Robust dynamics, invariant structures and topological classification

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An ilustrative example

M closed surface, $f : M \to M$ a diffeomorphism.

Theorem (Franks-Mañé)

f is robustly transitive if and only if it is Anosov.

f is *transitive* if it has a dense orbit (i.e. $\exists x \in M$ s.t. $\{f^n(x)\}_{n\geq 0}$ is dense in M).

f is *robustly transitive* if it has a C^1-neighborhood $\mathcal{U}$ s.t. $\forall g \in \mathcal{U}$ is transitive.

f is *Anosov* (or *hyperbolic*) if $TM = E^s \oplus E^u$ continuous Df-invariant splitting so that vectors in E^s are contracted and vectors in E^u are expanded.

An ilustrative example

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Theorem (Franks-Mañé)

f is robustly transitive if and only if it is Anosov.

Franks '70: $f : M \to M$ Anosov implies that $M = \mathbb{T}^2$ and f is conjugate to a linear hyperbolic automorphism. (Anosov is an open property, and that linear automorphisms are transitive, so this gives the converse.)

Mañé '78: $f : M \to M$ is robustly transitive then it is Anosov.

A proposal

Robust dynamical properties $\Longrightarrow$ Invariant geometric structures

Topological classification

Higher dimensions

M closed manifold and $f : M \to M$ diffeomorphism.

Theorem (Bonatti-Diaz-Pujals-Ures)

$f : M \to M$ robustly transitive $\Rightarrow$ f admits some weak form of hyperbolicity.

The converse is **false.** (But the result is sharp.)

Similar results in related context of *stable ergodicity*.

Ultimate goal

M closed manifold and $f : M \to M$ diffeomorphism.

Goal

Describe robustly transitive diffeomorphisms.

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Proposal

Attempt a topological classification of systems admitting the geometric structures imposed by robust transitivity (dominated splittings, partial hyperbolicity, etc...).

Note: This is largely insufficient! Even assuming topological classification, the characterisation is wide open. (But the problem is also interesting by itself :))

Potential byproduct: Understanding of finer dynamical properties.

Easiest possible context

M is a closed 3-manifold.

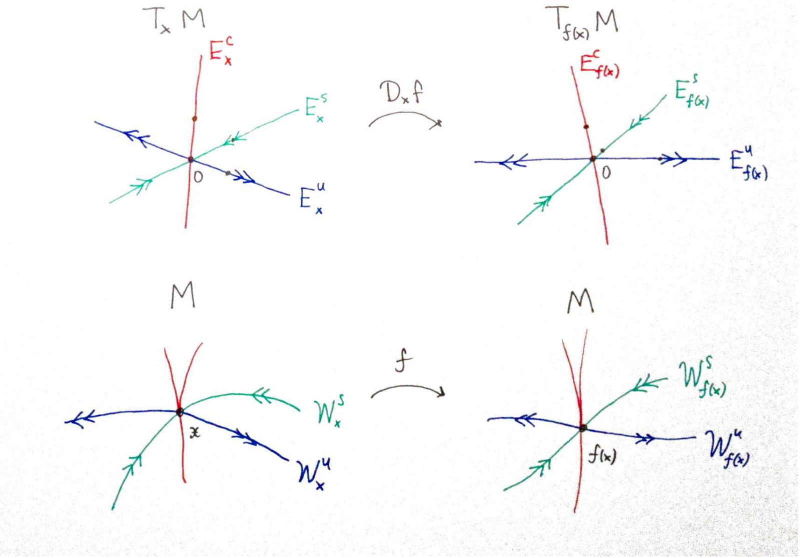
$f : M \to M$ diffeomorphism is *partially hyperbolic* (PH) if:

- $TM = E^s \oplus E^c \oplus E^u$ one-dimensional continuous and Df-invariant bundles.
- There is $n > 0$ so that for unit vectors $v^\sigma \in E^\sigma(x)$ one has:

$$\|Df^n v^s\| < \min\{1, \|Df^n v^c\|\}, \quad \text{and}$$

$$\|Df^n v^u\| > \max\{1, \|Df^n v^c\|\}.$$

Partial hyperbolicity



Examples

Known examples of PH diffeos in dimension 3.

1. Algebraic and geometric constructions (linear automorphisms of tori and nilmanifolds + some time one maps of Anosov flows).
2. Skew-products.
3. Surgery constructions (mostly for Anosov flows, whose time one maps are PH).
4. Deformations (both small and large, recently introduced h-transversalities).

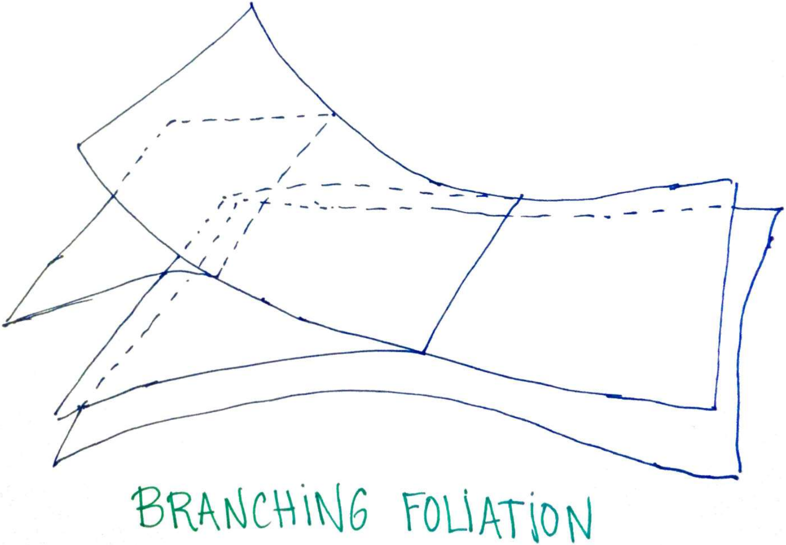
Question

Other ways to construct examples?

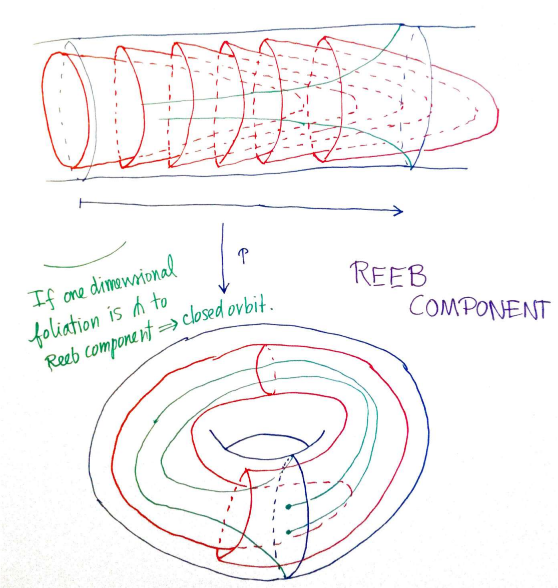
Why is there some hope?

- Reasonably good understanding of topology of 3-manifolds, more so in relation with foliations.
- There are 1D-foliations $\mathcal{W}^s$ and $\mathcal{W}^u$ tangent to E^s and E^u without closed orbits.
- There are 2D-*branching foliations* tangent to $E^{cs} = E^s \oplus E^c$ and $E^{cu} = E^c \oplus E^u$ (Burago-Ivanov).
- These branching foliations are approximated by *Reebless foliations* which have good structure (Novikov theorem, etc...).

Branching foliations



Reeb components



Difficulties

- No clear proposal for classification (recently several new examples started to appear, new ones may arise).
- In general E^{cs} and E^{cu} are not integrable to foliations (branching foliations are useful, but harder to work with).
- Anosov flows in 3-manifolds are far from being classified. Same with Reebless foliations.

The problem of classification

Divided in three main subproblems:

1. **Integrability of E^c:** Does E^c integrate into an f-invariant foliation? (*Dynamical Coherence*)
2. **Topological obstructions:** Which manifolds and which homotopy classes admit PH diffeos.
3. **Classification modulo E^c:** Describe the dynamics of the center 'leafs'. (*Leaf conjugacy*)

It turns out that all problems are related and in general, the attack is simultaneous.

The approach so far

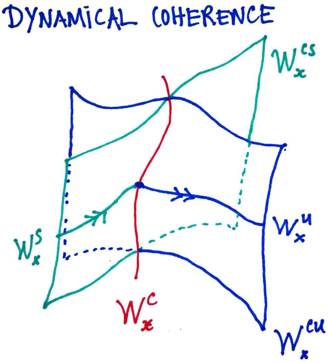
- Consider manifolds and isotopy classes separately, by increasing the complexity.
- Understand the structure of (branching) foliations, the possible dynamics of the foliations depending on the isotopy class of the diffeomorphism.
- Try to compare to model examples.

In the process one sometimes finds topological obstructions, or establishes integrability of the bundles.

What is classification?

Definition

$f : M \to M$ partially hyperbolic is *dynamically coherent* (DC) $E^{cs} = E^s \oplus E^c$ and $E^{cu} = E^c \oplus E^u$ integrate into f-invariant foliations $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ ($\Rightarrow E^c$ integrates into an f-invariant foliation $\mathcal{W}^c = \mathcal{W}^{cs} \cap \mathcal{W}^{cu}$).



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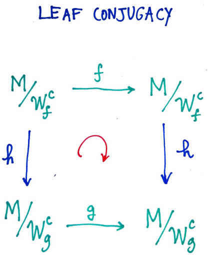
Definition

$f, g : M \to M$ dynamically coherent partially hyperbolic diffeomorphisms are *leaf conjugate* if there exists $h : M \to M$ homeomorphism sending $\mathcal{W}^c_f$ to $\mathcal{W}^c_g$ and conjugating the dynamics of center leaves (i.e. $h \circ f(\mathcal{W}^c_f(x)) = h(\mathcal{W}^c_f(f(x))) = \mathcal{W}^c_g(g \circ h(x)) = g(\mathcal{W}^c_g(h(x)))$.



$$\begin{tikzcd} M/\mathcal{W}^c_f \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\circlearrowright"] & M/\mathcal{W}^c_f \arrow[d, "h"] \\ M/\mathcal{W}^c_g \arrow[r, "g"] & M/\mathcal{W}^c_g \end{tikzcd}$$

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LEAF CONJUGACY

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Question

What should classification be in the non-DC context?

Simplest manifolds-torus bundles over the circle

When $\pi_1(M)$ is (virtually) solvable, one can reduce to study torus bundles over S^1.

Theorem (j.w. A.Hammerlindl)

Let $f : M \to M$ partially hyperbolic and $\pi_1(M)$ is (virtually) solvable. Then, either there is a torus tangent to E^{cs} or E^{cu} or f is dynamically coherent and up to finite lift and cover, either:

- *f is leaf conjugate to a linear hyperbolic automorphism of $\mathbb{T}^3$, or,*
- *f is a skew-product over an Anosov in $\mathbb{T}^2$, or,*
- *f is leaf conjugate to time one map of suspension Anosov flow.*

All these options are *algebraic*.

We can precisely classify the possible quotients and the examples having a torus tangent to either E^{cs} or E^{cu}.

Key ingredients

- Possible to get precise classification of (branching) foliations.
- Depending on isotopy class of f (almost) unique algebraic model.
- Foliations are related with model algebraic examples.
- Coarse dynamics forces infinite separation of leaves (this allows to show dynamical coherence).

(And of course, previous work by many other people, e.g. Brin-Burago-Ivanov, Bonatti-Wilkinson, Hertz-Hertz-Ures, Parwani,.....)

Larger manifolds

When M has fundamental group not virtually solvable ($\Rightarrow$ with exponential growth) one looses all those ingredients.

Possible questions:

Question

If M admits a partially hyperbolic diffeomorphism, does it admit an Anosov flow?

Question

Is it possible to compare a partially hyperbolic diffeomorphism in such a manifold to an Anosov flow in some way?

Seifert manifolds

Up to finite cover are circle bundles over higher genus surfaces. In this setting, the case of Anosov flow was well understood (Ghys, Barbot...), and Reebles foliations too (Thurston-Levitt-Britenhamm....)

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- (j.w. A. Hammerlindl, M. Shannon) If M admits (transitive) partially hyperbolic diffeomorphism then it admits an Anosov flow.
- (j.w. C. Bonatti, A. Gogolev, A. Hammerlindl) Several isotopy classes admit partially hyperbolic representatives. Moreover, some of them are not dynamically coherent. Also some obstructions.
- (j.w. T. Barthelmé, S. Fenley, S. Frankel) If f is homotopic to identity then f is dynamically coherent and (up to iterate) leaf conjugate to Anosov flow (always algebraic).

Hyperbolic 3-manifolds

$M = \mathbb{H}^3/_\Gamma$ where $\Gamma < \mathrm{Isom}(\mathbb{H}^3)$. 'Largest' class of 3-mfds (after Thurston-Perelman, etc..). Mostow theorem implies f homotopic to identity after iterate.

Theorem (j.w. T. Barthelmé, S. Fenley, S. Frankel)

Let $f : M \to M$ partially hyperbolic on M hyperbolic. Then, either f is dynamically coherent and leaf conjugate to (topological) Anosov flow, or f is not dynamically coherent and a double translation.

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A double translation is a potential example (we don't have examples, but we know they should look very similar to the non-dynamically coherent examples in Seifert manifolds).
Follows from a very detailed study of classification of partially hyperbolic diffeomorphisms homotopic to identity in general 3-manifolds, but also uses specific properties of foliations on hyperbolic 3-manifolds such as transverse pseudo-Anosov flows.

Dynamical consequences.

Theorem (j.w. T. Barthelmé, S. Fenley, S. Frankel)

Let $f : M \to M$ be a partially hyperbolic diffeomorphism of a hyperbolic 3-manifold M. Then f has no contractible periodic points.

More recently, we used the ideas developed for this classification and some new ones to get:

Theorem (j.w. S. Fenley)

Let $f : M \to M$ be a conservative C^{1+} partially hyperbolic diffeomorphism on a hyperbolic 3-manifold. Then, f is ergodic. (In fact, a K-system.)

Speculation

Question

Does the topological classification help to understand finer dynamical properties of partially hyperbolic dynamics?

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Does the topological classification help to understand finer dynamical properties of partially hyperbolic dynamics?

Question

Can it help understand the boundary of robustly transitive diffeomorphisms?

Thanks!