Partially hyperbolic diffeomorphisms and foliations in 3-manifolds

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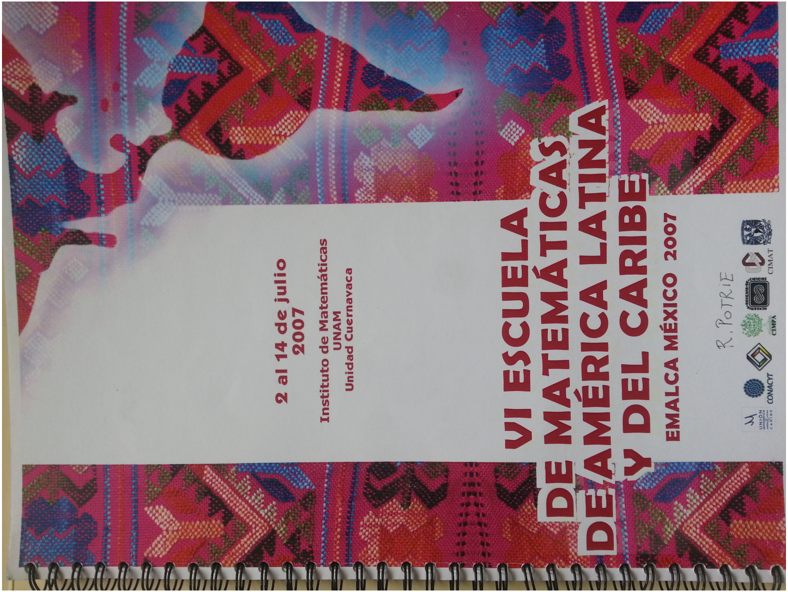
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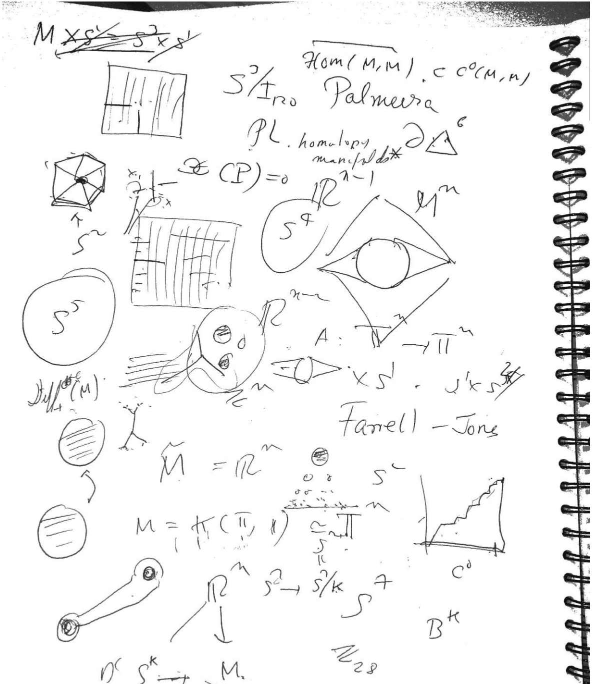
Hom(M,M) ⊂ C^0(M,M)
Palmeira
PL. homology manifolds
Farrell - Jones

Partially hyperbolic diffeomorphisms

Definition (Partially hyperbolic diffeomorphism)

$f : M \to M$ is *partially hyperbolic* (PH) if $TM = E^s \oplus E^c \oplus E^u$ continuous Df-invariant splitting so that $\exists n > 0$ so that if $v^\sigma \in E^\sigma(x)$ are unit vectors, then

$$\|Df^n v^s\| < \min\{1, \|Df^n v^c\|\} \leq \max\{1, \|Df^n v^c\|\} < \|Df^n v^u\|$$

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Relevant notion in: robust transitivity, stable ergodicity, smooth ergodic theory, homoclinic bifurcations, etc, etc....

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Proposal (Pujals/Bonatti-Wilkinson)

Compare partially hyperbolic diffeomorphisms to Anosov systems........

...... and leave the classification of the later to other people.

Partially hyperbolic diffeomorphisms: foliations

Definition (Dynamical coherence)

$f : M \to M$ partially hyperbolic is dynamically coherent (DC) if $\exists \mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$ f-invariant foliations tangent to $E^s \oplus E^c$ and $E^c \oplus E^u$ respectively. ($\Rightarrow \exists \mathcal{F}^c$ tangent to E^c.)

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Definition (Leaf conjugacy)

$f, g : M \to M$ partially hyperbolic and dynamically coherent are leaf conjugate if there exists $h : M \to M$ homeomorphism such that

$$\mathcal{F}^c_f(f \circ h(x)) = h(\mathcal{F}^c_g(g(x))).$$

Main goal, classify partially hyperbolic diffeomorphisms up to leaf conjugacy. (Includes deciding when they are DC)

Important technical tool: *branching foliations* introduced by Burago-Ivanov (we will ignore this and assume DC when needed).

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From now on M will be a closed 3-manifold. Advances in understanding of topology and geometry of 3-manifolds give hope that it is more within reach. Due to cumulative work of Brin-Burago-Ivanov, Bonatti-Wilkinson, Parwani, Hertz-Hertz-Ures, Hammerlindl-P. one knows that (see e.g. recent surveys Carrasco-Hertz-Hertz-Ures, Hammerlindl-P.)

Theorem

Let M be a 3-manifold with (virtually) solvable fundamental group and assume $\nexists T$ torus tangent to $E^s \oplus E^c$ or $E^c \oplus E^u$. Then, f is dynamically coherent and (modulo finite lift and iterate), f is leaf conjugate to one of the following:

- *linear Anosov with 3 different real eigenvalues on $\mathbb{T}^3$,*
- *skew-product over linear Anosov diffeomorphism on $\mathbb{T}^2$ (in this case M is $\mathbb{T}^3$ or nilmanifold),*
- *time one map of suspension of linear Anosov diffeomorphism on $\mathbb{T}^2$.*

'Big manifolds'

Recently, several new examples appeared:

Theorem (Bonatti-Gogolev-Hammerlindl-Parwani-P.)

There exists manifolds M with exponential growth of fundamental group so that they admit Anosov flows and partially hyperbolic diffeomorphisms not isotopic to identity.

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In particular, the following presents a challenge to the classification program:

Theorem (Bonatti-Gogolev-Hammerlindl-P.)

For any higher genus surface S there exists partially hyperbolic diffeomorphisms on T^1S so that: they are stably ergodic and robustly transitive but they are stably NON dynamically coherent.

Non transitive non-dynamically coherent examples were already known (Hertz-Hertz-Ures).

'Big manifolds'

The examples on the last theorem are not isotopic to identity. There is still hope that the following could admit a positive answer:

Question

Let M be a 3-manifold with fundamental group with exponential growth admitting a (transitive) partially hyperbolic diffeomorphism: Does M admit a (topologically) Anosov flow? If f is isotopic to identity, must it be dynamically coherent and leaf conjugate to such (topologically) Anosov flow?

'Big manifolds': Some results.

Theorem (Hammerlindl-P.-Shannon)

Let M be a Seifert 3-manifold and $f : M \to M$ a transitive partially hyperbolic diffeomorphism. Then M admits an Anosov flow.

Theorem (Ures)

Let $M = T^1 S$ with S a higher genus surface and $f : M \to M$ a PH diffeo isotopic through PH diffeos to the time one map of the geodesic flow for a metric of negative curvature in S. Then f is DC and leaf conjugate to the time one map of the geodesic flow in S.

'Big manifolds': Some results.

Theorem (Barthelemé-Fenley-Frankel-P. (work in progress))

Let M be either a Seifert or Hyperbolic 3-manifold and $f : M \to M$ DC, PH diffeo isotopic to identity. Then (up to finite lift or iterate) f is leaf conjugate to a topologically Anosov flow.

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- For Seifert manifolds there is no need to assume DC.
- For Hyperbolic manifolds there is no need to assume isotopic to identity (Mostow).
- We get very detailed (but more incomplete) information for general PH diffeos isotopic to identity on general 3-manifolds.

Some ideas

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$\tilde{f} : \tilde{M} \to \tilde{M}$ lift to universal cover at bounded distance from identity. ($\tilde{\mathcal{F}}^{cs}$, $\tilde{\mathcal{F}}^{cu}$ lifted foliations.)

IDEA: If leaves of $\tilde{\mathcal{F}}^{cs}$ and $\tilde{\mathcal{F}}^{cu}$ separate then they have to be fixed by $\tilde{f}$. ($\Rightarrow f$ 'looks like' an Anosov flow...)

Dichotomies

General results (Novikov+non-existence of compact leafs for $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$) imply that all leaves of $\tilde{\mathcal{F}}^{cs}$ and $\tilde{\mathcal{F}}^{cu}$ are planes.

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Either every leaf of $\tilde{\mathcal{F}}^{cs}$ is fixed by $\tilde{f}$ or the foliation $\tilde{\mathcal{F}}^{cs}$ is uniform and $\tilde{f}$ acts as a translation in the leafs.

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A symmetric statement holds for $\tilde{\mathcal{F}}^{cu}$, but in principle NOT simultaneously.

(The proof uses either that the foliations are minimal in M or that M is hyperbolic or Seifert.)

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1. Show that there is no 'mixed behaviour': if all leaves of $\tilde{\mathcal{F}}^{cs}$ are fixed by $\tilde{f}$, then, the same holds for $\tilde{\mathcal{F}}^{cu}$.

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The third item can be done without need of DC hypothesis in Seifert manifolds, but in Hyperbolic manifolds DC seems crucial for the moment (maybe $\exists$ examples??).

Blackboard.

Gracias!
......y por 20 más!!!!