DIMENSIONS OF SIMPLE $u_q(\mathfrak{sl}_3)$-MODULES

MARIANA PEREIRA

This is a summary of the correspondence between the simple $u_q(\mathfrak{sl}_3)$-modules described by Dobrev in [1] and those described using a method introduced by Radford ([4]) that appeared in [3].

First we will describe how the quantum groups in [3] can be seen as the quantum groups used by Dobrev. The parameter used when referring to Dobrev's modules will be called q and the parameter used when referring to modules constructed using Radford's method will be denoted by θ.
In [1] $U_q(\mathfrak{sl}_3)$ is the associative algebra over $\mathbb{C}$, with generators $X_i^{\pm}$, H_i, $i = 1, 2$ and relations

(R1) $[H_i, H_j] = 0$
(R2) $[H_i, X_j^{\pm}] = \pm a_{ij} X_j^{\pm}$
(R3) $[X_i^+, X_j^-] = \delta_{ij} \dfrac{q^{H_i/2} - q^{-H_i/2}}{q^{1/2} - q^{-1/2}} = \delta_{ij} [H_i]_q$.
(R4) $\sum_{k=0}^{2} (-1)^k \binom{2}{k}_q (X_i^{\pm})^k X_j^{\pm} (X_i^{\pm})^{2-k} = 0$, $i \neq j$,

where $(a_{ij}) = (2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i))$ is the Cartan matrix of $\mathfrak{sl}_3$. It follows from (R4) that $H_i^m X_j^{\pm} = X_j (H_i \pm a_{ij})^n$.

Lemma 1. *Changing the generating elements to*

$$K_i^{\pm 1} = q^{\pm H_i/2}, \quad E_i = X_i^+ K_i^{-1/2}, \quad F_i = X_i^- K_i^{1/2}$$

the relations are replaced by

(R1') $K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad K_1 K_2 = K_2 K_1$
(R2') $K_i X_j^{\pm} K_i = q^{\pm a_{ij}/2} X_j^{\pm}$
(R3') $[E_i, F_j] = \delta_{ij} q^{1/2} \dfrac{K_i - K_i^{-1}}{q^{1/2} - q^{-1/2}}$.
(R4') $\sum_{k=0}^{2} (-1)^k \binom{2}{k}_q E_i^k E_j E_i^{2-k} = 0 = \sum_{k=0}^{2} (-1)^k \binom{2}{k}_q F_i^k F_j F_i^{2-k}$, $i \neq j$,

The coalgebra structure is given by

- $\Delta(K_i^{\pm}) = K_i^{\pm} \otimes K_i^{\pm}$
- $\Delta(E_i) = E_i \otimes 1 + K_i^{-1} \otimes E_i$
- $\Delta(F_i) = F_i \otimes K_i + 1 \otimes F_i$

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Proof. We will show (R3') (the other relations are shown in [1]). First note that, if $m \in \mathbb{R}$, $K_i^m X_j^{\pm} = q^{\pm a_{ij} m/2} X_j^{\pm} K_i^m$: if $q = e^{\alpha}$ then

$$\begin{aligned} K_i^m X_j^{\pm} &= q^{mH_i/2} X_j^{\pm} = e^{\alpha m H_i/2} X_j^{\pm} = \sum \frac{1}{n!}\left(\frac{\alpha m}{2}\right)^n H_i^n X_j^{\pm} \\ &= \sum \frac{1}{n!}\left(\frac{\alpha m}{2}\right)^n X_j^{\pm} (H_i \pm a_{ij})^n = X_j^{\pm} \sum \frac{1}{n!}\left(\frac{\alpha m (H_i \pm a_{ij})}{2}\right)^n \\ &= X_j^{\pm} e^{\frac{\alpha m (H_i \pm a_{ij})}{2}} = X_j^{\pm} q^{mH_i/2 \pm a_{ij} m/2} = q^{\pm a_{ij} m/2} X_j^{\pm} K_i^m. \end{aligned}$$

In particular, if $m = 1$ we have (R2') and if $m = 1/2$ we have that $K_i^{1/2} X_j^{\pm} = q^{\pm a_{ij}/4} X_j^{\pm} K_i^{\pm 1/2}$.

$$\begin{aligned} [E_i, F_j] &= X_i^+ K_i^{-1/2} X_j^- K_j^{+1/2} - X_j^- K_j^{+1/2} X_i^+ K_i^{-1/2} \\ &= q^{-(-a_{ij}/4)} X_i^+ X_j^- K_i^{-1/2} K_j^{1/2} - q^{a_{ij}/4} X_j^- X_i^+ K_j^{1/2} K_i^{-1/2} \\ &= q^{a_{ij}/4} (X_i^+ X_j^- - X_j^- X_i^+) K_j^{1/2} K_i^{-1/2} \\ &= q^{a_{ij}/4} \delta_{ij} \frac{K_i - K_i^{-1}}{q^{1/2} - q^{-1/2}} K_j^{1/2} K_i^{-1/2} = q^{1/2} \frac{K_i - K_i^{-1}}{q^{1/2} - q^{-1/2}} \end{aligned}$$

□

Lemma 2. *Let* $e_i = q^{-1/4} E_i$, $f_i = q^{-1/4} F_i$, $\omega_i^{\pm 1} = K^{\mp 1}$ $i = 1, 2$, *and* $\theta = q^{-1/2}$, *we have that* $U_q(\mathfrak{sl}_3)$ *is generated by* $e_i, f_i, \omega_i^{\pm 1}$, $i = 1, 2$ *with relations*

(r1) $\omega_i e_j = \theta^{\langle \epsilon_i, \alpha_j \rangle} \theta^{-\langle \epsilon_{i+1}, \alpha_j \rangle} e_j \omega_i$.
(r2) $\omega_i f_j = \theta^{-\langle \epsilon_i, \alpha_j \rangle} \theta^{\langle \epsilon_{i+1}, \alpha_j \rangle} f_j \omega_i$
(r3) $[e_i, f_j] = \frac{\delta_{ij}}{\theta - \theta^{-1}} (\omega_i - \omega_i^{-1})$,
(r4) $e_1^2 e_2 - (\theta + \theta^{-1}) e_1 e_2 e_1 + e_2 e_1^2 = 0$, $e_1 e_2^2 - (\theta + \theta^{-1}) e_2 e_1 e_2 + e_2^2 e_1 = 0$,
(r5) $f_1^2 f_2 - (\theta + \theta^{-1}) f_1 f_2 f_1 + f_2 f_1^2 = 0$, $f_1 f_2^2 - (\theta + \theta^{-1}) f_2 f_1 f_2 + f_2^2 f_1 = 0$,

and the coalgebra structure is given by

- $\Delta(e_i) = e_i \otimes 1 + \omega_i \otimes e_i$
- $\Delta(f_i) + 1 \otimes f_i + f_i \otimes \omega_i^{-1}$
- $\Delta(\omega_i^{\pm}) = \omega_i^{\pm} \otimes \omega_i^{\pm}$.

Hence, the $U_q(\mathfrak{sl}_3)$ used in Dobrev's paper [1] is the quotient (identifying $\omega_i' = w_i^{-1}$) of the $U_{\theta,\theta^{-1}}(\mathfrak{sl}_3)$ used in [2] and [3] for $\theta = q^{-1/2}$.

1. Simple $u_q(\mathfrak{sl}_3)$-modules.

In [1], Dobrev describes all simple $U_q(\mathfrak{sl}_3)$ modules when q is a primitive ℓth root of unity. We will only look at the modules that are also modules for the finite-dimensional quotient $\mathfrak{u}_q(\mathfrak{sl}_3)$ (quotient by $f_i^{\ell} = e_i^{\ell} = 0$ and $\omega_i^{\ell} = 1$) since these are the ones that can be obtained using Radford's construction (see [4], [3]).

The simple modules are quotients of Highest Weight Modules. A highest weight module V^λ is given by a highest weight vector v_0 and a weight $\lambda \in \mathcal{H}^*$ ($\mathcal{H}$ is the subalgebra of $U_q(\mathfrak{sl}_3)$ generated by the elements H_i), such that

$$X_i^+ v_0 = 0,\ i = 1, 2, \quad H v_0 = \lambda(H) v_0,\ H \in \mathcal{H}.$$

If we let G_1 be the group generated by ω_1 and ω_2, elements $\lambda \in \mathcal{H}^*$ correspond to algebra maps from $\mathbb{K}G_1$ to $\mathbb{K}$ in the following way:

$$\lambda(w_i) = \lambda(K_i^{-1}) =^{\text{def}} q^{-\frac{1}{2}\lambda(H_i)} = \theta^{\lambda(H_i)}.$$

In the same way, if we let G_2 be the group in $U_{\theta,\theta^{-1}}(\mathfrak{sl}_3)$ generated by ω_1' and ω_2', elements $\lambda \in \mathcal{H}^*$ correspond to algebra maps from $\mathbb{K}G_2$ to $\mathbb{K}$ via

$$\lambda(w_i') = \theta^{-\lambda(H_i)}.$$

Let H be the co-opposite of the Hopf subalgebra of $\mathfrak{u}_{\theta,\theta^{-1}}(\mathfrak{sl}_3)$ generated by f_i and ω_i', $i = 1, 2$. In [3] it was shown that simple $\mathfrak{u}_\theta(\mathfrak{sl}_3)$-modules are of the form $H \bullet_\beta g$, with $g = \omega_1'^c \omega_2'^d \in G_2$ ($c, d \in \{0, \cdots, \ell - 1\}$) and β the algebra map from $\mathbb{K}G_2$ to $\mathbb{K}$ given by

$$\beta(\omega_1') = -2c + d \quad \text{and} \quad \beta(\omega_2') = c - 2d.$$

If $(\ell, 3) = 1$, taking $m_i \in \{1, \cdots, \ell\}, m_1 \equiv (2c - d + 1) \text{mod } \ell$ and $m_2 \equiv (2d - c + 1) \text{mod } \ell$, the simple modules are given by pairs $(m_1, m_2) \in \{1, \cdots, \ell\} \times \{1, \cdots, \ell\}$, where $\beta(\omega_i') = \theta^{-m_i+1}$ and $g = \omega_1'^c \omega_2'^d$ with

$$\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}^{-1} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}$$

In addition, in such a $H \bullet_\beta g$ the following are satisfied:

- $\omega_i' \bullet_\beta g = \beta(\omega_i') g = \theta^{-m_i+1} g$. Hence, under the correspondence of algebra maps from $\mathbb{K}G_2$ to $\mathbb{K}$ and elements in the dual of $\mathcal{H}$, we have that

$$H_i \cdot g = \beta(H_i) g, \qquad \text{with } \beta \in \mathcal{H}^* \text{ given by } \beta(H_i) = m_i - 1.$$

- $e_i \cdot g = 0$, hence $X_i^+ \cdot g = 0, \quad i = 1, 2.$

Therefore $H \bullet_\beta g$ is a highest weight module, with highest weight $\beta \in \mathcal{H}^*$ given by $\beta(H_i) = m_i - 1$, and highest weight vector g. In what follows, we show the correspondence between the modules $H \bullet_\beta g$ and the ones constructed by Dobrev in [1] (the correspondence depends on the choice of parameters m_1 and m_2).

Case 1. If $m_1 + m_2 < \ell$. In this case, $H \bullet_\beta g$ is isomorphic to $L^4_{m_1 m_2}$ (type 4 in [1]) which is the only simple quotient of the $U_q(\mathfrak{sl}_3)$ highest weight module V^λ, with $\lambda(H_i) = m_i - 1$. It follows from the formulas given in [1] that

$$\dim(H \bullet_\beta g) = \frac{m_1 m_2 (m_1 + m_2)}{2}$$

Case 2. If $m_1 + m_2 > \ell$.

a. If $m_i \neq \ell$ for $i = 1, 2$ and we let $n_1 = \ell - m_2$ and $n_2 = \ell - m_1$, we have that $0 < n_i < \ell$ and $n_1 + n_2 = 2\ell - (m_1 + m_2) < \ell$. In this case, $H{\bullet_\beta}g$ is isomorphic to $L'^{4}_{n_1 n_2}$ from [1], which is the only irreducible quotient of the $U_q(\mathfrak{sl}_3)$ highest weight module V^γ with $\gamma = \lambda - n_3\alpha_3$, where $\lambda(H_i) = n_i - 1$, $i = 1, 2$, $n_3 = n_1 + n_2$ and $\alpha_3 = \alpha_1 + \alpha_2$. This follows from

$$\gamma(H_i) = n_i - 1 - (n_1 + n_2)\alpha_3(H_i) = n_i - 1 - (n_1 + n_2) = -1 - (\ell - m_i) \equiv (m_i - 1) \bmod \ell.$$

It follows from the formulas given in [1] that

$$\dim(H{\bullet_\beta}g) = \frac{m_1 m_2(m_1 + m_2)}{2} - \frac{n_1 n_2(n_1 + n_2)}{2}.$$

b. If $m_1 = \ell$ and $m_2 < \ell/2$, taking $n_1 = m_2 < \ell/2$, we have that $H{\bullet_\beta}g$ is isomorphic to $L^{52}_{n_1}$ from [1], which is the irreducible quotient of the $U_q(\mathfrak{sl}_3)$ highest weight module V^γ with $\gamma = \lambda - n_2\alpha_2$, where $n_2 = \ell - n_1$ and $\lambda(H_i) = n_1 - 1$. This follows from

$$\gamma(H_1) = \lambda(H_1) - n_2\alpha_2(H_1) = n_1 - 1 + n_2 = \ell - 1 = m_1 - 1$$

and

$$\gamma(H_2) = \lambda(H_2) - n_2\alpha_2(H_2) = n_2 - 1 - 2n_2 = -n_2 - 1 = n_1 - \ell - 1 \equiv (m_2 - 1) \bmod \ell.$$

Hence, by the formula from [1] we have that

$$\dim(H{\bullet_\beta}g) = \dim(L^{52}_{n_1}) = \frac{\ell n_1(\ell + n_1)}{2} = \frac{m_1 m_2(m_1 + m_2)}{2}.$$

c. If $m_1 = \ell$ and $m_2 \geq \ell/2$, then

$$\dim(H{\bullet_\beta}g) = \dim(H{\bullet_{\beta'}}g'),$$

where (β', g') correspond to the choice of parameters $(m_2, m_1) = (n_1, \ell)$ with $n_1 \geq \ell/2$ (see Lemma 4). Hence, the dimension of $H{\bullet_\beta}g$ can be obtained from the next case.

d. If $m_2 = \ell$ and $m_1 \geq \ell/2$. Taking $n_1 = \ell - m_1 \leq \ell/2$, we have that $H{\bullet_\beta}g$ is isomorphic to $L^{51}_{n_1}$ from [1], which is the irreducible quotient of the $U_q(\mathfrak{sl}_3)$ highest weight module V^γ with $\gamma = \lambda - n_1\alpha_1$, where $\lambda(H_1) = n_1 - 1$ and $\lambda(H_2) = \ell - n_1 - 1$. This follows from

$$\gamma(H_1) = \lambda(H_1) - n_1\alpha_1(H_1) = n_1 - 1 - 2n_1 == -n_1 - 1 = m_1 - \ell - 1 \equiv (m_1 - 1) \bmod \ell$$

and

$$\gamma(H_2) = \lambda(H_2) - n_1\alpha_1(H_2) = \ell - n_1 - 1 + n_1 = \ell - 1 = m_2 - 1.$$

Hence, by the dimensions formulas from [1] we have that

$$\dim(H{\bullet_\beta}g) = \frac{\ell(\ell - n_1)(2\ell - n_1)}{2} = \frac{m_1 m_2(m_1 + m_2)}{2}.$$

e. If $m_2 = \ell$ and $m_1 < \ell/2$. Again, $\dim(H_{\bullet\beta}g) = \dim(H'_{\bullet\beta'}g')$, with (β', g') corresponding to $(m_2, m_1) = (\ell, n_2)$ with $n_2 < \ell/2$. Hence, by case 2.b. we have that

$$\dim(H_{\bullet\beta}g) = \dim(H_{\bullet\beta'}g') = \frac{m_1m_2(m_1+m_2)}{2}.$$

Case 3. If $m_1 + m_2 = \ell$.

a. If $m_1 \leq \ell/2$, $H_{\bullet\beta}g$ is isomorphic to $L^5_{m_1}$ from [1], which is the only irreducible quotient of the $U_q(\mathfrak{sl}_3)$ highest weight module V^λ with $\lambda(H_i) = m_i - 1$. It follows that

$$\dim(H_{\bullet\beta}g) = \frac{m_1(\ell - m_1)\ell}{2} = \frac{m_1m_2(m_1+m_2)}{2}.$$

b. If $m_1 > \ell/2$, taking $n_1 = m_2$ and $n_2 = m_1$ we have that $n_1 < \ell/2$ and if (β', g') correspond to the choice of parameters (n_1, n_2), we have that

$$\dim(H_{\bullet\beta}g) = \dim(H_{\bullet\beta'}g') = \dim(L^5_{n_1}) = \frac{n_1n_2(n_1+n_2)}{2} = \frac{m_1m_2(m_1+m_2)}{2}.$$

The above discussion shows that if $H_{\bullet\beta}g$ is a simple $\mathfrak{u}_\theta(\mathfrak{sl}_3)$-module, with $\beta(\omega_i') = \theta^{m_i-1}$, $m_i \in \{1, \cdots, \ell\}$, $i = 1, 2$, then

- If $m_1 + m_2 \leq \ell$,

$$\dim(H_{\bullet\beta}g) = \frac{m_1m_2(m_1+m_2)}{2}.$$

- If $m_1 + m_2 > \ell$ and we let $m_i' = \ell - m_i$, then

$$\dim(H_{\bullet\beta}g) = \frac{m_1m_2(m_1+m_2)}{2} - \frac{m_1'm_2'(m_1'+m_2')}{2}.$$

2. Conjugation

Let $\Phi : K \to L$ be an algebra homomorphism and $(M, \cdot)$ a left L-module. Then M is a left K-module, with the action given by $k \cdot_\Phi m = \Phi(k) \cdot m$. Moreover, if Φ is surjective and N is a K-submodule of M, N is also a L-submodule of M. If $\Psi : L \to K$ is a coalgebra homomorphism and (M, δ) is a right L-comodule, then (M, δ_Ψ) is a right K-comodule with $\delta_\Psi(m) = (\mathrm{id}_M \otimes \Psi)\delta(m)$. Moreover, if Ψ is injective and N is a right K-subcomodule of M, then N is also a L-subcomodule of M.

Lemma 3. *Let $\Phi : K \to L$ be a bialgebra isomorphism and $(M, \cdot, \delta) \in {}_L\mathcal{YD}^L$, then $M_\Phi = (M, \cdot_\Phi, \delta_{\Phi^{-1}}) \in {}_K\mathcal{YD}^K$. Moreover, M_θ is simple if M is simple.*

Proof. The only thing left to check is the compatibility between the action and coaction. Since $(M,\cdot,\delta)\in {}_L\mathcal{YD}^L$ we have that

$$\sum l_1\cdot m_0\otimes l_2m_1=\sum(l_2\cdot m)_0\otimes(l_2\cdot m)_1l_1,$$

for all $l\in L$, $m\in M$, where $\delta(n)=\sum n_0\otimes n_1$, for all $n\in M$. By the definitions of $\cdot_\Phi$ and $\delta_{\Phi^{-1}}$, we have to prove that

$$\sum\Phi(k_1)\cdot m_0\otimes k_2\Phi^{-1}(m_1)=\sum(\Phi(k_2)\cdot m)_0\otimes\Phi^{-1}((\Phi(k_2)\cdot m)_1)k_1.$$

Now,

$$\begin{aligned}\sum\Phi(k_1)\cdot m_0\otimes k_2\Phi^{-1}(m_1) &= (\mathrm{id}_M\otimes\Phi^{-1})\left(\sum\Phi(k_1)\cdot m_0\otimes\Phi(k_2)m_1\right)\\ &= (\mathrm{id}_M\otimes\Phi^{-1})\left(\sum\Phi(k)_1\cdot m_0\otimes\Phi(k)_2m_1\right)\\ &= (\mathrm{id}_M\otimes\Phi^{-1})\left(\sum(\Phi(k)_2\cdot m)_0\otimes(\Phi(k)_2\cdot m)_1\Phi(k)_1\right)\\ &= \sum(\Phi(k)_2\cdot m)_0\otimes\Phi^{-1}\left((\Phi(k)_2\cdot m)_1\right)k_1.\end{aligned}$$

□

As before, let H be the co-opposite of the Hopf subalgebra of $\mathfrak{u}_{\theta,\theta^{-1}}(\mathfrak{sl}_3)$ generated by f_i and ω_i', $i=1,2$. Let $\tau=(1,2)$ be the transposition that exchanges the indices 1 and 2. It is easy to see that there is Hopf algebra isomorphism $\Phi:H\to H$ such that $\Phi(f_i)=f_{\tau(i)}$ and $\Phi(\omega_i')=\omega_{\tau(i)}'$, $i=1,2$. For $(m_1,m_2)\in\{1,\cdots,l\}^2$, let $M(m_1,m_2)=H\bullet_\beta g$ be the simple $\mathfrak{u}_\theta(\mathfrak{sl}_3)$-module corresponding to the parameters (m_1,m_2). That is, $\beta(\omega_i')=\theta^{-m_i+1}$ and $g=\omega_1'^c\omega_2'^d$ with

$$\begin{pmatrix}c\\d\end{pmatrix}=\begin{pmatrix}-2&1\\1&-2\end{pmatrix}^{-1}\begin{pmatrix}m_1\\m_2\end{pmatrix}.$$

By the last lemma, $(M(m_1,m_2))_\Phi$ is also a simple H-Yetter-Drinfel'd module.

Lemma 4. *If $\Phi:H\to H$ is the Hopf algebra isomorphism that exchanges subindices, and $(m_1,m_2)\in\{1,\cdots,\ell\}^2$, then*

$$(M(m_1,m_2))_\Phi\simeq M(m_2,m_1).$$

In particular, $\dim(M(m_1,m_2))=\dim(M(m_2,m_1))$.

Proof. As before, $M(m_1,m_2)=H\bullet_\beta g$ with $g=\omega_1'^c\omega_2'^d$ and $\beta(\omega_i')=\theta^{m_i-1}$. Since $(M(m_1,m_2))_\Phi$ is a simple H-Yetter-Drinfel'd module, there exist $g'=\omega_1'^e\omega_2'^f\in G_2$ and β' and algebra map from $\mathbb{K}G_2$ to $\mathbb{K}$ given by $\beta'(\omega_i')=\theta^{n_i-1}$, $n_i\in\{1,\cdots,\ell\}$, $i=1,2$, such that $(M(m_1,m_2))_\Phi\simeq H\bullet_{\beta'}g'$ as Yetter-Drinfel'd modules. Let $T:(M(m_1,m_2))_\Phi\to H\bullet_{\beta'}g'$ be a Yetter-Drinfel'd module isomorphism. Since T is a right comodule homomorphism, $(T\otimes\mathrm{id}_H)\circ\delta_\Phi=\Delta\circ T$ and

$$T(g)\otimes\Phi^{-1}(g)=\Delta(T(g)).\tag{5}$$

Then $\mathbb{K}T(g)$ is a (non-zero) right-coideal of $H\bullet_{\beta'}g'$, but the only non-zero coideal contained in $H\bullet_{\beta'}g'$ is $\mathbb{K}g'$ (see [4]). Therefore $T(g)$ is a multiple of g, and we may assume that $T(g) = g'$. Applying $\epsilon \otimes \mathrm{id}$ to equation 5, we get that $\epsilon(g')\Phi^{-1}(g) = g'$. Since both $\Phi^{-1}(g)$ and g' are grouplike elements, and grouplike elements are linearly independent, we have that $g' = \Phi^{-1}(g) = \omega_1'^d \omega_2'^c$; therefore $(e, f) = (d, c)$, and so

$$\begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} d \\ c \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}^{-1} \begin{pmatrix} m_2 \\ m_1 \end{pmatrix}.$$

Since T is a homomorphism of left H-modules, $T(\omega_i' \cdot_\Phi g) = \omega_i' \bullet_{\beta'} T(g)$. Since

$$T(\omega_i' \cdot_\Phi g) = T(\omega_{\tau(i)}' \bullet_\beta g) = T(\theta^{m_{\tau(i)}-1} g) = \theta^{m_{\tau(i)}-1} g'$$

and

$$\omega_i' \bullet_{\beta'} T(g) = \theta^{n_i - 1} g',$$

it follows that $(n_1, n_2) = (m_2, m_1)$, and $H\bullet_{\beta'}g' = M(m_2, m_1)$.

□

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