Some dynamics in hyperbolic 3-manifolds

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based on joint work with T. Barthelmé, S. Fenley and S. Frankel.

Sample results

Partially hyperbolic diffeomorphisms in hyperbolic 3-manifolds:

- Only exist if the manifold admits an Anosov flow (*topological obstruction*).
- Are mixing with respect to volume whenever they are conservative (*dynamical consequence*).

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Goal

Given a structure on a dynamical system, impose topological restrictions to admit it. Deduce dynamical behaviour from it.

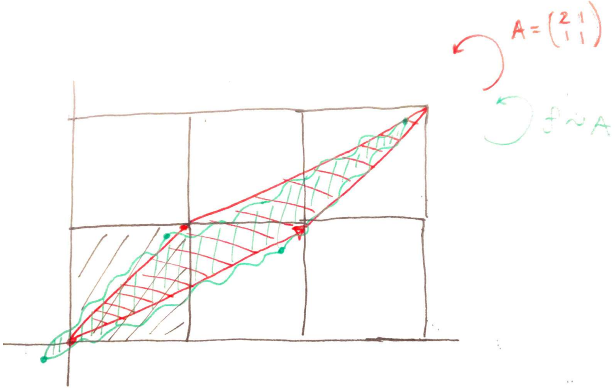
Tool

Understand the coarse behaviour imposed by the structure plus the topology of the manifold and the map.

Illustrative example I: Franks semiconjugacy

Proposition

Let $f : \mathbb{T}^d \to \mathbb{T}^d$ a homeomorphism homotopic to a linear hyperbolic matrix $A \in SL_n(\mathbb{Z})$. Then, there exists $h : \mathbb{T}^d \to \mathbb{T}^d$ continuous and homotopic to identity such that $h \circ f = A \circ h$.



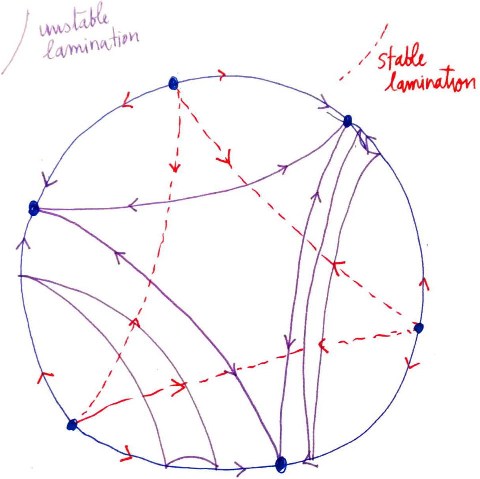
Illustrative example II: Handel/Thurston/Nielsen semiconjugacy

Let $f : S \to S$ be a homeomorphism of a surface of genus $g \geq 2$. We say f is *pseudo-Anosov* (pA) if it does not preserve any free homotopy class of simple curves in S.

Proposition (Lifts of pA maps)

Every lift $\tilde{f} : \tilde{S} \to \tilde{S}$ where $\tilde{S} \cong \mathbb{H}^2$ extends to $\partial \mathbb{H}^2$ with finitely many fixed points which are alternatingly attracting and repelling. Moreover, if there are more than 2 fixed points in $\partial \mathbb{H}^2$ there is a compact $\tilde{f}$-invariant set in $\mathbb{H}^2$ (of negative index).

Lift of pseudo-Anosov map



Hyperbolic 3-manifolds

$$\mathbb{H}^3 = \{(x,y,z) \in \mathbb{R}^3 \ : \ z > 0\} \qquad ds^2 = \frac{dx^2 + dy^2 + dz^2}{z^2}.$$

Has constant negative curvature. M closed manifold is hyperbolic if $M = \mathbb{H}^3/_\Gamma$ where $\Gamma < \mathrm{Isom}(\mathbb{H}^3)$.

Theorem

There exist hyperbolic 3-manifolds.

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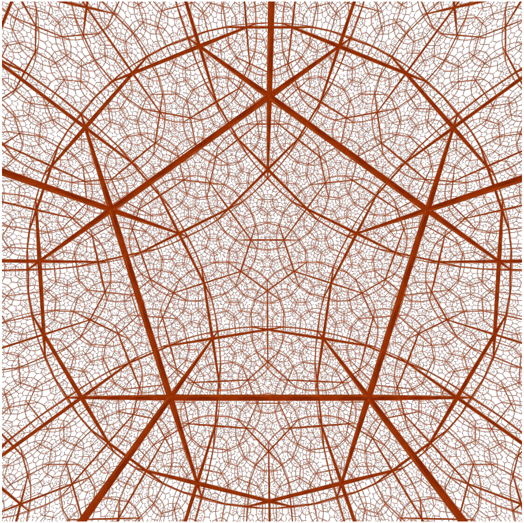
Theorem (Perelman)

Aspherical and homotopically atoroidal implies hyperbolic. (i.e. $\pi_n(M) = 0$ for $n = 2, 3$ and no $\mathbb{Z}^2$ in $\pi_1(M)$).

Hyperbolic manifolds



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Consequence

Every homeomorphism of a closed hyperbolic 3-manifold has an iterate homotopic to identity.

As in surfaces, dynamics homotopic to identity are the hardest to analyse, as there is no structure a priori.

We noticed that much can be said if one knows that the map preserves some additional structure.

Foliation preserving maps

Let $\mathcal{F}$ be a *taut* foliation on a closed 3-manifold M.
(taut= exists a closed transversal to the foliation through each point "=" exists a metric on M making each leaf a minimal surface)
Let $f : M \to M$ be a homeomorphism which preserves the foliation $\mathcal{F}$.

Goal

Understand dynamical consequences of preserving the foliation.

Upshot, when dealing with partially hyperbolic diffeomorphisms (whatever they are), one gets some f-invariant taut foliations[1].

[1] Technically, one gets an uglier object, but good enough to perform most of what will be discussed today.

Good (commuting) pairs

Taut foliation look 'nice' in the universal cover: One gets $\widetilde{\mathcal{F}}$ on $\tilde{M}$ where a lift $\tilde{f}$ and $\pi_1(M)$ (deck transformations) act preserving the foliation.

The quotient $\mathcal{L} = \tilde{M}/_{\widetilde{\mathcal{F}}}$ is called the *leaf space*. It is a one-dimensional simply connected manifold (possibly non-Hausdorff).

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Definition

A *good pair* is a pair $(\tilde{f}, \gamma)$ where $\tilde{f}$ is a lift of f to $\widetilde{M}$ and $\gamma \in \pi_1(M)$ such that they commute ($\gamma\tilde{f} = \tilde{f}\gamma$).

We will concentrate on the case where the leaf space $\mathcal{L}$ is homeomorphic to $\mathbb{R}$. Notice that since $f \sim \mathrm{id}$ there is a lift $\tilde{f}$ such that $(\tilde{f}, \gamma)$ is a good pair for every $\gamma \in \pi_1(M)$.

$\mathbb{R}$-covered foliations and pseudo-Anosov flows

The idea is to get coarse dynamical information of the action of $\tilde{f}$ using the transverse geometry of $\mathcal{F}$.

Theorem (Thurston, Calegari, Fenley)

Let $\mathcal{F}$ be an $\mathbb{R}$-covered foliation on a hyperbolic 3-manifold. Then, there exists a pseudo-Anosov flow Φ_t transverse and regulating to $\mathcal{F}$.

If γ is associated to a periodic orbit of Φ and $\tilde{f}$ does not fix any leaf of $\widetilde{\mathcal{F}}$ then the dynamics of the good pair $(\tilde{f}, \gamma)$ is *coarsely* the same as the one of the periodic orbit.

Coarse dynamics

Basic result:

Proposition

Let f be a homeomorphism of a hyperbolic 3-manifold preserving an $\mathbb{R}$-covered foliation $\mathcal{F}$ and let Φ be the transverse pseudo-Anosov flow. Assume that $\tilde{f}$ is a lift commuting with deck transformations and $\gamma \in \pi_1(M)$ associated to periodic orbit of Φ. Then, there is a closed $(\tilde{f}, \gamma)$-invariant set T_γ which intersects every leaf of $\widetilde{\mathcal{F}}$ in a compact set.

The set T_γ is 'what is left' from the periodic orbit.

Application to partially hyperbolic dynamics

$f : M \to M$ is *partially hyperbolic* if $TM = E^s \oplus E^c \oplus E^u$ continuous Df-invariant. E^s is contracted, and E^u is expanded.
We say f is *dynamically coherent* if $\exists \mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ invariant foliations tangent to $E^s \oplus E^c$ and $E^c \oplus E^u$.

Proposition

If f is dynamically coherent partially hyperbolic in hyperbolic 3-manifold M then up to iterate f has a lift that leafwise fixes every leaf of $\widetilde{\mathcal{W}^{cs}}$.

(This is an important step in the classification of partially hyperbolic diffeomorphisms in hyperbolic 3-manifolds.)

Idea of the proof

1. Let $\tilde{f}$ the lift which commutes with deck transformation (up to iterate so $f \sim \mathrm{id}$).
2. Assume that $\tilde{f}$ fixes one leaf of $\widetilde{\mathcal{W}^{cs}}$, then it fixes a closed subset of leaves whose complement is invariant under $\tilde{f}$ and deck transformations.
3. Show that if $\tilde{f}$ fixes one leaf of $\widetilde{\mathcal{W}^{cs}}$ then the foliation is minimal. (Uses hyperbolic M.)
4. If does not fix any leaf, then leaf space is $\mathbb{R}$ and $\tilde{f}$ acts as translation. We can then apply proposition to get T_γ.
5. Locally, in T_γ every point is *expanding* due to dynamics of E^u, impossible.

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