{bran}$ and $\tilde{\mathcal{F}}^{cu}_{bran}$ are fixed by $\tilde f$ we are able to deduce many things about how points advance in the universal cover.

The following lemma will be essential in the proof of dynamical coherence:

Lemma 8.10. *Assume that $\mathcal{F}^{cu}_{bran}$ is almost parallel to $\mathcal{F}^{cu}_A$. There exists $K > 0$ such that every point $x \in \tilde S_A$ verifies that for every $n \geq 0$ we have that $p_1(\tilde f^n(x)) \geq p_1(x) - K$. If $\mathcal{F}^{cu}_{bran}$ is almost parallel to $\mathcal{F}^{cs}_A$ then there exists $K > 0$ such that every point $x \in \tilde S_A$ verifies that for every $n \geq 0$ we have that $p_1(\tilde f^n(x)) \leq p_1(x) + K$.*

Symmetric statements hold for $\mathcal{F}^{cs}_{bran}$.

Proof. We will prove the Lemma in the case where $\mathcal{F}^{cu}_{bran}$ is almost parallel to $\mathcal{F}^{cu}_A$. The case where it is almost parallel to $\mathcal{F}^{cs}_A$ is analogous (or it may be reduced to this case by considering f as a diffeomorphism of $S_{A^{-1}}$). The key point is that leaves of $\tilde{\mathcal{F}}^{cu}_A$ get separated uniformly when $p_1(x)$ goes to $-\infty$ and the separation only depends on $p_1(x)$ (Proposition 4.17).

Consider $\alpha > 0$ such that for every $x \in \tilde S_A$ we have that the arc of $\tilde{\mathcal{W}}^s_{\tilde f}$ centered at x and of length α verifies that it intersects inside $B_\alpha(x)$ leaves of $\tilde{\mathcal{F}}^{cu}_{bran}$ such that the distance between the connected components of the leaf through x and those leaves in

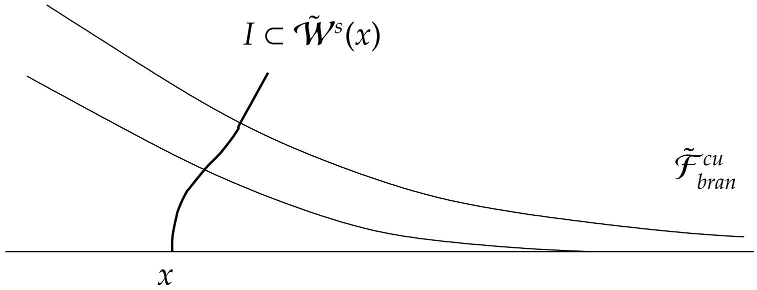

FIGURE 12. Points cannot move backwards because the leaves of $\tilde{\mathcal{F}}^{cu}_{bran}$ separate and are fixed.

$B_\alpha(x)$ is larger than ε. This property is a consequence of the uniform transversality of the bundles E^s and $E^c \oplus E^u$.

Since the length of stable arcs goes to zero at a uniform rate, we can choose β such that an arc of $\tilde{\mathcal{W}}^s$ of length α cannot reach length larger than $\beta/2$ by future iterates.

The proof then reduces to the following claim.

Claim. *There exists $K > 0$ such that if two leaves U_1 and U_2 of $\tilde{\mathcal{F}}^{cu}_{bran}$ have points x, y in the same strong stable leaf and whose connected components in $B_\alpha(x)$ are at distance larger than ε , then for every $z \in U_1$ such that $p_1(z) < p_1(x) - K$ we have that $d(z, U_2) > \beta$.*

PROOF. Let D be a fundamental domain in $\tilde{S}_A$. Cover D by finitely many boxes $\{B_j\}_j$ of radius α such that there is a well controlled local product structure with that size.

For each box B_j there exists K_j such that if two points in the box are in leaves of $\tilde{\mathcal{F}}^{cu}_{bran}$ whose connected components inside B_j are at distance larger than ε then if z is in one of the leaves and $p_1(z) < p_1(B_j) - K_j$ then the distance from z to the other leaf is larger than 10β. This follows from the fact that those leaves are almost parallel to different leaves of $\tilde{\mathcal{F}}^{cu}_A$, thus, by Proposition 4.17 we know that for K_j large enough, if $p_1(z) < p_1(B_j) - K_j$ then the distance between the leaves of $\tilde{\mathcal{F}}^{cu}_A$ is larger than $10\beta + 2R$ where R is given by the definition of being almost parallel.

Considering $K = \max\{K_j\} < \infty$ we prove the claim since deck transformations are isometries and respect the relative p_1 coordinates (i.e. for a deck transformation γ we have that $p_1(\gamma x) - p_1(\gamma y) = p_1(x) - p_1(y)$ for every $x, y \in \tilde{S}_A$).

◊

□

Corollary 8.11. *Let $f : S_A \to S_A$ be a partially hyperbolic diffeomorphism in the hypothesis of this section, then, either the homological rotation set of f consists only of positive points or it consists only of negative points.*

A more important (and more useful) consequence is the following. It will allow us to use Proposition 8.4.

Corollary 8.12. *Assume that both $\mathcal{F}^{cs}_{bran}$ and $\mathcal{F}^{cu}_{bran}$ are almost parallel to $\mathcal{F}^{cu}_{A}$. Then, there exists $K > 0$ such that for every $x \in \tilde{S}_A$ we have that for every $n \in \mathbb{Z}$ we have $p_1(x) - K < p_1(\tilde{f}^n(x)) < p_1(x) + K$.*

Proof. If there exists $x \in \tilde{S}_A$ such that either $p_1(\tilde{f}^n(x)) \to +\infty$ for $n \to +\infty$ or $n \to -\infty$ then we obtain a point having iterates in the other sense contradicting Lemma 8.10.

□

Finally, using the fact that there is no Denjoy's property, we can obtain a stronger result in the case where there is no branching.

Proposition 8.13. *Assume that $\mathcal{F}^{cu}_{bran}$ is a foliation (every point belongs to a unique leaf) and that it is almost parallel to $\mathcal{F}^{cu}_{A}$. Then, there exists $n > 0$ such that for every $x \in \tilde{S}_A$ we have that $p_1(\tilde{f}^n(x)) \geq p_1(x) + 1$. In particular, the homological rotation vector of f is contained in $[1/n, +\infty)$.*

Similar statements hold for $\mathcal{F}^{cs}_{bran}$ or when $\mathcal{F}^{cu}_{bran}$ is instead close to $\mathcal{F}^{cs}_{A}$.

Proof. It is enough to prove that for every point $x \in \tilde{S}_A$ we have that $\lim_{n\to+\infty} p_1(\tilde{f}^n(x)) = +\infty$. Then, using the fact that deck transformations do not change the relative p_1-coordinates and compactness we conclude.

Now, pick a point $x \in \tilde{S}_A$ in a leaf $L_1 \in \tilde{\mathcal{F}}^{cu}_{bran}$ and a stable arc J which intersects a different leaf L_2 of $\tilde{\mathcal{F}}^{cu}_{bran}$. We must show that given K there exists $\varepsilon > 0$ such that if $d(z, L_2) < \varepsilon$ for $z \in L_1$ then $p_1(z) > K$. This would conclude since the length of J goes to zero exponentially as it is iterated forward by $\tilde{f}$.

Now, since L_1 and L_2 are contained in R-neighborhoods of different leaves of $\tilde{\mathcal{F}}^{cu}_{A}$ (Proposition 8.8) we know that there exists $T \in \mathbb{R}$ such that if $p_1(z) < T$ and $z \in L_1$ then $d(z, L_2) > 1$. Now, given $K > 0$ using the continuity of foliations we get that if ε is small enough, if there is $z \in L_1$ such that $d(z, L_2) < \varepsilon$ then there will be a point $w \in L_1$ such that $p_1(w) < p_1(z) - K$ and such that $d(w, L_2) < 1$. This concludes.

□

Corollary 8.14. *Assume that $\mathcal{F}^{cu}_{bran}$ is a foliation, then $\mathcal{F}^{cu}_{bran}$ and $\mathcal{F}^{cs}_{bran}$ are not almost parallel to the same foliation.*

8.4. **Dynamical coherence.** We first prove the following Proposition which shows that if the foliations are close to the "correct ones" then branching is not possible.

Before we prove dynamical coherence, we will show the following result which gives conditions under which the branching foliations do not branch.

Proposition 8.15. *Assume that $\mathcal{F}^{cs}_{bran}$ and $\mathcal{F}^{cu}_{bran}$ are not almost parallel to each other. Then, f is dynamically coherent.*

Consider $\mathcal{F}^{c}_{bran}$ to be the collection of curves tangent to E^c obtained by intersecting pairs of leaves of $\mathcal{F}^{cs}_{bran}$ and $\mathcal{F}^{cu}_{bran}$. We will denote as $\mathcal{F}^{c}_{bran}(x)$ to the set of intersections between the leaves in $\mathcal{F}^{cs}_{bran}(x)$ and $\mathcal{F}^{cu}_{bran}(x)$. Notice that a priori, some of those curves may not contain x. It is clear however that $\mathcal{F}^{c}_{bran}$ is f-invariant: $f(\mathcal{F}^{c}_{bran}(x)) = \mathcal{F}^{c}_{bran}(f(x))$.

Using Corollary 8.9 we obtain moreover that the lift $\tilde{f}$ of f fixes every $\tilde{\mathcal{F}}^{c}_{bran}(x)$ the lift of $\mathcal{F}^{c}_{bran}(x)$ to the universal cover: We have that $\tilde{f}(\tilde{\mathcal{F}}^{c}_{bran}(x)) = \tilde{\mathcal{F}}^{c}_{bran}(x)$.

We can prove the following:

Lemma 8.16. *There exists $\nu > 0$ uniform such that if there exists a center curve γ_0 in $\tilde{\mathcal{F}}^{c}_{bran}$ such that $\tilde{f}^k(\gamma_0) \neq \gamma_0$ for some $k \in \mathbb{Z}$. Then for every center curve $\gamma \in \tilde{\mathcal{F}}^{c}_{bran}$ we have that $d(\gamma, \tilde{f}^k(\gamma)) > \nu$.*

Proof. First notice that given a surface L^{cs} in $\tilde{\mathcal{F}}^{cs}_{bran}$ and another L^{cu} in $\tilde{\mathcal{F}}^{cu}_{bran}$ we know by Corollary 4.9 (i) that there exists $\nu > 0$ such that two different connected components of $L^{cs} \cap L^{cu}$ must be at distance at least $\nu > 0$ which is uniform (notice that L^{cs} and L^{cu} intersect in curves tangent to E^c so ν is given by the size of local product structure between stable and center leaves inside L^{cs} which can be chosen uniform in $\tilde{S}_A$ by compactness).

Now, assume that there exists $k \geq 0$ such that $\tilde{f}^k(\gamma_0) = \gamma_0$, we must prove that every center curve in $\tilde{\mathcal{F}}^{c}_{bran}$ is fixed.

First, notice that the set of center leaves in $\tilde{\mathcal{F}}^{c}_{bran}$ which are fixed by $\tilde{f}^k$ is closed. Moreover, consider $\varepsilon > 0$ such that if two points are at distance smaller than ε then their image by $\tilde{f}^k$ is smaller than $\frac{\nu}{10}$, then, we know that since the center stable and center unstable surfaces of $\tilde{\mathcal{F}}^{cs}_{bran}$ and $\tilde{\mathcal{F}}^{cu}_{bran}$ are fixed by $\tilde{f}^k$ we know that every center curve in $\tilde{\mathcal{F}}^{c}_{bran}$ intersecting the ε-neighborhood of a fixed center curve of $\tilde{\mathcal{F}}^{c}_{bran}$ by $\tilde{f}^k$ is also fixed by $\tilde{f}^k$.

Now, by connectedness we obtain that every leaf of $\tilde{\mathcal{F}}^{c}_{bran}$ must be fixed by $\tilde{f}^k$: Let us expand this a little; Consider the set of points such that every curve of $\tilde{\mathcal{F}}^{c}_{bran}$ containing x is fixed by $\tilde{f}^k$, by the argument above, this set is open. Moreover, since the set of curves of $\tilde{\mathcal{F}}^{c}_{bran}$ containing x passes to the closure, this set is also closed, this allows to use the connectedness argument.

So, if one center curve of $\tilde{\mathcal{F}}^{c}_{bran}$ is not fixed by $\tilde{f}^k$ we know that no center curve of $\tilde{\mathcal{F}}^{c}_{bran}$ can be fixed by $\tilde{f}^k$. Since center curves which are not fixed are mapped into a different connected component of the intersection of the center stable and center unstable surfaces containing them, we deduce the Lemma.

□

Using the previous lemma we can use the following simple yet powerful remark inspired in [BoW]. This lemma, as the previous one, do not make use of the standing hypothesis of Proposition 8.15.

Lemma 8.17. *The lift $\tilde{f}$ has no periodic points. In particular, it cannot fix any leaf of $\tilde{\mathcal{W}}^s$ nor $\tilde{\mathcal{W}}^u$.*

Proof. Assume $\tilde{f}$ has a periodic point p. Considering an iterate, we can assume it is fixed. Let $\tilde{\mathcal{W}}^s(p)$ be the strong stable leaf through p.

Then, we have that $\tilde{f}(\tilde{\mathcal{W}}^s(p)) = \tilde{\mathcal{W}}^s(p)$. On the other hand, $\tilde{f}$ must fix the leaves of $\tilde{\mathcal{F}}^c_{bran}(p)$ through p which are transverse to $\tilde{\mathcal{W}}^s(p)$ so that each leaf of $\tilde{\mathcal{F}}^c_{bran}$ intersects $\tilde{\mathcal{W}}^s(p)$ in a discrete set of points. This implies that every center leaf is fixed thanks to Lemma 8.16.

On the other hand, using a Poincare-Bendixon's like type of argument in a leaf of $\tilde{\mathcal{F}}^{cs}_{bran}(p)$ one can see that each leaf of $\tilde{\mathcal{F}}^c_{bran}$ intersects $\tilde{\mathcal{W}}^s(p)$ in at most one point (this also follows from Corollary 4.9 (i)).

Thus, since both the center-leaves and $\tilde{\mathcal{W}}^s(p)$ are fixed, every point of $\tilde{\mathcal{W}}^s(p)$ would be fixed, a contradiction.

Since every fixed or periodic leaf of $\tilde{\mathcal{W}}^s$ or $\tilde{\mathcal{W}}^u$ implies the existence of a periodic point we conclude the proof of the Lemma.

□

When the branching foliations are close to different branching foliations we can moreover prove the following:

Lemma 8.18. *Assume that $\mathcal{F}^{cs}_{bran}$ is not almost parallel to $\mathcal{F}^{cu}_{bran}$ then some iterate of $\tilde{f}$ fixes every curve in $\tilde{\mathcal{F}}^c_{bran}$.*

Proof. Lemma 8.16 and the fact that the leaves L^{cs} and L^{cu} are almost parallel to different leaves which intersect in a unique connected component implies that they must intersect in finitely many connected components.

This implies that there is an iterate which fixes a center curve, and using again Lemma 8.16 we deduce that every center curve of $\tilde{\mathcal{F}}^c_{bran}$ is fixed by this iterate of $\tilde{f}$.

□

Now, we will show that branching is not possible to deduce dynamical coherence.

Proof of Proposition 8.15. Let $\mathcal{F}^{cs}_{bran}$ and $\mathcal{F}^{cu}_{bran}$ be the f-invariant branching foliations given by Theorem 3.12. We can assume without loss of generality (by using Theorem 4.16) that $\mathcal{F}^{cs}_{bran}$ is almost parallel to $\mathcal{F}^{cs}_A$ (and thus $\mathcal{F}^{cu}_{bran}$ is almost parallel to $\mathcal{F}^{cu}_A$).

By Proposition 3.16 we know that to show dynamical coherence it is enough to show that there is a unique leaf of $\mathcal{F}^{cs}_{bran}$ and $\mathcal{F}^{cu}_{bran}$ through each point. To do this, one can work in the universal cover and consider an iterate (thus one can assume that f is isotopic to the identity due to Proposition 8.1 and that $\tilde{f}$ fixes every curve of $\tilde{\mathcal{F}}^{c}_{bran}$ by Lemma 8.18).

Now, assume that there are two leaves L_1 and L_2 of $\tilde{\mathcal{F}}^{cs}_{bran}$ that share a point x. Since leaves of $\tilde{\mathcal{F}}^{cs}_{bran}$ are saturated by the foliation $\tilde{\mathcal{W}}^s$ and in particular, both L_1 and L_2 contain $\tilde{\mathcal{W}}^s(x)$.

Let $\gamma_i \in \tilde{\mathcal{F}}^{c}_{bran}(x)$ $(i = 1, 2)$ a center leaf through x which is fixed by $\tilde{f}$ and contained in L_i. From Lemma 8.17 we know that there are no fixed points in γ_i for $\tilde{f}$ so we get that we can orient γ_i in such a way that the image by $\tilde{f}$ of every point in γ_i moves forward.

Motivated by Lemma 3.16 of [BoW] we can prove that that the union of strong stable manifolds of the points in γ_i cover L_i. In fact, it is not hard to prove that the union of strong stable manifolds of γ_i is open and closed in L_i: Openness follows from local product structure, and if you consider a sequence $x_n \to x$ of points in the stable manifolds of γ_i we get that the stable manifold of x intersects the center manifold of x_n for large enough n (again by local product structure) so, since centers are fixed we get that eventually the center stable of x is arbitrarily close to a center manifold of x_n, the proof concludes by noticing that the fact that center curves are fixed implies that the union of strong stables through γ_i is saturated by center curves due to a Poincare-Bendixon's like type of argument (see Lemma 3.13 of [BoW]).

Moreover, the leaf L_i is obtained by considering the union of the iterates of the strips S_i obtained by saturating the subarc of γ_i from x to $\tilde{f}(x)$. These strips have uniformly bounded width (this means, the length in the center direction), see Proposition 3.12 of [BoW].

This implies that every point in L_1 and L_2 is at bounded distance from a point of branching implying that L_1 and L_2 remain at bounded distance and contradicting Proposition 8.8.

We have proved that the branching foliations under the assumptions of this Proposition cannot be branched, this concludes by Proposition 3.16.

□

Now we are ready to prove dynamical coherence.

Theorem 8.19. *Let $f : S_A \to S_A$ be a partially hyperbolic diffeomorphism without f-periodic two-dimensional torus tangent to either $E^s \oplus E^c$ nor $E^c \oplus E^u$. Then, f is dynamically coherent.*

Proof. As mentioned before, it is enough to show this for an iterate, so we can assume that the hypothesis in the results of this section all hold.

Claim. *If f is not dynamically coherent, then there exists K such that every point $x \in \tilde{S}_A$ verifies that*

$$p_1(x) - K < p_1(\tilde{f}^n(x)) < p_1(x) + K$$

Proof. If f is not dynamically coherent, then both $\tilde{\mathcal{F}}^{cs}_{bran}$ and $\tilde{\mathcal{F}}^{cu}_{bran}$ must be almost parallel to the same invariant foliation of the suspension of A because of Proposition 8.15.

This implies that the claim holds due to Corollary 8.12.

$\diamondsuit$

Now, consider an unstable arc J of $\tilde{\mathcal{W}}^u$ in $\tilde{S}_A$ such that it has a point with $p_1(x) = 0$ and length smaller than K. We get that $\tilde{f}^n(J)$ is contained in the set of points such that $|p_1(x)| < 2K + 1$, this implies, via Lemma 8.3 that the sets $\tilde{f}^n(J)$ remains is contained in a set of the form $B_{nD,2K+1}(p)$ for some $p \in \tilde{S}_A$, using Proposition 8.4 we obtain that the volume of a neighborhood of $\tilde{f}^n(J)$ is bounded by a polynomial in n.

However, the length of $\tilde{f}^n(J)$ grows exponentially fast, so, after Proposition **??** so does the volume of an δ-neighborhood. This is a contradiction and completes the proof of the Theorem.

$\square$

As a consequence, we obtain the following result about dynamical coherence for absolutely partially hyperbolic diffeomorphisms of S_A:

Corollary 8.20. *Let $f : S_A \to S_A$ be an absolutely partially hyperbolic diffeomorphism. Then f is dynamically coherent.*

Proof. Assume f is not dynamically coherent. Then, by the previous Theorem one has a center-stable or center-unstable incompressible two-dimensional torus which is f-invariant (modulo considering an iterate). We can cut S_A along this torus in order to obtain a partially hyperbolic diffeomorphism of $\mathbb{T}^2 \times [0, 1]$ which is not dynamically coherent. By regluing with the identity, we obtain an absolutely partially hyperbolic diffeomorphism of $\mathbb{T}^3$ which is not dynamically coherent contradicting [BBI$_2$] (see also Proposition 5.8).

$\square$

As we mentioned, in order that our assumption that the branching foliations exist not to affect our result, we must now prove that there is a unique f-invariant foliation tangent to the bundles (we emphasize that this does not imply that the bundles are uniquely integrable which we do not know if it is true).

Proposition 8.21. *There is a unique f-invariant foliation tangent to $E^s \oplus E^c$. The same holds for $E^s \oplus E^c$.*

Proof. We have shown that in the universal cover all leaves of these foliations should be fixed by the lift f. If they separate, then we would find a point x having two different center leaves $\tilde{\mathcal{F}}^c_1(x)$ and $\tilde{\mathcal{F}}^c_2(x)$ and points $y_1 \in \tilde{\mathcal{F}}^c_1(x)$ and $y_2 \in \tilde{\mathcal{F}}^c_2(x)$ such that $\tilde{\mathcal{W}}^s(y_1) \cap \tilde{\mathcal{W}}^u(y_2) \neq \emptyset$. By iteration, the points $\tilde{f}^n(y_1)$ and $f^n(y_2)$ remain in $\tilde{\mathcal{F}}^c_1(x)$ and $\tilde{\mathcal{F}}^c_2(x)$ respectively, by an angle argument, one deduces that $y_2 \in \tilde{\mathcal{W}}^s(y_1)$ since the points must approach the future iterates of x where the branching takes place.

This means that there is a first point of branching, and as above, this will imply the existence of a fixed strong unstable leaf leading to a contradiction with Lemma 8.17.

□

8.5. **Leaf conjugacy.** In this section we complete the proof of Theorem 2.17.

We first state the following lemma which summarizes what we have already proved:

Lemma 8.22. *Let $f : S_A \to S_A$ be a dynamically coherent partially hyperbolic diffeomorphism with center stable and center unstable foliations $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ without torus leaves. Assume moreover that $\mathcal{W}^{cu}$ is almost parallel to $\mathcal{F}^{cu}_A$. Then, the lift $\tilde{f}$ of f which is at bounded distance from the identity fixes every leaf of $\tilde{\mathcal{W}}^c$, the lift of the center-foliation obtained by intersecting $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$. Moreover, there exists $n > 0$ such that for every point $x \in \tilde{S}_A$ we have that $p_1(\tilde{f}^n(x)) > p_1(x) + 1$.*

Proof. Since f is dynamically coherent, we can apply Proposition 8.13 and Corollary 8.14 to obtain that points are moving forward in the universal cover as well as that both foliations are almost parallel to different foliations. From the hypothesis of the lemma and Theorem 4.16 we obtain that $\mathcal{W}^{cs}$ is almost parallel to $\mathcal{F}^{cs}_A$.

Now, we are in the hypothesis of Lemma 8.18 so we obtain that $\tilde{f}$ has an iterate which fixes every center leaf. This concludes.

□

We will use the following result which we borrow from page 74 of [V] (see also [Sc]):

Proposition 8.23. *Let $\phi_t : M \to M$ be a flow and let $P : M \to S^1$ be a continuous function. Let $\tilde{P} : \tilde{M} \to \mathbb{R}$ the lift of P and assume that for every $x \in \tilde{M}$ we have that $\lim_{t\to+\infty} \tilde{\phi}_t(x) = +\infty$. Then, there exists a differentiable surface $S \subset M$ which is transverse to the flow.*

Proof of Theorem 2.17. We consider the function $P : S_A \to S^1$ given by the fact that S_A is a torus bundle over the circle. Notice that $\tilde{P} : \tilde{S}_A \to S^1$ is exactly p_1.

Using Lemma 8.22 we know that an iterate of f (which we assume is f) fixes every center leaf. So, we can define a flow ϕ_t by choosing an orientation on E^c and to prove leaf conjugacy to the suspension of the linear Anosov it is enough to show that this flow is orbit equivalent to the suspension of the linear Anosov automorphis.

To prove this, notice that the flow verifies the hypothesis of Proposition 8.23 so there exists a surface of section S for ϕ_t. Since being transverse to $\mathcal{W}^c$ implies it is transverse to $\mathcal{W}^{cs}$ this implies that S admits a non-vanishing vector field and thus $S \cong \mathbb{T}^2$. Moreover, from the properties of $\mathcal{F}^{cs}_A$ and $\mathcal{F}^{cu}_A$ given in Proposition 4.17 we get that the return map to S is expansive. Using the main result of [L] we deduce then that the flow is orbit equivalent to the suspension of a linear Anosov automorphism of $\mathbb{T}^2$ as desired.

Finally, from the work done in the previous sections this concludes the proof.

□

Remark 8.24. We remark that we have not used the classification of Anosov flows in solvmanifolds [V] so that our proof allows us to recover it (even if the proof once one obtains coherence is much based on the ideas for that Theorem).

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