Partially hyperbolicity and leaf conjugacy in small 3-manifolds

Rafael Potrie

CMAT-Universidad de la Republica

ICTP-Trieste

rpotrie@cmat.edu.uy

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Motivation

Theorem (Mañe-Franks)

Let M^2 be a closed surface. The following are equivalent:

- *f is C^1-robustly transitive*
- *f is Anosov*
- *f is (C^1-robustly) conjugated to a linear Anosov automorphism*

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Pointwise partial hyperbolicity!

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Try to obtain a similar picture as in dimension 2.

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(2) Given an isotopy class and a PH diffeo in the isotopy class, can we say something about its dynamics?

- Progress in (1): Bonatti-Wilkinson. Brin-Burago-Ivanov. Hammerlindl.
- Progress in (2): Hertz-Hertz-Ures. Hammerlindl-Ures (Conservative case only).

Definitions

Definition

We say that $f : M^3 \to M^3$ is *strongly partially hyperbolic* (SPH) if $TM = E^s \oplus E^c \oplus E^u$ a Df-invariant splitting such that $\exists N > 0$ and for every $x \in M^3$:

$$\|Df^N|_{E^s(x)}\| < \|Df^N|_{E^c(x)}\| < \|Df^N|_{E^u(x)}\|$$

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The last definition is NOT ADDAPTED to Diaz-Pujals-Ures result and it seems unsuitable to treat robust dynamical behaviour.

Classification Conjecture

Conjecture (Pujals)

$f : M^3 \to M^3$ a transitive SPH diffeomorphism. Then (modulo finte covers), f is leaf conjugate to one of the following models:

- *A linear Anosov in $\mathbb{T}^3$ with 3 real different eigenvalues.*
- *A skew product over an Anosov on $\mathbb{T}^2$ ($M = \mathbb{T}^3$ or nilmanifold).*
- *The time one map of an Anosov flow ($M =$???).*

Related conjectures by Hertz-Hertz-Ures, mainly concerning *dynamical coherence* (to be defined...)

Transitivity is necessary hypothesis (or at least dynamical coherence) due to example of Hertz-Hertz-Ures.

Progress by: Bonatti-Wilkinson, Brin-Burago-Ivanov, Hammerlindl...

Coherence and leaf conjugacy

Definition

A SPH diffeo f is *dynamically coherent* if there exists f-invariant foliations $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$ tangent to $E^s \oplus E^c$ and $E^c \oplus E^u$ respectively.

It implies that there exists $\mathcal{F}^c$ tangent to E^c (by intersecting).
There always exists $\mathcal{F}^s$ and $\mathcal{F}^u$ f-invariant foliations tangent to E^s and E^u called strong manifolds.

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Definition

Two SPH diffeos $f, g : M \to M$ are *leaf conjugate* if $\exists\ h : M \to M$ homeomorphism which sends leaves of $\mathcal{F}^c_f$ to leaves of $\mathcal{F}^c_g$ and verifies that:

$$h(\mathcal{F}^c_f(f(x))) = \mathcal{F}^c_g(g(h(x)))$$

Main Result

Theorem (joint with A. Hammerlindl)

Let $f : M \to M$ be a SPH such that either:

- $M = \mathbb{T}^3$ *and* f *has no attracting nor repelling torus, or*
- M *is a non-toral nilmanifold*

Then, f *is leaf conjugate to its linear part (which is an algebraic SPH diffeo). In particular,* f *is dynamically coherent.*

Previous Results

- Brin-Burago-Ivanov proved that if $f : M \to M$ is SPH and $\pi_1(M)$ is abelian, then $f_* : H_1(M) \to H_1(M)$ has eigenvalues larger and smaller than 1. (In particular, no SPH on S^3 nor $S^2 \times S^1$).

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These results are based on a remarkable result by Burago and Ivanov showing that there always exist Reebless foliations transverse to E^s and to E^u (possibly not-invariant).

The absolute case

- For ASPH, Dynamical coherence always holds in $\mathbb{T}^3$ (Brin-Burago-Ivanov) and Nilmanifolds (Parwani and Hammerlindl).
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- There is a recent example of Hertz-Hertz-Ures of a non-dynamically coherent SPH in $\mathbb{T}^3$.
- I proved that if there are no attracting or repelling torus in $\mathbb{T}^3$ then f is dynamically coherent.

What is left?

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- In nilmanifolds we still have to prove dynamical coherence.
- We must put ourselves in the hypothesis used by Hammerlindl to get leaf conjugacy.

Strategy of the proof

When the linear part has one eigenvalue of modulus 1 or M is a nilmanifold. (The Anosov case uses different arguments).

(1) Burago-Ivanov provide f-invariant branching foliations $\mathcal{F}^{cs}_{bran}$ and $\mathcal{F}^{cu}_{bran}$.

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(7) Finally, it also gives that if two points in the universal cover have iterates at bounded distance, they belong to the same center-leaf. This implies that all center leaves are circles.

Thanks!