Mini-course Numerical Methods for SDEs, May 2006. Course Instructors: Ernesto Mordecki and Raúl Tempone.

Homework exercises for the course.

Método de aprobación del Curso

Se espera que un 25 por ciento de los ejercicios (a elegir por los estudiantes) sean entregados hasta el miércoles 17 de mayo. La entrega final es el 1^o de julio de 2006.

1 Ejercicios correspondientes al 26 y 28 de abril

1. Prove the following identities using limits of the Forward Euler method:

$$\int_0^T t dW(t) = TW(T) - \int_0^T W(t)dt$$

(integration by parts)

$$\int_0^T W(t)dW(t) = \frac{W(T)^2}{2} - \frac{T}{2}$$

2. The Ornstein-Uhlenbeck process is defined by

$$X(t) = x_\infty + e^{-at}\big(x_0 - x_\infty\big) + b\int_0^t e^{-a(t-s)}dW(s).$$

Compute the expected value and the variance of $X(t)$, and the limits of these functions as $t \to \infty$. Interpret the results.

3.[Trayectorias del Proceso de Wiener] El siguiente ejercicio muestra que las trayectorias de un proceso de Wiener presentan un comportamiento diferente al de las funciones diferenciables: su variación en un intervalo $[0, T]$ no existe, es infinita; y su variación cuadrática en un intervalo $[0, T]$ es igual a T.

Consideremos para cada N la partición diádica $\{t_0^N = 0, t_1^N = 1/2^N, \dots, t_{2^N}^N = 1\}$. Demostrar que se verifica:

$$V_N = \sum_{k=1}^{2^N} \big|W_{t_k^N} - W_{t_{k-1}^N}\big| \to \infty \quad (N \to \infty) \quad c.s.$$

Sugerencias: (i) Demostrar que $V_N(\omega) \leq V_{N+1}(\omega)$ y concluir que existe el límite (casi seguro) de $\{V_N\}$. (ii) Estudiar el límite en probabilidad de V_N y utilizar la unicidad del límite en probabilidad.

Demostrar que se verifica:

$$Q_N = \sum_{k=1}^{2^N} \big(W_{t_k^N} - W_{t_{k-1}^N}\big)^2 \to T \quad (N \to \infty)$$

en media cuadrática, es decir demostrar que

$$\mathbf{E}\big(Q_N - T\big)^2 \to 0 \quad (N \to \infty) \tag{1}$$

Sugerencia: considerar las variables $Y_k = \big(W_{t_k^N} - W_{t_{k-1}^N}\big)^2 - (t_k^N - t_{k-1}^N)$, observar que $Q_N - T$ es la suma de las Y_k, y calcular el segundo momento en (1).

2 Solve the exercises below from the lecture notes [GMS+06]

1. Ejercicios 3.6, 3.8, 3.11, 3.12 3.13, 3.15, 3.17, (corresponden a la clase del 5 de mayo).

2. Ejercicios 4.2, 4.6, 4.10 (corresponden a la clase del 8 de mayo)

3. Ejercicios 5.3, 5.11, 5.12, 5.13 (corresponden a la clase del 12 de mayo)

3 After May 12, solve the following exercises:

Exercise 1 The geometric Brownian motion S, solves

$$\begin{aligned} dS(t) =& rS(t)dt + \sigma S(t)dW(t), \\ S(0) =& S_0, \end{aligned} \tag{2}$$

where in the context of option pricing r and σ are constants representing the short interest rate and the volatility. Show that

$$S(T) = \exp(rT - \sigma^2/2T + \sigma W(T))\ S_0 \tag{3}$$

and compute an approximate of the option value, cf. [BS73],

$$\Pi \equiv e^{-rT} E[g(S(T))],$$

by the Monte Carlo Method for the following cases:

- $g(x) = \max(x - 100, 0)$,
- $g(x) = \begin{cases} 1, & \text{if } 50 < x < 100, \\ 0 & \text{otherwise}, \end{cases}$
- $g(x) = \sqrt{\max(x - 100, 0)}$.

Here we use $r = 0.04$, $\sigma = 0.4$, $T = \frac{1}{4}$ and $S_0 = K = 100$. You may use the Matlab program *mc1d.m* for this purpose. Determine the required values with a precise accuracy.

□

Exercise 2 Solve Exercise 1 by instead approximating the corresponding Black-Scholes PDE. You may use the Matlab Finite Difference program *fd1d.m*. Determine the values with a precise accuracy and compare the new results with the Monte Carlo results.

□

Exercise 2.5 Consider the deterministic differential equation

$$dZ(t) = a(Z(t)), \qquad Z(0) = x_0, \qquad 0 \le t \le T,$$

and a perturbation of it, the Ito stochastic differential equation

$$dX(t) = a(X(t)) + b dW_t, \qquad X(0) = x_0, \qquad 0 \le t \le T,$$

and $b > 0$ a positive constant. The aim of this excercise is to compare the solution of both equations. Define then the difference

$$e(t) = X(t) - Z(t).$$

(a) Consider $a(x) = ax$ (linear case) and compute $E(e(t))$, and $var(e(t))$. Hint: Use Ito's formula when necessary.
(b) Assume now that $|a(x) - a(y)| \le C_a|x - y|$ with a positive constant C_a. Find bounds for the expectation $E(|e(t)|)$ and the variance $var(e(t))$. Discuss the asymptotic behaviour as $b \to 0$.
(c) Implement a simulation scheme with the uniform time step forward Euler discretization of the above equations taking $a(x) = \cos(x)$, $b = 0.1$ and $T = 2\pi$. Plot the estimation of $var(e(t))$, and compare it with the bound obtained in part (b). Use different number of time steps in your simulations: $N = 10, 20, 40$.
Exercise 3 The following stochastic volatility model [FPS00]generalizes the well known Black-Scholes geometric Brownian motion model [BS73], improving some aspects of option pricing. A simplified version of the model reads

$$\begin{aligned} dS(t) =& rS(t)dt + e^{Y(t)} S(t)\; dW(t), \\ dY(t) =& \left(-\alpha(1 + Y(t)) + 0.4\sqrt{\alpha}\sqrt{1-\rho^2}\right) dt + 0.4\sqrt{\alpha}\; d\hat{Z}(t), \end{aligned}$$

where W and Z are independent Wiener process and

$$\hat{Z}(t) \equiv \rho W(t) + \sqrt{1-\rho^2} Z(t).$$

Here the correlation coefficient is $\rho = -0.3$. Compare the following explicit method

$$\begin{aligned} S_{n+1} - S_n =& rS_n\Delta t + e^{Y_n} S_n\; \Delta W_n, \\ Y_{n+1} - Y_n =& \left(-\alpha(1 + Y_n) + 0.4\sqrt{\alpha}\sqrt{1-\rho^2}\right) \Delta t + 0.4\sqrt{\alpha}\Delta\hat{Z}_n, \\ \hat{Z}_n =& \rho W_n + \sqrt{1-\rho^2} Z_n \end{aligned}$$

with the implicit method

$$\begin{aligned} S_{n+1} - S_n =& rS_n\Delta t + e^{Y_n} S_n\; \Delta W_n, \\ Y_{n+1} - Y_n =& \left(-\alpha(1 + Y_{n+1}) + 0.4\sqrt{\alpha}\sqrt{1-\rho^2}\right) \Delta t + 0.4\sqrt{\alpha}\Delta\hat{Z}_n, \\ \hat{Z}_n =& \rho W_n + \sqrt{1-\rho^2} Z_n \end{aligned}$$

for the computation of the option value

$$e^{-rT}E[\max(S(T)-K,0)].$$

Here we use $\alpha = 200$, $r = 0.04$, $T = \frac{3}{4}$ and $S_0 = K = 100$. Motivate which of the above numerical methods is best in this case. In the limit when $\alpha \to \infty$ one obtains the results from the geometric Brownian motion. Can you verify this by numerical experiments? For the purposes of this exercise you may use the Matlab program *stochvol.m*.

□

Exercise 4

a . In the risk neutral formulation a stock solves the SDE (2) with solution (3). Simulate the price

$$f(0, S_0) = e^{-rT}E[\max(S(T)-K,0)|S(0) = S_0],$$

of an European call option by a Monte Carlo method, where

$$S_0 = K = 35, \quad r = 0.04, \quad \sigma = 0.2, \quad T = \frac{1}{2}.$$

Compute also the corresponding delta

$$\Delta \equiv \frac{\partial f(0,s)}{\partial s},$$

by approximating it with a finite difference quotient and determine a good choice of your "∂s". Estimate the accuracy of your results and suggest a better method to solve this problem.

b . Assume that a system of stocks follows

$$dS_i/S_i(t) = rdt + \sum_{j=1}^{d} \sigma_{ij} dW_j(t), \quad i = 1, \dots, d, \tag{4}$$

where $W_j,\ j = 1, \dots, d$ are independent Brownian motions. Show that the solution of (4) is given by

$$S_i(T) = \exp(rT + \sum_{j=1}^{d}(\sigma_{ij}W_j(T) - \sigma_{ij}^2 T/2))S_i(0),\ i = 1, \dots, d.$$

Let

$$S_{av} \equiv \sum_{i=1}^{d} S_i/d,$$

and simulate the option price

$$\Pi_{av} \equiv e^{-rT}E[\max(S_{av}(T)-K,0)],$$

where

$$d = 10,\ r = 0.04,\ S_i(0) = 40,\ i = 1, \dots 10,\ T = \frac{1}{2}$$

and some ad hoc non diagonal choice of the volatility matrix σ. Estimate the accuracy of your results. Can you find a better method to solve this problem?

□

References

[BS73] F. Black and M. Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81:637–659, 1973.

[FPS00] Jean-Pierre Fouque, George Papanicolaou, and K. Ronnie Sircar. *Derivatives in financial markets with stochastic volatility.* Cambridge University Press, Cambridge, 2000.

[GMS+06] J. Goodman, K.S. Moon, A. Szepessy, R. Tempone, and Z. Zouraris. Stochastic and Partial Differential Equations with Adapted Numerics. *Lecture Notes*, 2006.

Optional Excercises

4. Let the Brownian bridge be

$$W^o(s) \equiv E[W_s|W(1)], \text{ for } s \in (0,1).$$

Use the properties of the Wiener process to find out the distribution of $W^o(s)$ also called "pinned Brownian motion".

[**a**] Compute

$$P(W(s) \in A|W(1) \in B)$$

where A and B are a real intervals.
Hint: Use the definition of conditional probability and independence of increments of W.
[**b**] Compute

$$P(W(s) \in A|W(1) = x) = \lim_{\epsilon \to 0} P(W(s) \in A|W(1) \in B)$$

with $B = (x - \epsilon, x + \epsilon)$

5. Show that the weak limit of Gaussian random variables is also Gaussian. Hint: Consider a sequence $X_n \sim N(\mu_n, \sigma_n^2)$ and assume that $\mu_n \to \mu$ and $\sigma_n \to \sigma$.

6. Consider X that solves for $t > 0$

$$dX(t) = a(t, X(t))dt + b(t, X(t))dW(t)$$

with random initial condition, $X(0)$. Assume that $X(0)$ is independent of W. Moreover, assume that $X(0)$ has a probability density function ρ_0.

Find the Focker Planck equation for X. Justify your answer.