UNIVERSITY OF CHICAGO	COCYCLES OF DIFFEOMORPHISMS OVER A HYPERBOLIC BASIS.	November 18 2013

Let $f: \mathbb{T}^2 \to \mathbb{T}^2$ be an Anosov diffeomorphism: i.e. $T\mathbb{T}^2 = E^s \oplus E^u$

[Thm (Franks) $\exists\, h: \mathbb{T}^2 \to \mathbb{T}^2$ homeomorphism such that $f = h f_* h^{-1}$, $f_* \in GL(2,\mathbb{Z})$.]

Question: When does f preserve an absolutely continuous invariant measure?
(say with density bounded from zero and infinity)

An obvious obstruction is to have that $\forall x \in Fix(f^n)$ one should have $|\det(Df^n_x)| = 1$.

[Thm (Livsic-Sinai '72) Let $f: \mathbb{T}^2 \to \mathbb{T}^2$ be a C^2-Anosov diffeomorphism such that $\forall x \in Fix(f^n)$ one has $|\det(Df^n_x)| = 1 \Rightarrow f$ preserves an absolutely continuous invariant measure with C^1-density.]

In fact, if $u: \mathbb{T}^2 \to \mathbb{R}$ is a bounded function and $\tilde{\omega} = e^u \omega$ where ω = volume form then we get that $\tilde{\omega}$ is f-invariant if and only if $e^{u(f(x))} \cdot Jac(Df_x) = e^{u(x)}$ for every $x \in \mathbb{T}^2$ or equivalently

$$-\log(Jac(Df_x)) = u(f(x)) - u(x).$$

(Here the obstruction in periodic orbits becomes evident).

[Thm (Livsic '71) $f: \mathbb{T}^2 \to \mathbb{T}^2$ C^1-Anosov diffeo and $\Phi: \mathbb{T}^2 \to \mathbb{R}$ an α-Hölder function such that $\sum_{i=0}^{n-1} \Phi(f^i(x)) = 0 \quad \forall x \in Fix(f^n)$. Then, $\exists\, u: \mathbb{T}^2 \to \mathbb{R}$ α-Hölder continuous s.t. $\Phi(x) = u(f(x)) - u(x)$.]

Regularity: Guillemin-Kazhdan, de la Llave-Marco-Moriyon, Journée, Hurder-Katok, Wilkinson....

Rmk: Results are more general, one needs $f: M \to M$ be a hyperbolic homeomorphism. For this talk, enough to think $f: \mathbb{T}^2 \to \mathbb{T}^2$ linear Anosov.

The problem I am interested in is to extend this kind of results to other (non-abelian) groups.

Let $f: M \to M$ be a hyperbolic homeo, $\Phi: M \to G$ a continuous function and G a topological group.

Question: Under what assumptions we can ensure that the vanishing of the trivial obstruction

$$(\text{POO}) \quad \forall x \in \mathrm{Fix}(f^n) \qquad \Phi^{(n)}(x) = \Phi(f^{n-1}(x)) \circ \dots \Phi(x) = id_G$$

is enough to guarantee $\exists\, u: M \to G$ such that $\Phi(x) = u(f(x)) \circ (u(x))^{-1}$?

Rmk: In the non-abelian context $\exists$ harder question of whether conjugate periodic data implies conjugacy.

This question appeared naturally in the study of rigidity properties of higher rank lattice actions on compact manifolds:

Hurder, Katok, Lewis, Spatzier, Zimmer, ... Lie groups.

Nitica, Török then de la Llave Windsor ... diffeomorphism groups. Katok Nitica

For the general question we have the following recent result:

always required.

Thm (Kalinin) $\Phi: M \to G$ α-Hölder satisfying the (POO) where G=Lie Group $\Rightarrow \exists\, u: M \to G$ α-Hölder such that $\Phi(x) = u(f(x)) \circ u(x)^{-1}$.

~~always~~

We have:

Thm (j.w. A. Kocsard) $\Phi: M \to \mathrm{Diff}^r(S^1)$ with $r \geq 1$ is α-Hölder and verifies the (POO) $\Rightarrow \exists\, u: M \to \mathrm{Diff}^r(S^1)$ α-Hölder such that

$$u(f(x)) \circ (u(x))^{-1} = \Phi(x).$$

This follows from a more general result for every manifold. First I will explain Livsic's classical proof.

Proof of classical Livsic thm: Consider $x_0 \in M$ with dense forward orbit.

Define $u(f^n(x_0)) = \sum_{j=0}^{n-1} \Phi(f^j(x_0))$ and $u(x_0) = 0$.

We have that $\Phi(x) = u(f(x)) - u(x)$ in $\mathcal{O}^+(x_0)$ so we want to see that u is uniformly continuous (in fact Hölder)

This will be a consequence of the following important result:

$\boxed{\text{Anosov Closing Lemma: If } d(f^k(x), x) \text{ is small} \Rightarrow \exists\, p \in \mathrm{Fix}(f^k) \text{ such that } d(f^i(x), f^i(p)) \leq C\lambda^{\min\{i, k-i\}} d(x, f^k(x))}$

Assume then that $d(f^n(x_0), f^m(x_0))$ is small and let p be such that
$$d(f^{m+i}(x_0), f^i(p)) \leq C\lambda^{\min\{i, k-i\}} d(f^n(x_0), f^m(x_0)).$$

$$|u(f^n(x_0)) - u(f^m(x_0))| = \left|\sum_{i=0}^{n-1} \Phi(f^i(x_0)) - \sum_{i=0}^{m-1} \Phi(f^i(x_0))\right| =$$

$$= \left|\sum_{i=m}^{n-1} \Phi(f^i(x_0)) - 0\right| = \left|\sum_{i=m}^{n-1} \Phi(f^i(x_0)) - \sum_{i=0}^{n-m-1} \Phi(f^i(p))\right| \leq$$

$$\leq \sum_{i=0}^{n-m-1} |\Phi(f^{m+i}(x_0)) - \Phi(f^i(p))| \leq K d(f^n(x_0), f^m(x_0))^\alpha$$

$\blacksquare$

Previous approaches (including Kalinin's) try to adress this problem by solving those problems.

Our approach uses Hölder continuity to lift foliations to a skew-product which in retrospective resembles more to an approach used for classical Livsic in partially hyperbolic systems (Katok-Kononenko/Wilkinson)

Consider $\hat{f}: M \times N \circlearrowleft$ as $\hat{f}(x,t) = (f(x), \Phi(x)(t))$
where $\Phi: M \to \mathrm{Diff}^1(N)$ is an α-Hölder cocycle.

Given an invariant measure μ for $\hat{f}$ we define $\lambda^c(\mu)$ to be the maximum modulus of the Lyapunov exponents of μ along the N direction.

We say $\hat{f}$ is dominated if $\forall \mu$ $\hat{f}$-invariant one has $\lambda^c(\mu) < \lambda^\alpha$ where λ is the hyperbolicity constant for f.

Thm (j.w. Kocsard) Let $f: M \to M$ be a transitive hyperbolic homeomorphism and $\Phi: M \to \mathrm{Diff}^r(N)$ an α-Hölder function such that:

- (I) Φ satisfies the (POO)
- (II) The diffeomorphism $\hat{f}$ is dominated

Then, $\exists u: M \to \mathrm{Diff}^r(N)$ α-Hölder such that $\Phi(x) = u(f(x))(u(x))^{-1}$.

Remark: If $\exists$ such a $u \Rightarrow$ both (I) and (II) are satisfied.

Question: Does (I) $\Rightarrow$ (II)? Answer: Yes for $\mathrm{Diff}^1(S^1)$ ($\Rightarrow$ hyperbolic periodic orbits)
Yes for $\mathrm{Diff}^{1+\alpha}_{vol}$(Surface).
Yes for Proyective cocycles (recovers Kalinin)

Main Ideas:

1) Domination allows one to lift W^s and W^u of f as Hölder graphs invariant under $\hat{f}$.
2) Using weak version of ACL (only shadowing) and the POO condition one sees that $\forall (x,t) \in M \times N$ one has that $\overline{\mathcal{O}_{\hat{f}}(x,t)}$ is a Hölder graph. (in principle it might not be saturated by $\hat{W}^s$ and $\hat{W}^u$)
3) We use Kalinin's result to show that every measure μ such that is invariant under $\hat{f}$ verifies that $\lambda^c(\mu) = 0$.
4) Invariance principle ~~(Furstenberg Ledrappier~~ (Avila-Viana) implies that the graphs are saturated by $\hat{W}^s$ and $\hat{W}^u$. As a consequence, the graphs form a "foliation" and we obtain a solution in Homeo(N)
5) To promote to $\mathrm{Diff}^1(N)$ we ~~use~~ show that Kalinin's solutions vary continuously and give the derivative.

Furstenberg
Ledrappier
Bonatti & Mont-Viana