Adaptive Weak Approximation of Diffusions with Jumps

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Contents

1. Adaptive

- Task: Estimate an unknown parameter θ.
- By adaptive we mean an algorithm that

  automatically produces an approximation $\widehat{\theta}$ such that

$$|\theta - \widehat{\theta}| \leq \mathit{TOL}$$

- *TOL* is a maximum tolerance given in advance.

Adaptive weak approximation

By weak approximation we mean that

$$\theta = \mathbf{E}\left[g\left(X(1)\right)\right]$$

where

- $\{X(t) \colon 0 \leq t \leq 1\}$ is a stochastic process in $\mathbb{R}^d$
- $g \colon \mathbb{R}^d \to \mathbb{R}$ is a regular function

Adaptive weak approximation of diffusion with jumps

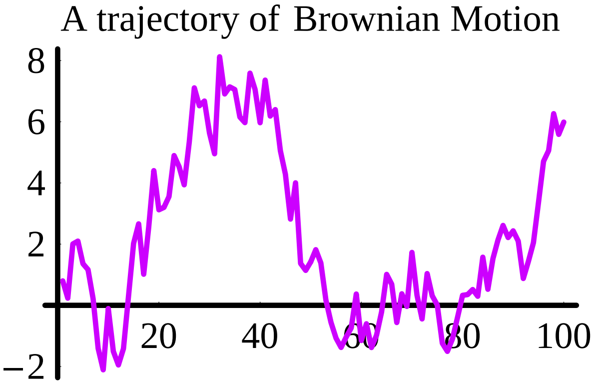
$\{X(t)\colon 0 \le t \le 1\}$ is the solution of a stochastic differential equation with jumps (SDEJ) i.e.

$$dX(t) = a(t, X(t))dt + b(t, X(t))dW(t) + \int_{\mathbf{Z}} c(t, X(t^-), z)p(dt, dz)$$

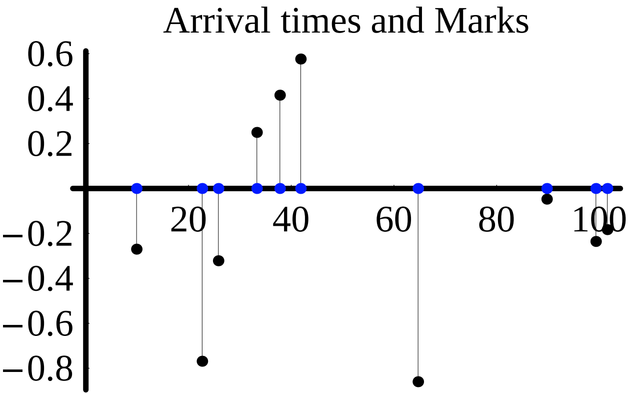
driven by

- $\{W(t)\colon 0 \le t \le 1\}$: standard brownian motion in $\mathbb{R}^{\ell_0}$
- $p(dt, dz)$: Poisson random measure on the (auxiliary) mark space $\mathbf{Z}$
- Initial condition is $X(0) = x_0$ constant (for simplicity)

Sources of randomness: Brownian motion



Sources of randomness: Marked Poisson Process



- Interarrival exponential times, with expectation 10.
- Normal marks: $N(-0.1, 0.5)$
- Brownian motion and Marked process are independent

2. SDEJ in detail

$$
\begin{aligned}
X(t) = x_0 &+ \int_0^t a(s, X(s))ds && \text{(L)} \\
&+ \int_0^t b(s, X(s))dW(s) && \text{(I)} \\
&+ \int_0^t \int_{\mathbf{Z}} c(s, X(s^-), z)p(ds, dz) && \text{(RM)}
\end{aligned}
$$

- (L) is an usual Lebesgue integral
- (I) is an Itô integral
- (RM) is an integral against a random measure, the simplest integral:

$$\int_0^t \int_{\mathbf{Z}} c(s, X(s-), z) p(ds, dz) = \sum_{k=1}^{\nu(t)} c\left(\tau_k, X(\tau_k^-), Z_k\right)$$

Here:

- $\nu(t)$ is the number of arrivals up to t (Poisson process)
- τ_k are the arrival times (sums of exponentials)
- $X(\tau_k^-)$ value of the process just before the k-th jump
- Z_k are the marks (normal variables in the picture)

One possible motivation: Finance

Compute the price of a call option written on a basket of d assets $(S_1, S_2, \dots, S_d)$

- Each asset follow a diffusion with jumps (i.e. affine processes)
- Usually they display correlations (modelled by $b(t, x)$)
- $g(x) = (x - K)^+$
- A prescribed precision is required.

Problem: Compute $E\left[g(X(T)\right]$ where

$$X(T) = \pi_1 S_1(T) + \cdots + \pi_d S_d(T).$$

Our framework and technical assumptions

- $a\colon [0,1]\times\mathbb{R}^d\to\mathbb{R}^d$,
- $b\colon [0,1]\times\mathbb{R}^d\to\mathbb{R}^{d\times\ell_0}$, and
- $c\colon [0,1]\times\mathbb{R}^d\times\mathbf{Z}\to\mathbb{R}^d$
- All derivatives up to order 8 bounded
- $q(dt,dz)=\lambda(t)\mu(t,dz)dt$ is the compensator (time dependent)
- $g(x)$ derivatives up to 8 with polynomial growth
- Curse of dimensinality[2]: Monte Carlo methods are efficient for high dimensions.

[2] coined by Richard Bellman

3. Main result. Basic algorithm: Jump-augmented Euler

- Give a deterministic partition $0 = \hat{t}_0 < \hat{t}_1 < \ldots < \hat{t}_N = 1$
- Sample ν jumps τ_k and marks Z_k,
- Construct jump-augmented partition $\{t_k\}_{k=0}^{N+\nu} = \{\hat{t}_k\} \cup \{\tau_k\}$
- Set $\overline{X}(0) = x_0$
- For $n = 1$ to $N + \nu$

$$\overline{X}_{n+1}^{-} = \overline{X}_n + a(t_n, \overline{X}_n)\Delta t_n + b(t_n, \overline{X}_n)\Delta W,$$

$$\overline{X}_{n+1} = \overline{X}_{n+1}^{-} + c(t_{n+1}, \overline{X}_{n+1}^{-}, Z_{n+1})\ 1_{\{t_{n+1} \text{is a jump time}\}}$$

- In this way we construct our approximation $\overline{X}$.

Adaptive algorithm: Error Splitting

Our estimation is

$$\widehat{\theta} = \frac{1}{M}\sum_{j=1}^{M} g(\overline{X}(T;\omega_j))$$

and we split the error:

$$\mathcal{E} = E[g(X(T))] - \frac{1}{M}\sum_{j=1}^{M} g(\overline{X}(T;\omega_j)) = \mathcal{E}_T + \mathcal{E}_S$$

where

$$\mathcal{E}_T = E[g(X(T))] - E[g(\overline{X}(T))]$$

$$\mathcal{E}_S = E[g(\overline{X}(T))] - \frac{1}{M}\sum_{j=1}^{M} g(\overline{X}(T;\omega_j))$$

Main Result

Theorem: Error expansion in computable a posteriori form.

$$\mathcal{E}_T = E\left[\sum_{n=0}^{N+\nu-1} \tilde{\rho}(t_n, \overline{X})(\Delta t_n)^2\right] + \mathcal{O}\left((\Delta t_{\max})^2\right)$$

$$\begin{aligned}\tilde{\rho}(t_n, \overline{X}) \equiv \frac{1}{2}\Big(& \big(\frac{\partial}{\partial t} a_k + \partial_j a_k a_j + \partial_{ij} a_k d_{ij}\big)\varphi_k(t_{n+1^-}) \\ & + \big(\frac{\partial}{\partial t} d_{km} + \partial_j d_{km} a_j + \partial_{ij} d_{km} d_{ij} + 2\partial_j a_k d_{jm}\big)\varphi'_{km}(t_{n+1^-}) \\ & + \big(2\partial_j d_{km} d_{jr}\big)\varphi''_{kmr}(t_{n+1^-})\Big),\end{aligned}$$

Here

- We use Einstein conventio for summation
- $\partial_\alpha a \equiv \partial_\alpha a(t_n, \overline{X}(t_n))$,
- $\varphi, \varphi', \varphi''$ are the duals, constructed in a $d + d^2 + d^3 + d^4$ dimensional backwards auxiliary algorithm (expensive).

Main References

- **KP** P. E. Kloeden und E. Platen. *The Numerical Solution of Stochastic Differential Equations*, Springer (1992)
- **TT** D. Talay and L. Tubaro (Stoc. An. Appl. 1990), provide a priori error estimate:

$$E[g(X_T) - g(\overline{X}_T)] \simeq \int_0^T E[\Delta t(s)\Psi(X_s, s)]ds = \mathcal{O}(\Delta t_{max}).$$

- **LL** X. Q. Liu and C. W. Li (SINUM, 2000) analyze the weak order of several different schemes
- **STZ** A. Szepessy, R. Tempone and G. Zouraris (Comm. Pure Appl. Math., 2001) provide a posteriori error estimate for diffusions
- **MSTZ** *Adaptive weak approximation of diffusions with jumps* SINUM (2008)

4. Adaptive Algorithm: Tolerance splitting

Our total work is $N \times M$. The precision achieved is

- By the Euler method: $\mathcal{E}_T \sim 1/N$
- By the CLT: $\mathcal{E}_S \sim 1/\sqrt{M}$

Then we solve

$$\text{Minimize:} \qquad N \times M$$

$$\text{subject to :} \qquad 1/N + 1/\sqrt{M} = TOL$$

obtaining

$$TOL_T = \frac{1}{3} TOL, \quad TOL_S = \frac{2}{3} TOL.$$

Algorithm

Our algorithm has the following structure:

1. First: M_T runs to determine the mesh according to TOL_T
2. Second: M runs to construct $\widehat{\theta}$ according to TOL_S

Observe:

- First runs are in dimension $d + d^2 + d^3 + d^4$, M_T "small"
- Second runs is $d-$dimensional, allows larger M

First step of AA: construct the mesh

1. Input M_T (realizations to construct the mesh)
2. Input N (to construct initial uniform mesh)
3. Sample ν jumps and insert them in the deterministic mesh
4. **For** $n = 1$ to $N + \nu$ estimate the local error

$$r_n = \frac{1}{M_T} \sum_{j=1}^{M_T} \tilde{\rho}(t_n, \overline{X}(\omega_j))(\Delta t_n)^2$$

   - **If** $r_n > TOL_T/(N + \nu)$ divide Δt into two equal subintervals
   - **Else** continue ,

5. **End For**
6. **If** global statistical error of $\{r_n\}$ is large, enlarge M_T and go to 4, **Else** End

Second step of AA: Estimation and Statistical error control

We have the mesh and proceed to estimation. By the CLT:

$$\mathbf{E}_S = \frac{1.65\bar{\sigma}}{\sqrt{M}}$$

where $\bar{\sigma}^2$ is the empirical variance

1. Input M (realizations to estimate θ)
2. Input the deterministic non-uniform mesh and
3. Produce M trajectories with the basic algorithm
4. **If** the statistical error $\mathbf{E}_S < TOL_s$, **End**
5. **Else** enlarge M and go to 3

Obs: $M_T << M$ (as the error estimation is more expensive)

5. Example A: Time variation and many jumps

- Test function: $g(x) = x$, initial value: $x_0 = 1$.
- Drift: $a(t,x) = a(t)x$ where the time-varying drift is

$$a(t) = \begin{cases} 0, & \text{if } t < 1/3, \\ \frac{1}{2\sqrt{t-1/3+TOL^4}}, & \text{if } 1/3 \leq t \leq 1. \end{cases}$$

- Diffusion $b(t,x) = x/\sqrt{2}$
- Jump measure compensator:

$$q(dt, dz) = \lambda(t)\mu(dz)dt,$$

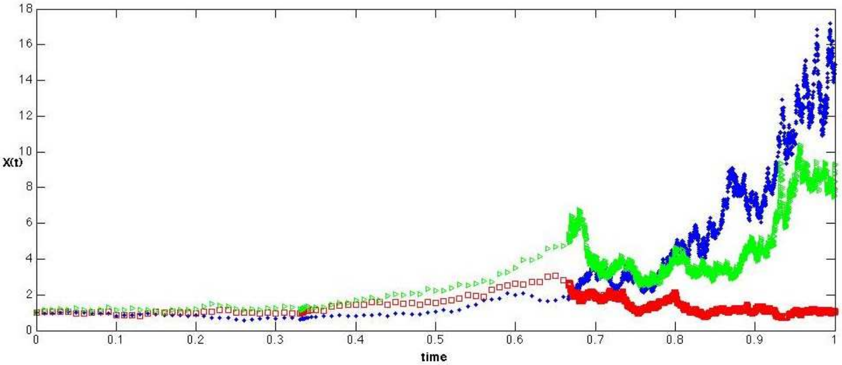
Where:

- $\mu(dz)$ is Uniform distribution in $[-\sqrt{3}, \sqrt{3}]$,
- Jump intensity exhibits two different regimes

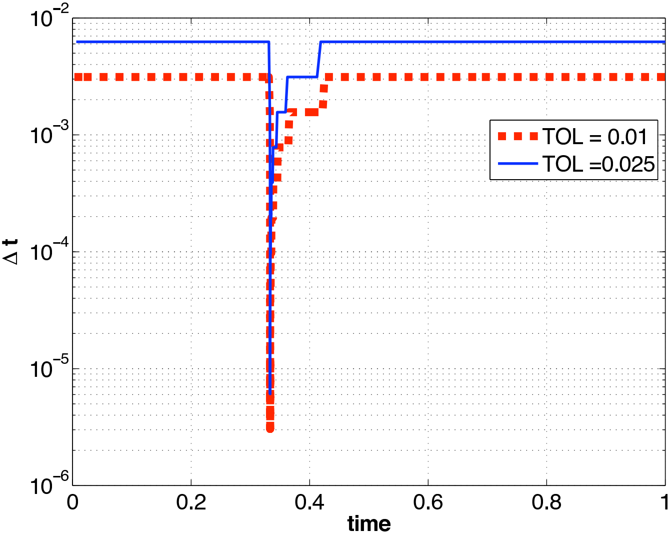
$$\lambda(t) = \begin{cases} 0, & \text{if } 0 \leq t \leq 2/3, \\ 3N_J, & \text{if } 2/3 < t \leq 1, \end{cases}$$

with (in the mean) $N_J = 1024$ jumps per realization.

Simulated trajectories of our example



Mesh construction of our example (jumps not included)



Numerical results

- We have an exact solution:

$$E[g(X(1))] = \exp\left(\sqrt{2/3 + TOL^4} - TOL^2\right).$$

	step 1	grid	step 2	error	uniform
TOL	M_T	N	M	$\mathcal{E}$	N_U
0.025	1.7×10^3	220	2.0×10^5	2.0×10^{-2}	2×10^4
0.01	1.7×10^3	450	1.45×10^6	-1.4×10^{-2}	6×10^4

Table: Adaptive choice of M and Δt

6. Example B: Dimension $d = 2$

Consider $X = (X_1, X_2)$, $\ell_0 = 1$ (i.e. $W = W_1$), and coefficients

- $a(t,x) = \left(-x_2, x_1 + \frac{1}{2}\lambda(t)x_2\right)$
- $b(t,x) = \left(\sqrt{\frac{\lambda(t)}{1+t}}\sin x_1, 0\right)$
- $c(t,x,z) = \left(0, z\frac{\cos x_1}{\sqrt{1+t}} - x_2,\right)$
- $\lambda(t) = (1+t)^{-1}$ is also the jumps intensity.
- Marks are time dependent: Take U_k i.i.d $U[-1/2, 1/2]$ and

$$Z(\tau_k) = \cos(2\pi\tau_k) + 2\sqrt{3}\sin(2\pi\tau_k)U_k.$$

- $g(x_1, x_2) = x_1^2 + x_2^2 = \|x\|^2$
- $x(0) = 0$.
- The example has a closed solution:

$$E\|X(1)\|^2 = \|X(0)\|^2 + \int_0^1 \frac{\lambda(t)}{1+t} dt = \frac{1}{2}.$$

This example is adapted from [LL] where the intensity λ is constant, and the jumps are also constant (i.e. we have a Poisson process), and originated in [TT], where is presented without jumps.

Numerical results for Example B

Iter.	N	M	$\mathcal{E}$	$\mathbf{E}_S$
1	5	100	-2.66×10^{-2}	1.44×10^{-1}
2	10	100	-9.19×10^{-3}	1.35×10^{-1}
3	20	100	9.16×10^{-2}	1.30×10^{-1}
4	20	1000	-1.78×10^{-2}	4.54×10^{-2}
5	20	10000	-1.77×10^{-2}	1.50×10^{-2}
6	20	14088	-1.29×10^{-2}	1.23×10^{-2}

- $TOL = 0.02$, then $TOL_S = 1.33 \times 10^{-2}$
- Start with $N = 5$ and $M = 100$
- Overkilling runs where performed to estimate the accuracy of $\mathcal{E}_T$.

7. Discussion

- Trajectory dependent mesh division (useful for irregularity in space). Dividing criteria: If

$$\tilde{\rho}(t_n, \overline{X}(\omega_j))(\Delta t_n)^2 > TOL_T/N$$

  then divide the interval for this trajectory.

- Use Large deviations theory instead of the Central Limit Theorem

$$P(|\mathcal{E}_S| \geq c) \leq 2\exp(-H(c)M).$$

- Other jump models:
  - state dependent intensity $\lambda = \lambda(t, X)$ (default risk problems)
  - infinite activity models (Lévy processes)