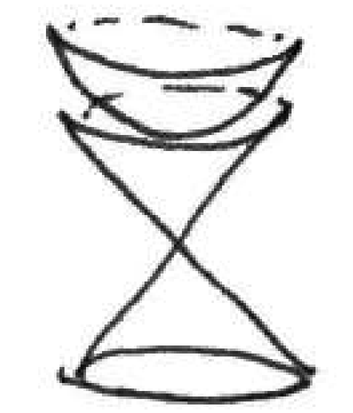
ENTROPY RIGIDITY FOR SURFACE GROUP REPRESENTATIONS (j.w. A. SAMBARINO)

Σ surface of genus ≥ 2 and $\Gamma = \pi_1(\Sigma)$ its fundamental group.

$$\mathrm{Teich}(\Sigma) = \{\text{metrics of curvature } -1\}/_{\mathrm{Diff}_0(\Sigma)} \cong \mathrm{Hom}_{\underset{\uparrow}{FD}}(\Gamma, \mathrm{Isom}(\mathbb{H}^2))/_{\mathrm{Isom}(\mathbb{H}^2)} \leftarrow \text{by conjugation}$$

faithful & discrete

one has that $\mathrm{Isom}(\mathbb{H}^2) \longrightarrow PSL(2,\mathbb{R})$ (upper half model)
$\hookrightarrow SO^+(1,2)$ (Klein model)



Question: ¿Can one deform $\rho : \Gamma \to SO^+(1,2) \hookrightarrow SL(3,\mathbb{R})$ outside $\mathrm{Teich}(\Sigma)$?

Answer: Yes: Goldman-Choi, Benzecri, Benoist, Hitchin...

More generally: if $i : PSL(2,\mathbb{R}) \hookrightarrow PSL(d,\mathbb{R})$ irreducible repr. (unique up to conjugacy. When $d=3$ it is $SO^+(1,2) \hookrightarrow SL(3,\mathbb{R})$)

One calls $\rho : \Gamma \to PSL(d,\mathbb{R})$ Fuchsian if it factors as

$$\Gamma \xrightarrow{\rho_F} PSL(2,\mathbb{R}) \overset{i}{\hookrightarrow} PSL(d,\mathbb{R}) \quad (\text{up to conj}) \quad \text{with } \rho_F \in \mathrm{Teich}(\Sigma)$$

Thm (Hitchin 80's) The connected components of Fuchsian representations in $\mathrm{Hom}(\Gamma, PSL(d,\mathbb{R}))/_{PSL(d,\mathbb{R})}$ is homeomorphic to a ball of dimension $(2g-2)\underbrace{\dim PSL(d,\mathbb{R})}_{d^2-1}$

One denotes $\mathcal{H}_d(\Sigma)$ this component (the Hitchin component), much larger than $\mathrm{Teich}(\Sigma)$.

Recent years lot of work, e.g: Labourie, Fock-Goncharov, Guichard-Wienhard, ... as a higher rank analogue of Teichmuller space ("Higher teichmuller theory")

Thm (Labourie) $\forall \rho \in \mathcal{H}_d(\Sigma)$ one has ρ is faithful (inyective) & discrete ($\rho(\Gamma)$ discrete in $PSL(d,\mathbb{R})$) and moreover, $\rho(\gamma)$ is (uniformly) loxodromic $\forall \gamma \in \Gamma$.

loxodromic $\to$ simple spectrum

(uniformly) $\to$ gaps proportional to transl. length of $[\gamma] \in [\Gamma]$

2 natural questions (in study of discrete subgroups of Lie groups in general)

1) Stability: when is discreteness a "robust" property

2) Rigidity: How to detect that ρ factors as $\Gamma \hookrightarrow H \to G$ (equivalent, when is the Zariski closure of $\rho(\Gamma)$ smaller than G). That is, how to detect "genuine" deformations.

For Hitchin representations, Labourie's thm adresses the first question completely via the use of "Anosov representations" (related with dominated splittings & stability conjecture - j.w. J. Bochi & A. Sambarino)

We will adress the second problem. (Notice $\mathrm{Teich}(\Sigma) \simeq \mathcal{H}_2(\Sigma) \subset \mathcal{H}_d(\Sigma)$ has smaller dimensions, Guichard classified all possible Zariski closures)

§ CRITICAL EXPONENT

$$h_\rho = \limsup_T \frac{1}{T} \log \#\{\gamma \in \Gamma : d_X(o, \rho(\gamma)\cdot o) \le T\}$$

$\sim \log \|\rho(\gamma)\|$

symm. space.

One can normalize so that $h_\rho = 1$ for ρ Fuchsian (for Fuchsians always equal by Gauss-Bonnet thm...)

$$\boxed{\underline{\text{Thm}} \text{ (j.w. A. Sambarino) } h_\rho \le 1 \text{ and equality holds} \Leftrightarrow \rho \text{ is Fuchsian}} \leftarrow \rho \in \mathcal{H}_d(\Sigma)$$

I will give some ingredients of our proof when $d=3$ (For $d=3$ this was first shown by Crampon with different techniques).

Previous related results: → Bowen, Patterson-Sullivan, Besson-Courtois-Gallot, Bourdon, Yue, ... rank 1

→ Bishop-Steger, Burger, Crampon, ... higher rank.

§ PROOF FOR $SL(3,\mathbb{R})$:

$\mathfrak{a} = \{(a_1, a_2, a_3) : a_1 + a_2 + a_3 = 0\}$

$\mathfrak{a}^+ = \{(a_1, a_2, a_3) \in \mathfrak{a} : a_1 \ge a_2 \ge a_3\}$ ← Weyl chamber (boundaries - walls)

For $g \in SL(3,\mathbb{R})$

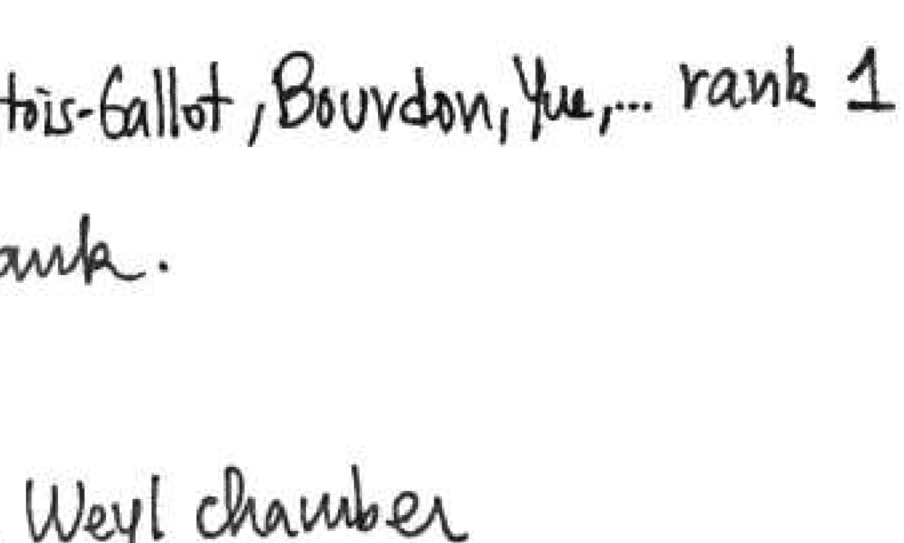
$\rightarrow \mu(g) = \underbrace{(\mu_1(g), \mu_2(g), \mu_3(g))}^{\text{log of singular values}} \in \mathfrak{a}^+$ Cartan projection ← $\|\mu(g)\| \sim d_X(o, g\cdot o)$

$\rightarrow \lambda(g) = (\lambda_1(g), \lambda_2(g), \lambda_3(g)) \in \mathfrak{a}^+$ Jordan projection

log of eigenvalues (not Lyap. exp)

The idea is to "count $\underline{\text{eigenvalues}}$ in different directions"

or sing values which is "easier" to believe is related with $d_X(o, g\cdot o)$



One defines $\mathcal{L}_\rho = \overline{\mathbb{R}\lambda(\rho(\Gamma))}$ limit cone $\subseteq \mathfrak{a}^+$ (Benoist)

By Labourie's result $\mathcal{L}_\rho$ is $\underline{\text{far from the walls}}$ of $\mathfrak{a}^+$ $\forall \rho \in \mathcal{H}_3(\Sigma)$

(we shall come back to this)

For $\varphi \in \mathfrak{a}^*$ which is positive in $\mathcal{L}_\rho$ one has

$$\exists\, h_\rho^\varphi = \lim_T \frac{1}{T} \log \#\{\underbrace{[\gamma] \in [\Gamma]}_{\text{conj. classes}} : \varphi(\lambda(\rho(\gamma))) \leq T\} \text{ and } h_\rho^\varphi \in (0,\infty)$$

(Entropy of certain reparametrization of Geodesic flow on $T^1\Sigma$ - Ledrappier)

One looks at $\mathcal{D}_\rho = \{\varphi \in \mathfrak{a}^* \text{ such that } h_\rho^\varphi \leq 1\}$ and it is a convex set whose boundary $\partial\mathcal{D}_\rho$ is a codimension 1 <u>analytic</u> submanifold of $\mathfrak{a}^*$

(Sambarino - Using thermodynamic formalism & properties of pressure function)

The importance is seen in:

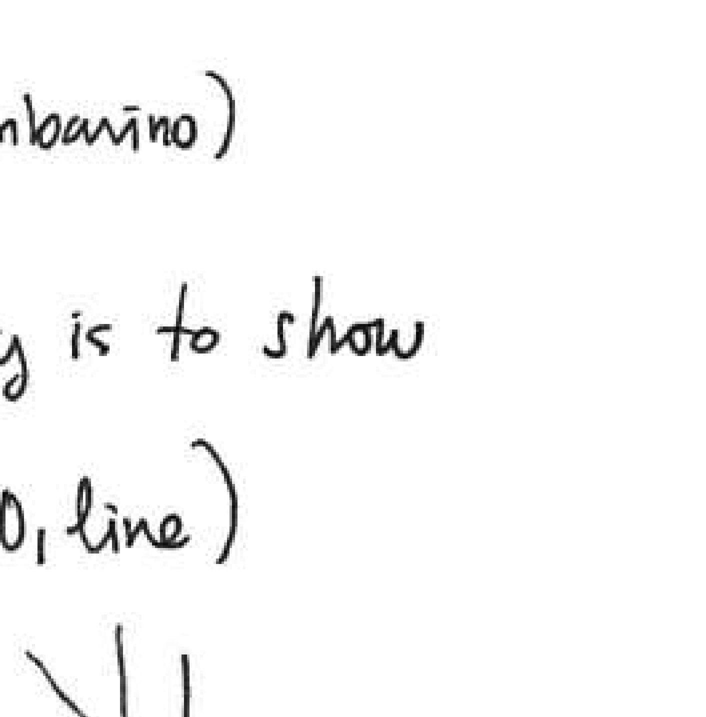
<u>Thm</u> (Quint/Sambarino) $h_\rho = d_{\mathfrak{a}^*}(0, \partial\mathcal{D}_\rho)$

Quint for Cartan proj. Sambarino for eigenvalues (Obviously after appropriate choice of $d_{\mathfrak{a}^*}$ dep. on normalizations...)

We will also use (though in $d=3$ is not necessary) that for $\varphi \in \mathfrak{a}^*$ one has that $\rho \mapsto h_\rho^\varphi$ is analytic (Bridgeman, Canary, Labourie, Sambarino)

The point is to estimate $d(0,\partial\mathcal{D}_\rho)$ and for this, the strategy is to show that $\partial\mathcal{D}_\rho$ ALWAYS contains certain points $\Rightarrow h_\rho \leq d(0,\text{line})$

By analiticity, equality will imply $\partial\mathcal{D}_\rho = \text{line} \Rightarrow$ algebraic relation between eigenvalues ($\Rightarrow$ Smaller Zariski closure by Benoist)



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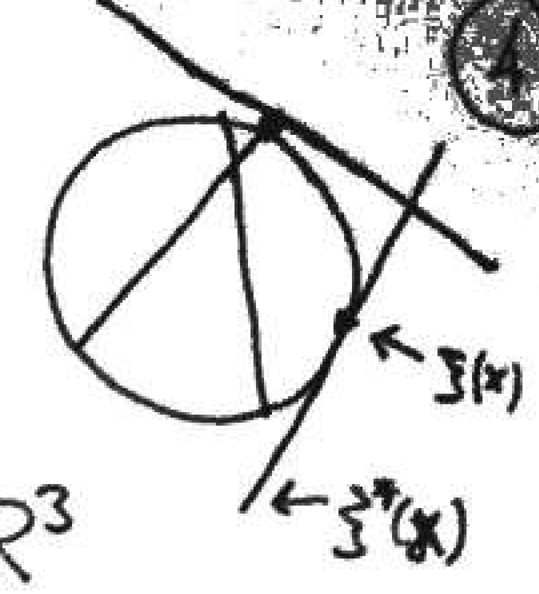
The rest of the talk I'll try to give some ideas on how can one "block" the entropy of certain linear forms in $\mathfrak{a}^*$.

<u>Main idea</u>: Construct a ($C^{1+\alpha}$) Anosov flow for which these linear forms are the stable & unstable Jacobians $\rightsquigarrow$ SRB theorem implies that the entropy of the corresponding reparametrizations (h_ρ^φ) equals 1 (the Anosov flow is 3-dimensional) $\Rightarrow$ belong to $\partial\mathcal{D}_\rho$.

For $\rho:\Gamma \to SO(1,2) \hookrightarrow SL(3,\mathbb{R})$ Fuchsian $\rightsquigarrow$ Klein model



there exist equivariant maps:

$\xi: \partial\Gamma \to \mathbb{P}(\mathbb{R}^3)$

$\xi^*: \partial\Gamma \to \mathbb{P}((\mathbb{R}^3)^*)$

such that if $x \neq y \Rightarrow \xi(x) \oplus \underbrace{\xi^*(y)}_{\substack{\wr \\ \mathrm{Ker}\,\xi^*(y)}} = \mathbb{R}^3$

These correspond to the bundles of a DOMINATED SPLITTING for a linear cocycle over the geodesic flow on $T^1\Sigma = (\partial\Gamma)^{(2)} \times \mathbb{R}/_{\Gamma}$ "so":

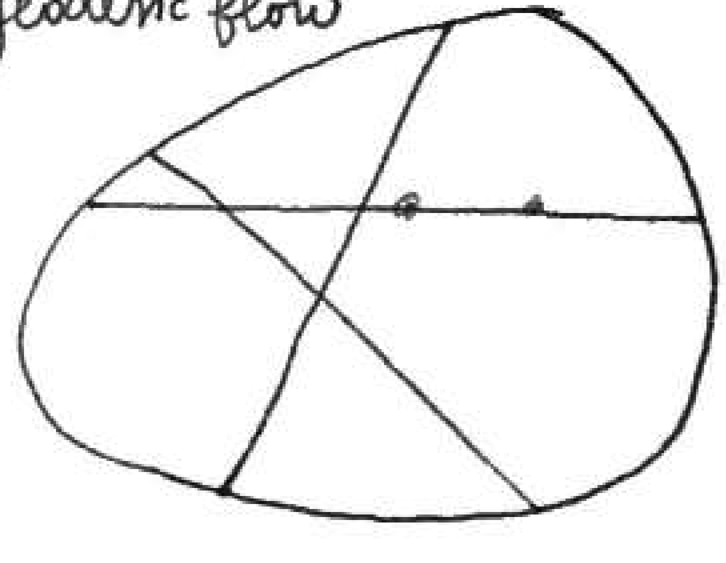
$\rightsquigarrow$ these maps exist after perturbation & continue to be injective (and verify $\xi(x) \oplus \xi^*(y) = \mathbb{R}^3$ for $x \neq y$)

$\rightsquigarrow$ their trace is a C^1 curve (with Hölder derivatives),

and using duality bounds a strictly convex set (Benoist, Convexes Divisibles I. In fact, the argument is other way around; the convex set is strictly convex $\Rightarrow C^1$)

Remarks: $\to$ If $d=3$ this holds in all the component & the Anosov flow is smooth. For $d>3$ we can only do something like this in nbhd of Fuchsian locus and the flow is only $C^{1+\alpha}$ (technical core of the proof ...)

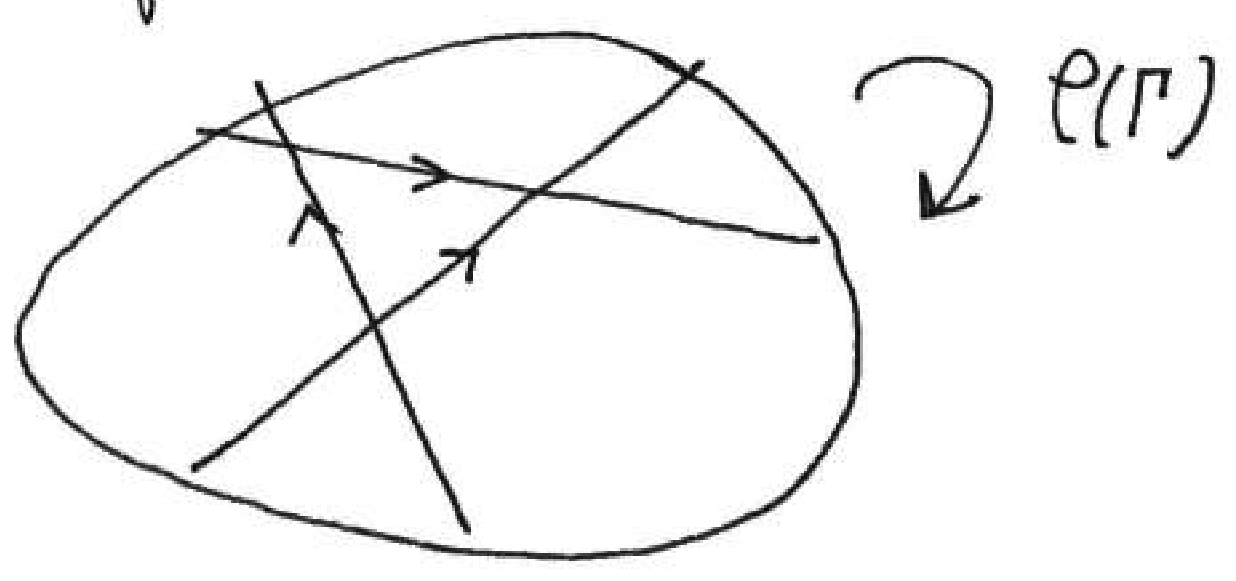
Hilbert metric (log cross ratio) is Finsler with Anosov geodesic flow



$\to$ Crampon uses instead measure of maximal entropy, calculates Lyapunov exponents & uses Ruelle's inequality & Pesin-Ledrappier-Young for rigidity.

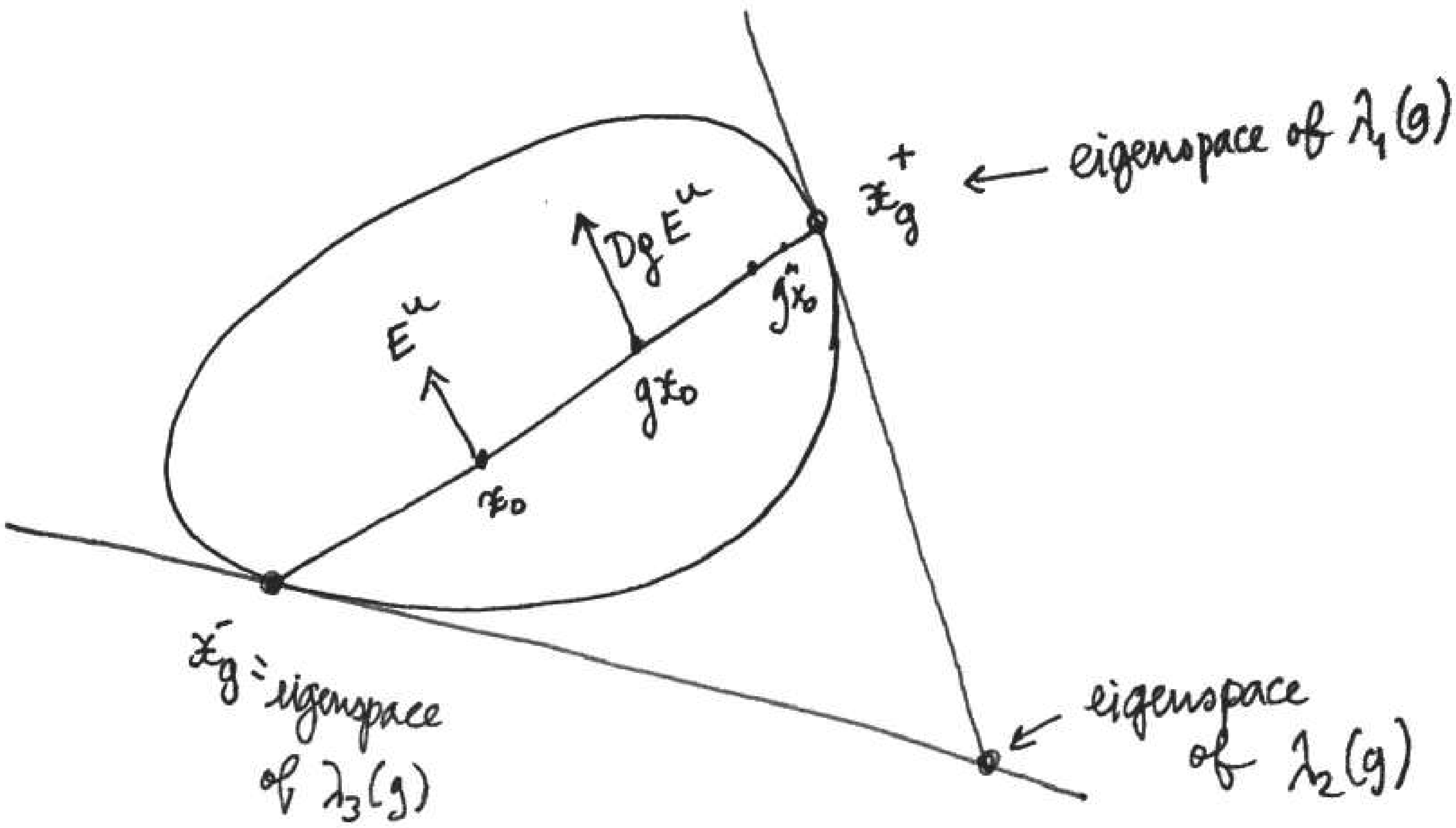
$\to$ Notice that this type of arguments give that eigenvalues are far from walls in nbhd of Fuchsian (difficult part in Labourie is show that works in all component)

The Anosov flow is the quotient of



We want to calculate unstable (resp. stable) Jacobian (via Livsic, enough in periodic orbits)

Let $\gamma \in \Gamma$ and $g = \rho(\gamma)$



One computes $J^u_g = \lambda_1(g) - \lambda_2(g)$

So, via SRB thm the form $\varphi_{12}(a_1, a_2, a_3) = a_1 - a_2$

verifies $h_\rho^{\varphi_{12}} = 1$ $\forall \rho$ in nbhd of Fuchsian.

$\Downarrow$

analiticity $\Rightarrow \forall \rho \in \mathcal{H}_3(\Sigma)$