LOCAL IMPLICATIONS OF ALMOST GLOBAL STABILITY

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ABSTRACT. We prove that for autonomous differential equations with the origin as a fixed point and with one eigenvalue with negative real part we have that the existence of a density function implies local asymptotical stability. We present examples of a system admitting a density function for which the origin is not locally asymptotically stable and an almost globally stable system for which no function is a density function.

1. INTRODUCTION

Consider the autonomous differential equation

$$\dot{x} = f(x) \tag{1}$$

where $f : \mathcal{R}^n \to \mathcal{R}^n$ is a C^1 field, which enough to ensure the existence and uniqueness of solutions to the initial value problem. Let the origin be an equilibrium point of (1), i.e. $f(0) = 0$. In this context, the idea of *density function* was recently introduced [1]: a function $\rho \in C^1(\mathcal{R}^n \backslash \{0\}, \mathcal{R})$ which is positive almost everywhere and verifies the first order condition

$$\nabla \cdot (\rho f) > 0 \quad m\text{-a.e.}$$

where m is the Lebesgue measure and it is integrable outside any arbitrary neighborhood of the origin. The relevance of this kind of functions is that their mere existence implies that almost every trajectory of the system (1) converges to the origin. This means that the complement of the set

$$R = \left\{ x \in \mathcal{R}^n \mid \lim_{t \to \infty} f^t(x) = 0 \right\}$$

is a zero Lebesgue measure set, where $f^t(x)$ denotes the time t of the trajectory of the system that starts at x. This was called *almost global stability* (a.g.s.). The initial results in [1] triggered several lines of research: converse results were addressed in [2, 3, 4, 5], synthesis of nonlinear controllers using a convex approach were presented in [6] and several theoretical results aiming a deeper knowledge of these new ideas, including topological restrictions and sign definition of density functions and

Lyapunov and density functions relationships were analyzed in [7, 8, 9, 10, 11].

Particularly, in [12], the idea of *monotone measure* was blended with the classical Poincaré-Bendixson theory to obtain a relation between a.g.s. and local stability of the origin for planar systems. If the system, in some way, *shrinks* the bounded sets and the origin is an isolated equilibrium, then a.g.s. is equivalent to local asymptotical stability of the origin. The proof of this result is deeply based on the topological properties of the plane that are the core of the Poincaré-Bendixson theory [13]. In this article we wonder what happens in higher dimensions. Here, we present a particular approach that tries to extend the planar results of [12] to higher dimensions, which does not involve generalizing Poincaré-Bendixson theory. We establish relationships between the existence of density functions, the almost global stability property and local (not necessarily asymptotical) stability of the origin. The main result of this note is:

Theorem 1. *Let $\dot{x} = f(x)$ with $f \in C^1(\mathcal{R}^n, \mathcal{R}^n)$, such that $\frac{\partial f}{\partial x}(0)$ has at least one eigenvalue with negative real and such that the system admits a density function. Then, the origin is a.g.s. and locally asymptotically stable.*

The hypothesis of having one negative eigenvalue is optimal in the following sense, we provide here an example of a system admiting a density function for which the origin is not locally asymptotically stable (see Example 3.3). The techniques used in the proof of this theorem involve the use of center manifolds and some properties of Lyapunov functions.

We remark that results obtained in [2, 3, 11, 4, 5] ensure the existence of density functions for a.g.s. systems with local asymptotical stability. The main result of this note is the converse under some mild hypothesis that are necessary for it to be true. This result generalizes in some sense the results of [12] to higher dimensions and also generalizes the result of Rantzer in [1] showing that density functions also ensure local asymptotical stability if one eigenvalue of the derivative of f in the origin has negative real part. Also, we present a new example providing a situation that was not known so far: an a.g.s. system which does not admit a density function. This system was derived using the main theorem (see Example 3.2).

This paper is concerned in local properties of a.g.s. systems so we shall work in

$\mathcal{R}^n$. Of course, all the results hold in the context of manifolds. The article is organized as follows: in Section 2, we review the basic results of [1] that will be used along the article; in Section 3, we present several examples that will give us an idea of the problem we are dealing with and the hypothesis that we should add in order to obtain the desired result; in Section 4 we present the main result of this article; finally, we present some conclusions and commentaries.

2. Densities and Lyapunov functions

We start this Section with the definition of a density function:

Definition: given a dynamical system $\dot{x} = f(x)$, a density function for this system is a scalar function $\rho : \mathcal{R}^n \setminus \{0\} \to [0, +\infty)$, of class C^1, integrable on the outside of a ball centered at the origin, and such that the following divergence condition is satisfied

$$\nabla.(\rho f)(x) > 0 \ \textit{almost everywhere}\,(a.e.) \tag{2}$$

The main result in [1] states that the existence of a density function implies almost global stability. Several relationships between densities and Lyapunov functions can be derived. Some of them have appeared in [1]. A more detailed analysis between Lyapunov and density functions can be found in [10, 11].

Proposition 1. [1] *Given $V(x) > 0 \ \forall x \neq 0$ satisfying*

$$\alpha \nabla V \cdot f < V \nabla \cdot f \quad m - ctp$$

for some $\alpha > 0$. Then for $\rho(x) = V(x)^{-\alpha}$ we have $\nabla \cdot (\rho f) > 0 \ m - ae$.

Proposition 2. [1] *If for every $x \neq 0$, ρ satisfies*

$$\nabla \cdot (\rho f) > 0 \quad \nabla \cdot f \leq 0 \quad \rho > 0$$

Then $V(x) = \rho(x)^{-1}$ satisfies $\nabla V \cdot f < 0$.

The proof of this properties are very simple and are based on the formula

$$\nabla \cdot (\rho f) = \nabla \rho \cdot f + \rho \nabla \cdot f$$

We present here the proof of Proposition 2 since it is possible to generalize it for the case when the hypothesis are satisfied locally. (Recall that Lyapunov functions only need to be defined locally).

Proof: Consider the expression

$$\nabla \cdot (\rho f) = \nabla \rho \cdot f + \rho \nabla \cdot f > 0 \quad m - a.e.$$

Using positivity of ρ and that in a neighborhood of the origin $\nabla \cdot f \leq 0$, we have that $V = \rho^{-1} > 0$ implies $\nabla \rho f \geq 0$. So

$$\dot{V} = \nabla V \cdot f = -\rho^{-2}(\nabla \rho \cdot f) \leq 0$$

$\triangle$

3. Some examples

We attempt to make a step towards the generalization of the results in [12] to dimensions higher than 2. We look forward to find reasonable hypothesis which should ensure that almost global stability implies local asymptotical stability. Also, as already discussed, we pretend to relate the existence of density functions with local asymptotical stability generalizing the main result in [1]. These problems are clearly related since under local asymptotical stability and very mild hypothesis, almost global stability is equivalent to the existence of a density function (see [2, 3, 11, 4, 5]).

We will present several examples showing different aspects we must look at. We look for conditions that should be required to the system (1) to relate a.g.s., density functions and local asymptotical or, less ambitiously, mere local stability. It is easy to construct examples of a.g.s. systems which are not locally asymptotically stable. In fact, every time we have a globally stable field we can multiply it with a function which is zero in a curve through zero and this new vector field will give rise to a flow which is a.g.s. but not locally asymptotically stable (of course it will be locally stable). Figure 1 shows an almost global stable but not asymptotically stable system, introduced in [6] for synthesis of nonlinear controllers. It is natural then, to study systems for which the origin is an isolated fixed point of the flow.

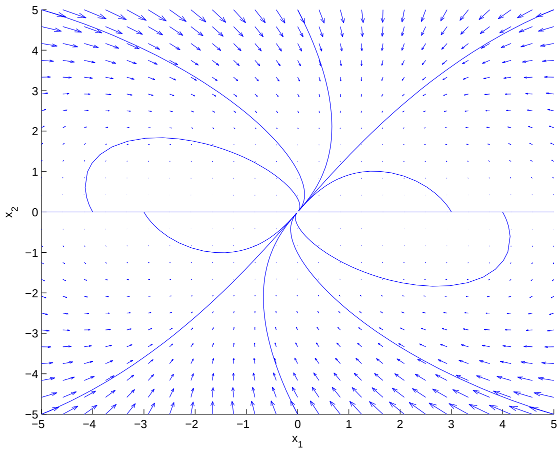

FIGURE 1. *An a.g.s. system without local asymptotical stability. The system's equations are* $\dot{x}_1 = x_2.(-6x_1x_2 - x_1^2 + 2x_2^2)$, $\dot{x}_2 = x_2.(2.229x_1^2 - 4.8553x_2^2)$ [6].

The next two well known examples show how a system may be a.g.s. but not even locally stable.

Example 3.1. Consider the field

$$\begin{cases} \dot{x}_1 &= x_1^2 - x_2^2 \\ \dot{x}_2 &= 2x_1x_2 \end{cases}$$

The trajectories are shown in figure 2. In the complex plane, the velocity at each point is the square of its magnitud. Almost every trajectory converges to the origin. The only exceptions are the trajectories that start on the positive half of the x_1-line. They have finite time escape. It seems that a density function should not exist, since points arbitrarily close to the *right* of the origin evolve to points arbitrarily close to the *left* and this implies that a particular set should grow a lot and then shrink. We can only show that the field does not accept $\rho(x) = \|x\|^{-\alpha}$ as a density function. We observe that for such a ρ.

$$\nabla \cdot (\rho f) = \|x\|^{-\alpha} x_1(4 - 2\alpha)$$

So, this kind of ρ can not be a density function for this system.

△

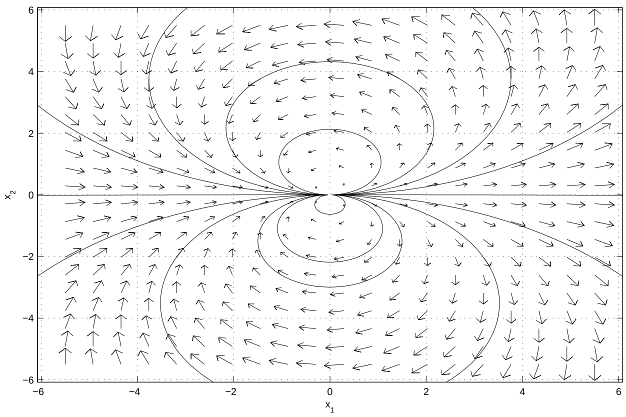

FIGURE 2. *Trajectories for the system of Example* 3.1

Example 3.2. Consider the system

$$\begin{cases} \dot{x}_1 &=& x_1^2 - x_2^2 \\ \dot{x}_2 &=& 2x_1x_2 \\ \dot{x}_3 &=& -x_3 \end{cases}$$

This case is similar to Example 3.1, but we have added a third contractive direction. We still have the almost global stability property, since the system dynamics can be decoupled. In the plane $x_3 = 0$, we recover the previous system.

We remark that this systems is a.g.s. and not locally asymptotically stable as the previous example. However, the existence of a negative eigenvalue will allow us to prove that it can not admit a density function (see Corollary 4.1).

$\triangle$

The next example shows how the hypothesis of possessing a negative eigenvalue of the field is necessary in order to obtain local asymptotical stability from the existence of a density function.

Example 3.3. Consider the system

$$\begin{cases} \dot{x}_1 &=& x_2 - 2x_1x_3^2 \\ \dot{x}_2 &=& -x_1 - 2x_2x_3^2 \\ \dot{x}_3 &=& -x_3^3 \end{cases}$$

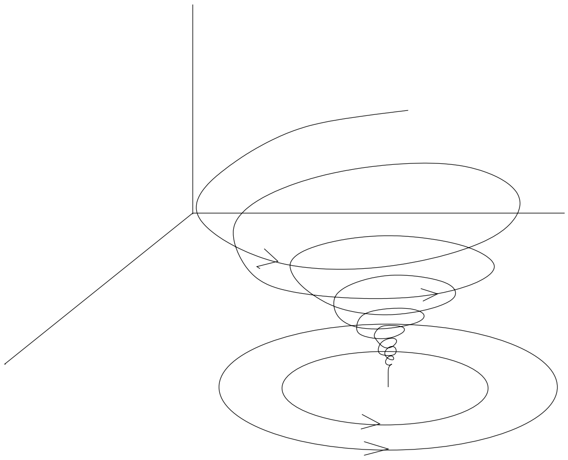

FIGURE 3. *Some trajectories of the Example* 3.3

Some representative trajectories are shown in figure 3. The linearization at the origin has zero as its only eigenvalue. The system is a.g.s., since the function

$$\rho(x) = (x_1^2 + x_2^2 + x_3^2)^{-4}$$

verifies

$$\nabla \cdot (\rho f) = x_3^2(x_1^2 + x_2^2 + x_3^2)^{-4}$$

which is positive almost everywhere. It is not hard to prove that the origin is locally stable. Observe that at the plane $x_3 = 0$, we have a harmonic oscillator, while the x_3-line is a contractive direction. So, we have a density function, the system is a.g.s., the origin is an isolated not asymptotically stable equilibrium and the linearization at the origin has only the null eigenvalue.

△

4. Classification of fixed points

In this Section, we try to classify the possible behaviors of the origin as a fixed point depending on the eigenvalues of the derivative of the field in the origin. We can conclude that if there is at least one eigenvalue with positive real part, then almost global stability is not possible. On the other hand the existence of a eigenvalue with negative real part added to the existence of a density function implies local asymptotical stability. This will imply the main theorem of this note.

Proposition 3. *Let $\dot{x} = f(x)$ with $f \in C^1(\mathcal{R}^n, \mathcal{R}^n)$ such that $\frac{\partial f}{\partial x}(0)$ has at least one eigenvalue λ with $Re(\lambda) > 0$. Then, the set of points such that $\lim_{t\to\infty} f^t(x) = 0$ has zero Lebesgue measure.*

Proof: This proof is based on the existence of local invariant manifolds [16]. In this case we are interested on the unstable manifold (given by the expanding eigenvalues of the derivative of f) and the center stable manifold ($W^{cs}_{loc}(0)$). These manifolds exist only locally and have zero Lebesgue measure since their dimensions are strictly smaller than the dimension of the space. Also, we know that if a point in $\mathcal{R}^n$ is to converge to the origin as $t \to \infty$ then for some $n \in \mathcal{N}$ $f^n(x) \in W^{cs}_{loc}$. We do not know if this manifolds are invariant, but we know that the points that stay close to the origin in the future belong to all posible center stable manifolds. So, we have that the region of attraction of the origin is contained in $\bigcup_{n\in\mathcal{N}} f^{-n}(W^{cs}_{loc})$ which is a countable union of sets with zero measure, so it has zero Lebesgue measure.

△

Proposition 4. *Let $\dot{x} = f(x)$ with $f \in C^1(\mathcal{R}^n, \mathcal{R}^n)$ such that all the eigenvalues of the Jacobian matrix $\frac{\partial f}{\partial x}(0)$ have non positive real part and at least one of them has negative real part. Then, the existence of a differentiable density function implies local (asymptotical) stability.*

Proof: The fact that the eigenvalues are as said implies that $\nabla \cdot f \leq 0$. Proposition 2 implies the existence of a Lyapunov function and we have local stability. Asymptotical stability is given by the fact that $\nabla \cdot f$ is strictly negative due to the eigenvalue with negative real part.

△

This propositions allow us to prove the main theorem of this note.

Theorem 2. *Let $\dot{x} = f(x)$ with $f \in C^1(\mathcal{R}^n, \mathcal{R}^n)$ such that $\frac{\partial f}{\partial x}(0)$ has at least one eigenvalue with negative real and such that the system admits a density function. Then, the origin is a.g.s. and locally asymptotically stable.*

Proof: The origin is a.g.s., because of Rantzer Theorem [1]. Proposition 3 implies that there is no eigenvalue with positive real part. After this remark, it is easy to notice that we are in the hypothesis of Proposition 4 and it concludes the proof.

△

As we have already mentioned, the converse of this theorem, local asymptotical stability plus a.g.s. implies the existence of a density function, is also true [2, 3, 11, 4, 5]. Also, this theorem implies what was already mentioned in Example 3.2.

Corollary 4.1. The system of Example 3.2 is a.g.s. and does not admit a density function.

5. Conclusions and future work

We achieved a result which guaranties local asymptotical stability under the existence of a density function. This result is optimal in the sense that it is not true if we remove the hypothesis of having one eigenvalue with negative real part (see Example 3.3). This has particular interest in the context of nonlinear control since a.g.s. without local stability is not very desirable for a system. In particular, we believe that this result makes the finding of a density function a worthwhile task considering that a.g.s. plus local asymptotical stability is very similar to global stability in terms of the realization of the system but is much more general (for example, there can not be globally stable systems in compact manifolds, and globally stable systems can not admit other equilibriums). Together with results of [1] about convexity of simultaneous finding of density functions and control actions this could help in the task of stabilizing unstable systems. However, it is important to recall the work of Angeli ([8]) which studies some restrictions on this approach.

In future works we should try to study the remaining case of zero divergence as motivated by Example 3.3 attempting to ensure that a density function should imply local not asymptotical stability of the equilibrium. We believe that a major step in that direction would be to prove that the system of Example 2 does not admit a density function, since the behavior appearing in that example is the rule for a.g.s. systems with no local stability.

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