7. Statistical Fitting with linear models.

MA6622, Ernesto Mordecki, CityU, HK, 2006.

References for Lecture 5:

Time Series: Theory and Methods. P.J. Brockwell and R. A. Davis Springer Verlag (1991)

Statistical Methods for Stochastic Models in Finance and Actuarial Science (MA5622)
http://math.cityu.edu.hk/∼macucker/ma5622/index_5622.html

Main Purpose: determine whether we can model the returns of a certain financial time series

$$X(0), X(1), \ldots, X(n)$$

through a linear time series model.

We begin with the Auto Regresive models (AR), continue with Moving Average Models (MA), discuss the (ARMA) model, in the stationary case, and finally consider the integrated ARMA or the ARIMA model in the case of non-stationarity.

The idea is to review the main procedures in the simplest cases. In practice, the statistical work is done with the help of statistical software.

7a. Fitting AR(1) time series

An Auto Regresive time series of order $p = 1$, or AR(1), is a stochastic process that satisfies the equation

$$X(t) - \phi X(t-1) = \varepsilon(t), \qquad t = 1, 2, \ldots$$

where

- $|\phi| < 1$
- $\{\varepsilon(t)\}$ is a strict white noise with variance σ_ε^2.

The correlogram of an AR(1) process is

$$\begin{cases} \rho(0) = 1 \\ \rho(h) = \phi^{|h|} \quad \text{when } h \neq 0 \end{cases}$$

Observe that the behaviour of the correlation strongly differs according to whether $-1 < \phi < 0$ or $0 < \phi < 1$.

The case $\phi = 0$ corresponds to the strict white noise process.

In order to determine whether it is plausible to model a given time series with an AR(1), the first thing to do is visual analysis to see if the empirical correlogram of the data resambles the theoretical model.

After this, the estimators of the parameter ϕ and the variance σ_ε^2 are simply computed by the formulas

$$\bar{\phi} = \frac{\sum_{k=1}^{n} X(t)X(t-1)}{\sum_{k=1}^{n} X(t)^2}$$

$$\bar{\sigma_\varepsilon}^2 = \bar{\sigma}^2\left(1 - \bar{\phi}^2\right)$$

where $\bar{\sigma}^2 = \overline{\mathbf{cov}}(0)$ is the estimator of the variance sample.

To fit an AR(1) model one sholud perform the following main steps.

STEP 1. Plot the correlogram and compare with the corresponding theoretical autocorrelation function.

STEP 2. Estimate the parameters ϕ and σ_ε^2

STEP 3. Compute the residuals

$$\bar{\varepsilon}(1) = X(1) - \bar{\phi}X(0), \ldots, \bar{\varepsilon}(n) = X(n) - \bar{\phi}X(n-1)$$

STEP 4. Test whether the residuals conform a strict white noise sequence with visual analysis and/or the Portmanteau test.

To construct confidence intervals for our estimated correlations we apply the KEY Theorem. Under the hypothesis of an AR(1) the series in Bartlett's formula can be summed to give the variance of

$$W_{ii} = \frac{(1-\phi^{2i})(1+\phi^2)}{1-\phi^2} - 2i\phi^{2i}$$
$$\sim \frac{1+\phi^2}{1-\phi^2} \quad \text{for big values of } i$$

In particular

$$W_{11} = 1, \quad W_{22} = (1+\phi^2)^2 - 4\phi^4, \ldots$$

make possible to construct individual confidence intervals for $\bar{\rho}(1)$ and $\bar{\rho}(2)$ and so on, based on the estimated value of ϕ.

Caution: One must be aware that, due to presence of correlation in the estimations of the correlations, this test is only to reject the AR(1) hyphotesis.

The following situations should be taken into account:

- If the estimation $\bar{\phi} \sim 0$, and and the different previous steps give affirmative answer, one should check directly for the possibility of the original series to be a white noise.

- If the absolute value of our estimated parameter $|\bar{\phi}| \sim 1$, then it is possible that we are trying to fit the wrong model, as this latter fact suggests that the initial data follow a random walk. A further differentiation of the returns should be tried:

$$Y(1) = X(1) - X(0), \ldots, Y(n) = X(1) - X(0).$$

  If you fit an ARMA(p,q) model to $\{Y(t)\}$ then it is said that the original data $\{X(t)\}$ follows an Integrated ARMA(p,q), or an ARIMA(p,q) model.

- If your estimation of $|\bar{\phi}|$ is significatively greater than 1, then the model definitively does not fit the data.

7b. Fitting MA(1) time series

A Moving Average time series of order $q = 1$, or MA(1), is a stochastic process that satisfies the equation

$$X(t) = \varepsilon(t) - \theta\varepsilon(t-1), \qquad t = 1, 2, \ldots$$

where

- θ is a real valued constant,
- $\{\varepsilon(t)\}$ is a strict white noise with variance σ_ε^2.

The value $\theta = 0$ gives a white noise model, furthermore, there is no restriction in the value of θ in order to have a stationary process (but $|\theta| < 1$ is necessary for the process to be invertible).

The autocorrelation of an MA(1) process is

$$\rho(h) = \begin{cases} 1 & \text{when } h = 0, \\ \frac{-\theta}{1+\theta^2} & \text{when } h = 1, \\ 0 & \text{when } h \geq 2. \end{cases}$$

giving a simple way to find out whether it is feasible to fit a MA(1) model to the data.

Confidence intervals for $\bar{\rho}(h)$ can be constructed for $h \geq 2$. Intervals are centered and estimated variances, according to Bartlett's formula, are:

$$\bar{W}_{hh} = 1 + 2\bar{\rho}(1)^2,$$

confidence intervals (with 95% probability) have the form

$$(-1.96\sqrt{\bar{W}_{hh}/n},\ 1.96\sqrt{\bar{W}_{hh}/n}).$$

The estimated parameter $\bar{\theta}$ solve the equation

$$\bar{\rho}(1)\bar{\theta}^2 + \bar{\theta} + \bar{\rho}(1) = 0$$

that has two solutions, and we take the one with $|\bar{\theta}| < 1$ for the process to be invertible. The estimation of the variance of the white noise is

$$\bar{\sigma_\varepsilon}^2 = \bar{\rho}(0)/\left(1 + \bar{\theta}^2\right).$$

The steps in order to fit a MA(1) model are similar, with a main difference:

- In AR models one first estimates the parameters, and then construct the confidence intervals for the estimated correlogram,
- In MA models, one begins by visual and statistical analysis of the correlograms, as the construction of the confidence intervals for estimated correlations with $h > 1$ does not require the knowledge of the parameters.

7c. Comments on AR(p) models

The equation of an AR(p) process is

$$X(t) - \phi_1 X(t-1) - \cdots - \phi_p X(t-p) = \varepsilon(t),$$

where $\{\varepsilon(t)\}$ is a strict white noise, with variance σ_ε^2, and the complex valued polynomial

$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p$$

should have its p roots outside the unit circle. Observe that for $p = 1$ the polynomial reduces to

$$\phi(z) = 1 - \phi z$$

the root is $1/\phi$ that should have absolute value bigger that one, i.e. $|\phi| < 1$.

The autocorrelation function is a linear combination of damped exponential and damped sines, so there is no a priori visual analysis to perform.

On the other side, the estimation of the parametrs is simple, as parameters can be estimated as solution of the Yule-Walker equations

$$
\begin{array}{rclcccc}
\rho(1) & = & \phi_1 & +\phi_2\rho(1) & + & \ldots & +\phi_p\rho(p-1) \\
\rho(2) & = & \phi_1\rho(1) & +\phi_2 & + & \ldots & +\phi_p\rho(p-2) \\
\vdots & = & \vdots & + \quad \vdots & + & \ldots & + \quad \vdots \\
\rho(p) & = & \phi_1\rho(p-1) & +\ldots & + & \ldots & + \quad \phi_p
\end{array}
$$

that is proved to have solutions, with estimated $\bar{\rho}(i)$ in place of $\rho(i)$. The variance of the white noise, can be estimated by the equation

$$\bar{\sigma_\varepsilon}^2 = \bar{\sigma}^2\big(1 - \bar{\phi_1}\rho(\bar{1}) - \cdots - \bar{\phi_p}\rho(\bar{p})\big).$$

7d. Comments on MA(q) models

The equations of an MA(q) process is

$$X(t) = \varepsilon(t) - \theta_1 \varepsilon(t-1) - \cdots - \theta_q \varepsilon(t-p),$$

where $\{\varepsilon(t)\}$ is a strict white noise, with variance σ_ε^2.

The process is stationary for all values of the parameters $\theta_1, \ldots, \theta_q$, but the complex valued polynomial

$$\theta(z) = 1 - \theta_1 z - \theta_2 z^2 - \cdots - \theta_p z^p$$

should have its q roots outside the unit circle in order to be invertible.

The autocorrelation function is simple, as it vanishes for values larger than q, more precisely

$$\rho(h) = \begin{cases} \frac{-\theta_h + \theta_1 \theta_{h+1} + \cdots + \theta_{q-h} \theta_q}{1 + \theta_1^2 + \cdots + \theta_q^2} & \text{when } h = 1, \ldots, q \\ 0 & \text{when } h > q \end{cases}$$

This give both the possibility of visual fitting, and furthermore, provides equal confidence intervals (with 95% probability) for $\bar{\rho}(h)$ with $h > q$ of the form:

$$(-1.96\sqrt{\bar{W}_{hh}/n}, -1.96\sqrt{\bar{W}_{hh}/n}).$$

where Bartlett's formula gives the estimated variances

$$\bar{W}_{hh} = 1 + 2\big(\bar{\rho}(1)^2 + \cdots + \bar{\rho}(q)\big)$$

On the other side, the estimation of the parametrs is not direct, and done through algorithmic methods.

7e. Final Comments on ARMA(p,q) models

An ARMA(p,q) process satisfies the equation

$$X(t) - \phi_1 X(t-1) - \cdots - \phi_p X(t-p)$$
$$= \varepsilon(t) - \theta_1 \varepsilon(t-1) - \cdots - \theta_q \varepsilon(t-p),$$

where $\{\varepsilon(t)\}$ is strict with noise and the parameters satisfy simultaneously the conditions for AR(p) and MA(q) models.

The combination of AR and MA produces an autocorrelation function that has no simple form, and algorithmic estimation procedures for the parameters, implemented in statistical software.

One final remark:

All procedures explained assumed that p and q are known. In fact, to find the best order for fitting an ARMA model, one can use the Akaike criteria, that consists in choosing the model that minimizes the statistic

$$AIC(p,q) = \log \bar{\sigma_\varepsilon}^2 + 2(p+q)/T.$$

Statistical software performs this order selection in a systematic way.

7f. Nonstationarity and ARIMA models

In the following situations, one should try to differentiate the data, constructing

$$Y(1) = X(1) - X(0), \dots, Y(n) = X(1) - X(0).$$

- When visually some non-stationarity is detected in the time series of the returns.
- When roots with modulus near one are find in the corresponding polynomials of the ARMA process. (For instance, when $|\phi| \sim 1$ in an AR(1) process.)
- When in the correlogram the first estimated correlations do not decay rapidly. This phenomena is found in the autocorrelation

of a random process.

If it is possible to fit the differentiated process $\{Y(t)\}$ with and ARMA(p,q) model, we say that the process $\{X(t)\}$ follows and ARIMA process. In some exceptional cases, it may be necessary to differentiate twice, i.e. to differentiate the process $\{Y(t)\}$.