Central Limit Theorem for the number of real roots of Kostlan Shub Smale random polynomial systems

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Abstract:
We obtain a Central Limit Theorem for the normalized number of real roots of a square Kostlan-Shub-Smale random polynomial system of any size as the degree goes to infinity. The asymptotic variance of the number of roots is needed, the corresponding result was obtained in [2]. Afterwards, we represent the number of roots as an explicit non linear functional belonging to the Itô-Wiener chaos. This representation provides a tool for applying the Fourth Moment Theorem and henceforth the asymptotic normality.

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1. Introduction

The real roots of random polynomials, or systems of such equations, have been intensively studied from the point of view of several branches of mathematics and physics.

The investigation on this subject were initiated by Bloch and Pólya [8], and Littlewood and Offord [18, 19], where polynomials in one real variable with random coefficients have been studied. The first asymptotically sharp result on the expected number of real roots is due to M. Kac [13]. (For more details on the case of random polynomials see the textbook by Barucha-Reid and Sambandham [7]).

In Maslova [21], for the first time, the asymptotic variance of the number of real roots of random polynomials is analyzed. Nevertheless, most of the research is concerned with partial characteristics, as the moments of first and second order, of the distribution of the number of roots.

In fact, up to 2010, the only result on the (asymptotic) distribution is the Central Limit Theorem for the standardized number of zeros of Kac Polynomials, see Maslova [20].

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The present decade witnesses a rapidly increasing series of results on the asymptotic distribution of the number of real roots of random polynomials. In 2011-2012 Granville and Wigman [12] and Azaïs and León [4] established the CLT in the case of Gaussian Qualls' trigonometric polynomials; in 2016 Azaïs, Dalmao and León [3] extended this result to classical trigonometric polynomials; in 2015 Dalmao [9] did the same for elliptic or Kostlan-Shub-Smale polynomials; in 2017 Do and Vu [10] proved a Central Limit Theorem for the number of real roots of Weyl polynomials.

Another important extension deals with systems of polynomials equations. In Shub and Smale [25], as suggested by Kostlan [14], the expected number of real roots of certain random systems of polynomials equations has been studied for the first time. Additional cases are considered in [5]. Similarly, Wschebor [27] investigated the asymptotic variance, as the dimension goes to infinity, of the normalized number of real roots of the Kostlan-Shub-Smale random polynomial system. Letendre [16] computed the expectation and the variance of the measure of the nodal sets of rectangular systems.

Concerning the complex version of Kostlan-Shub-Smale polynomials it is worth to mention that Sodin and Tsirelson [26] established a Central Limit Theorem for linear statistics of the complex zeros (i.e.: a sum of a test function over the set of zeros) using techniques closely related with our method.

In the present paper we establish a Central Limit Theorem (CLT for short) for the number of real roots of a square system of polynomial equations as their common degree tends to infinity. Up to our knowledge this is the first result about the asymptotic distribution of the number of real roots of systems of random polynomials.

2. Main result

Consider a square system $\mathbf{P}_d = 0$ of m polynomial equations in m variables with common degree $d > 1$. More precisely, let $\mathbf{P}_d = (P_1, \dots, P_m)$ with

$$P_\ell(t) = \sum_{|\boldsymbol{j}|\leq d} a_{\boldsymbol{j}}^{(\ell)} t^{\boldsymbol{j}}; \quad \ell = 1, \dots, m,$$

where

1. $\boldsymbol{j} = (j_1, \dots, j_m) \in \mathbb{N}^m$ and $|\boldsymbol{j}| = \sum_{k=1}^m j_k$;
2. $a_{\boldsymbol{j}}^{(\ell)} = a_{j_1 \dots j_m}^{(\ell)} \in \mathbb{R}$, $\ell = 1, \dots, m$, $|\boldsymbol{j}| \leq d$;
3. $t = (t_1, \dots, t_m)$ and $t^{\boldsymbol{j}} = \prod_{k=1}^m t_k^{j_k}$.

We say that $\mathbf{P}_d$ has the Kostlan-Shub-Smale (KSS for short) distribution if the coefficients $a_{\boldsymbol{j}}^{(\ell)}$ are independent centered normally distributed random variables with variances

$$\mathrm{Var}\left(a_{\boldsymbol{j}}^{(\ell)}\right) = \binom{d}{\boldsymbol{j}} = \frac{d!}{j_1! \dots j_m! (d - |\boldsymbol{j}|)!}.$$

We are interested in the number of roots of $\mathbf{P}_d$ in $\mathbb{R}^m$ that we denote by $N_{\mathbf{P}_d}$. Shub and Smale [25] proved that $\mathbb{E}\left(N_{\mathbf{P}_d}\right) = d^{m/2}$. The authors in [2], see also Letendre [17], proved that

$$\lim_{d\to\infty} \frac{\mathrm{Var}(N_{\mathbf{P}_d})}{d^{m/2}} = V_\infty, \tag{2.1}$$

where $0 < V_\infty < \infty$. We now establish a CLT.

Theorem 1. *Let $\mathbf{P}_d$ be an $m \times m$ KSS system, its standardized number of roots*

$$\bar{N}_d = \frac{N_{\mathbf{P}_d} - \mathbb{E}\left(N_{\mathbf{P}_d}\right)}{d^{m/4}}$$

converges in distribution as $d \to \infty$ towards a normal random variable with positive variance.

In order to obtain the CLT we use an Hermite (or chaotic) expansion of the standardized number of roots of the system and then we use an extension of the celebrated Fourth moment theorem by Peccati, Nualart and Tudor, see Theorem 11.8.3 in [24].

3. Outline of the proof

For the sake of readability, we present now a brief outline of the forthcoming proof.

As a first step, it is convenient to homogenize the system. The roots of the original system $\mathbf{P}_d$ are easily identified with the roots of the homogeneous version $\mathbf{Y}_d$ on the sphere S^m. Besides, the covariance structure of $\mathbf{Y}_d$ is simple and invariant under the action of the orthogonal group in $\mathbb{R}^{m+1}$. See the details in the next section.

In order to get the CLT we expand the standardized number of roots of $\mathbf{Y}_d$ on S^m in the L^2 sense in a convenient basis, this is called an Hermite or chaotic expansion in the literature. Taking advantage of the structure of chaotic random variables the CLT is easily obtained for each term in the expansion as well as for any finite sum of them.

The difficult part is to prove the negligeability (of the variance) of the tail of the expansion due to the degeneracy of the covariance of $\mathbf{Y}_d$ at the diagonal $\{(s,t) \in S^m \times S^m : s = t\}$. To deal with this degeneracy we adapt a trick used under stationarity on the Euclidean case which consists of covering a neighbourhood of the diagonal with isometric small regions. The variance of the number of roots on each such small region is handled with Rice formula or some other rough method. Then a balance between the number of such regions and the bound is needed.

On the sphere the regions can not be chosen to be isometric, though a careful construction allows to cover an essential part of the sphere by regions such that their projections on the tangent space at a convenient point are isometric in the limit. The diagonal is covered by products of these regions. Provided the existence of a common local limit process on the tangent spaces we can bound uniformly the tail of the variance of the number of roots on each region by approximating it with the corresponding tail of the number of roots of the local limit process.

Outline of the paper: The rest of the paper is organized as follows. Section 4 deals with some preliminaries. In Section 5 the proof of the main result is presented. Some technical or minor parts of the proof are postponed to Section 6.

Some remarks on the notation: We denote by S^m the unit sphere in $\mathbb{R}^{m+1}$ and its volume by κ_m. The variables s and t denote points on S^m and ds and dt denote the corresponding geometric measure. The variables u and v are in $\mathbb{R}^m$, and du and dv are the associated Lebesgue measure. The variables z and θ are reals, and dz and $d\theta$ are the associated differentials.

As usual we use the Landau's big O and small o notation. The set $\mathbb{N}$ of natural numbers contains 0. Also, $Const$ will denote a universal constant that might change from a line to another.

4. Preliminaries

Homogeneous version of $\mathbf{P}_d$: Let $\mathbf{Y}_d = (Y_1, \dots, Y_m)$ with

$$Y_\ell(t) = \sum_{|\boldsymbol{j}|=d} a_{\boldsymbol{j}}^{(\ell)} t^{\boldsymbol{j}}, \quad \ell = 1, \dots, m,$$

where this time $\boldsymbol{j} = (j_0, \dots, j_m) \in \mathbb{N}^{m+1}$; $|\boldsymbol{j}| = \sum_{k=0}^m j_k$; $a_{\boldsymbol{j}}^{(\ell)} = a_{j_0 \cdots j_m}^{(\ell)} \in \mathbb{R}$; $t = (t_0, \dots, t_m) \in \mathbb{R}^{m+1}$ and $t^{\boldsymbol{j}} = \prod_{k=0}^m t_k^{j_k}$. Note that $\mathbf{P}_d(t_1, \dots, t_m) = \mathbf{Y}_d(1, t_1, \dots, t_m)$.

Since $\mathbf{Y}_d$ is homogeneous, namely for $\lambda \in \mathbb{R}$ it verifies $\mathbf{Y}_d(\lambda t) = \lambda^d \mathbf{Y}_d(t)$, its roots consist of lines through 0 in $\mathbb{R}^{m+1}$. Then, each root of $\mathbf{P}_d$ in $\mathbb{R}^m$ corresponds exactly to two (opposite) roots of $\mathbf{Y}_d$ on the unit sphere S^m of $\mathbb{R}^{m+1}$. Furthermore, one can prove that the subset of homogeneous polynomials $\mathbf{Y}_d$ with roots lying in the hyperplane $t_0 = 0$ has Lebesgue measure zero. Then, denoting by $N_{\mathbf{Y}_d}$ the number of roots of $\mathbf{Y}_d$ on S^m, we have

$$N_{\mathbf{P}_d} = \frac{N_{\mathbf{Y}_d}}{2} \text{ almost surely.}$$

Hyperspherical coordinates: For $\theta = (\theta_1, \dots, \theta_{m-1}, \theta_m) \in [0, \pi)^{m-1} \times [0, 2\pi)$ we write $x^{(m)}(\theta) = (x_1^{(m)}(\theta), \dots, x_{m+1}^{(m)}(\theta)) \in S^m$ in the following way

$$x_k^{(m)}(\theta) = \prod_{j=1}^{k-1} \sin(\theta_j) \cdot \cos(\theta_k), \ k \le m \text{ and } x_{m+1}^{(m)}(\theta) = \prod_{j=1}^{m} \sin(\theta_j); \tag{4.1}$$

with the convention that $\prod_1^0 = 1$.

We use repeatedly in the sequel that for $h : [-1,1] \to \mathbb{R}$ it holds that

$$\int_{S^m \times S^m} h(\langle s,t \rangle) ds dt = \kappa_m \kappa_{m-1} \int_0^\pi \sin^{m-1}(\theta) h(\cos(\theta)) d\theta, \tag{4.2}$$

being $\langle \cdot,\cdot \rangle$ the usual inner product in $\mathbb{R}^{m+1}$ and κ_m the m-volume of the sphere S^m, see [2].

Covariances: From now on we work with the homogenized version $\mathbf{Y}_d$. Direct computation yields

$$r_d(s,t) := \mathbb{E}\left(Y_\ell(s) Y_\ell(t)\right) = \langle s,t \rangle^d; \quad s,t \in \mathbb{R}^{m+1}.$$

As a consequence, the distribution of the system $\mathbf{Y}_d$ is invariant under the action of the orthogonal group in $\mathbb{R}^{m+1}$.

For $\ell = 1,\dots,m$, we denote by $Y'_\ell(t)$ the derivative (along the sphere) of $Y_\ell(t)$ at the point $t \in S^m$ and by $Y'_{\ell k}$ its k-th component on a given basis of the tangent space of S^m at the point t. We define the standardized derivative as

$$\overline{Y}'_\ell(t) := \frac{Y'_\ell(t)}{\sqrt{d}}, \quad \text{and} \quad \overline{\mathbf{Y}}'_d(t) := (\overline{Y}'_1(t),\dots,\overline{Y}'_m(t)),$$

where $\overline{Y}'_j(t)$ is a row vector. For $t \in S^m$, set also

$$\mathbf{Z}_d(t) = (Z_1(t),\dots,Z_{m+m^2}(t)) = (\mathbf{Y}_d(t), \overline{\mathbf{Y}}'_d(t)). \tag{4.3}$$

The covariances

$$\rho_{k\ell}(s,t) = \mathbb{E}\left(Z_k(s) Z_\ell(t)\right), \quad k,\ell = 1,\dots,m+m^2, \tag{4.4}$$

are obtained via routine computations, see Section 6.3. These computations are simplified using the invariance under isometries, For instance, if $k = \ell \leq m$

$$\rho_{k\ell}(s,t) = \langle s,t \rangle^d = \cos^d(\theta), \quad \theta \in [0,\pi),$$

where θ is the angle between s and t.

When the indexes k or ℓ are larger than m the covariances involve derivatives of r_d. In fact, in [2] is shown that $\mathbf{Z}_d$ is a vector of $m+m^2$ standard normal random variables whose covariances depend upon the quantities

$$\begin{aligned}
\mathcal{A}(\theta) &= -\sqrt{d}\cos^{d-1}(\theta)\sin(\theta), \\
\mathcal{B}(\theta) &= \cos^d(\theta) - (d-1)\cos^{d-2}(\theta)\sin^2(\theta), \\
\mathcal{C}(\theta) &= \cos^d(\theta), \\
\mathcal{D}(\theta) &= \cos^{d-1}(\theta).
\end{aligned} \tag{4.5}$$

See also Section 6.3. Furthermore, when dealing with the conditional distribution of $(\overline{\mathbf{Y}}'_d(s), \overline{\mathbf{Y}}'_d(t))$ given that $\mathbf{Y}_d(s) = \mathbf{Y}_d(t) = 0$ the following expressions appear for the common variance and the correlation

$$\sigma^2(\theta) = 1 - \frac{\mathcal{A}(\theta)^2}{1 - \mathcal{C}(\theta)^2}; \quad \rho(\theta) = \frac{\mathcal{B}(\theta)(1 - \mathcal{C}(\theta)^2) - \mathcal{A}(\theta)^2\mathcal{C}(\theta)}{1 - \mathcal{C}(\theta)^2 - \mathcal{A}(\theta)^2}.$$

After scaling $\theta = z/\sqrt{d}$, we have the following bounds.

Lemma 4.1 ([2]). *There exist $0 < \alpha < \frac{1}{2}$ such that for $\frac{z}{\sqrt{d}} < \frac{\pi}{2}$ it holds that*

$$\begin{aligned} |\mathcal{A}| &\leq z \exp(-\alpha z^2), \\ |\mathcal{B}| &\leq (1 + z^2) \exp(-\alpha z^2), \\ |\mathcal{C}| &\leq |\mathcal{D}| \leq \exp(-\alpha z^2), \\ 0 \leq 1 - \sigma^2 &\leq Const \cdot \exp(-2\alpha z^2), \\ |\rho| &\leq Const \cdot (1 + z^2)^2 \exp(-2\alpha z^2). \end{aligned}$$

All the functions on the lhs are evaluated at $\theta = z/\sqrt{d}$ and $Const$ stands for some unimportant constant, its value can change for a line to other. □

Rice formula and variance: In [2] the variance $\text{Var}(N_{\mathbf{Y}_d})$ is written as an integral over the interval $[0, \sqrt{d}\pi/2]$ and a domination is found in order to pass the limit wrt d under the integral sign. More precisely, Rice formula, see [6], states that

$$\begin{aligned} &\text{Var}(N_{\mathbf{Y}_d}) - \mathbb{E}\,(N_{\mathbf{Y}_d}) = \mathbb{E}\,(N_{\mathbf{Y}_d}(N_{\mathbf{Y}_d} - 1)) - (\mathbb{E}\,(N_{\mathbf{Y}_d}))^2 \\ &\quad = d^m \int_{S^m \times S^m} \Big[\mathbb{E}\,(|\det \overline{\mathbf{Y}}'_d(s) \det \overline{\mathbf{Y}}'_d(t)| \mid \mathbf{Y}_d(s) = \mathbf{Y}_d(t) = 0) p_{s,t}(0,0) \\ &\quad -\mathbb{E}\,(|\det \overline{\mathbf{Y}}'_d(s)| \mid \mathbf{Y}_d(s) = 0)\mathbb{E}\,(|\det \overline{\mathbf{Y}}'_d(t)| \mid \mathbf{Y}_d(t) = 0) p_s(0) p_t(0)\Big]\, ds dt, \end{aligned}$$

being $p_{s,t}$ the joint density of $\mathbf{Y}_d(s)$ and $\mathbf{Y}_d(t)$, and p_s and p_t the densities of $\mathbf{Y}_d(s)$ and $\mathbf{Y}_d(t)$ respectively. The factor d^m comes from the normalization of $\mathbf{Y}'_d$ and the properties of the determinant.

By the invariance under isometries of the distribution of $\mathbf{Z}_d$ the integrand depends on (s,t) only through $\langle s,t \rangle$ and thus we can reduce the integral as in (4.2). The conditional expectation $\mathbb{E}\,(|\det \overline{\mathbf{Y}}'_d(s) \det \overline{\mathbf{Y}}'_d(t)| \mid \mathbf{Y}_d(s) = \mathbf{Y}_d(t) = 0)$ can be reduced to an ordinary expectation using the so called Gaussian regression, see Section 6.3 and [6]. This computations show that $\mathbb{E}\,(|\det \overline{\mathbf{Y}}'_d(s) \det \overline{\mathbf{Y}}'_d(t)| \mid \mathbf{Y}_d(s) = \mathbf{Y}_d(t) = 0)$ and $p_{s,t}(0,0)$ depend on (s,t) only through $\sigma^2, \rho, \mathcal{D}$ and $\mathcal{C}$ respectively. Hence, we can write

$$\mathbb{E}\,(|\det \overline{\mathbf{Y}}'_d(s) \det \overline{\mathbf{Y}}'_d(t)| \mid \mathbf{Y}_d(s) = \mathbf{Y}_d(t) = 0) p_{s,t}(0,0) = \mathcal{H}_d(\sigma^2, \mathcal{C}, \rho, \mathcal{D}).$$

In particular, it holds that

$$\begin{aligned} &\mathbb{E}\,(|\det \overline{\mathbf{Y}}'_d(s)| \mid \mathbf{Y}_d(s) = 0)\mathbb{E}\,(|\det \overline{\mathbf{Y}}'_d(t)| \mid \mathbf{Y}_d(t) = 0) p_s(0) p_t(0) \\ &\qquad = \mathcal{H}_d(1,0,0,0). \end{aligned}$$

Thus, we can write

$$\frac{\operatorname{Var}(N_{\mathbf{Y}_d}) - \mathbb{E}\,(N_{\mathbf{Y}_d})}{d^{m/2}}$$
$$= Const \cdot d^{\frac{m-1}{2}} \int_0^{\sqrt{d}\pi/2} \sin^{m-1}\left(\frac{z}{\sqrt{d}}\right) [\mathcal{H}_d(\sigma^2, \mathcal{C}, \rho, \mathcal{D}) - \mathcal{H}_d(1,0,0,0)] dz.$$

In [2] is shown that

$$|\mathcal{H}_d(\sigma^2, \mathcal{C}, \rho, \mathcal{D}) - \mathcal{H}_d(1,0,0,0)| \le Const \cdot (1 - \sigma^2 + |\mathcal{C}| + |\rho| + |\mathcal{D}|). \tag{4.6}$$

Using the bounds in Lemma 4.1 we obtain a domination for the integrand of $\operatorname{Var}(N_{\mathbf{Y}_d})$ in order to pass to the limit in d under the integral sign.
In the same way we have the following lemma

Lemma 4.2. *If $\mathcal{G}$ is a Borel set of S^m with m-dimensional volume* $\operatorname{Vol}(\mathcal{G})$ *and if $N_{\mathbf{Y}_d}(\mathcal{G})$ is the number of zeros belonging to $\mathcal{G}$, we have*

$$\frac{\operatorname{Var}(N_{\mathbf{Y}_d}(\mathcal{G})) - \mathbb{E}\,(N_{\mathbf{Y}_d}(\mathcal{G}))}{d^{m/2}}$$
$$\le (Const) Vol(\mathcal{G}) \int_0^{\sqrt{d}\pi/2} \sin^{m-1}\left(\frac{z}{\sqrt{d}}\right) [\mathcal{H}_d(\sigma^2, \mathcal{C}, \rho, \mathcal{D}) - \mathcal{H}_d(1,0,0,0)] dz$$
$$\le (Const) \operatorname{Vol}(\mathcal{G}).$$

Wiener Chaos: We introduce now the Wiener chaos in a form that is suited to our purposes. For the details of this construction see [24]. Let $\mathbf{B} = \{B(\lambda) : \lambda \ge 0\}$ be a standard Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ being $\mathcal{F}$ the σ-algebra generated by $\mathbf{B}$. The Wiener chaos is an orthogonal decomposition of $L^2(\mathbf{B}) = L^2(\Omega, \mathcal{F}, \mathbb{P})$:

$$L^2(\mathbf{B}) = \bigoplus_{q=0}^{\infty} \mathcal{C}_q,$$

where $\mathcal{C}_0 = \mathbb{R}$ and for $q \ge 1$, $\mathcal{C}_q = \{I_q^{\mathbf{B}}(f_q) : f_q \in L_s^2([0,\infty)^q)\}$ being $I_q^{\mathbf{B}}$ the q-folded multiple integral wrt $\mathbf{B}$ and $L_s^2([0,\infty)^q)$ the space of kernels $f_q : [0,\infty)^q \to \mathbb{R}$ which are square integrable and symmetric, that is, if π is a permutation then $f_q(\lambda_1, \dots, \lambda_q) = f_q(\lambda_{\pi(1)}, \dots, \lambda_{\pi(q)})$. Equivalently, each square integrable functional F of the Brownian motion $\mathbf{B}$ can be written as a sum of orthogonal random variables

$$F = \mathbb{E}\,(F) + \sum_{q=1}^{\infty} I_q^{\mathbf{B}}(f_q),$$

for some uniquely determined kernels $f_q \in L_s^2([0,\infty)^q)$.

We need to introduce the so-called contractions of the kernels. Let $f_q, g_q \in L^2_s([0,\infty)^q)$, then for $n = 0, \dots, q$ we define

$$\begin{aligned} f_q \otimes_n g_q(\lambda_1, \dots, \lambda_{2q-2n}) \\ &= \int_{[0,\infty)^n} f_q(z_1, \dots, z_n, \lambda_1, \dots, \lambda_{q-n}) \\ &\quad \cdot g_q(z_1, \dots, z_n, \lambda_{q-n+1}, \dots, \lambda_{2q-2n}) dz_1 \dots dz_n. \end{aligned} \tag{4.7}$$

Now, we can state the generalization of the Fourth Moment Theorem.

Theorem 2 ([24] Theorem 11.8.3)**.** *Let F_d be in $L^2_s(\mathbf{B})$ admit chaotic expansions*

$$F_d = \mathbb{E}(F_d) + \sum_{q=1}^{\infty} I_q(f_{d,q})$$

for some kernels $f_{d,q}$. Then, if $\mathbb{E}(F_d) = 0$ and

1. *for each fixed $q \geq 1$, $\mathrm{Var}(I_q(f_{d,q})) \to_{d\to\infty} V_q$;*
2. *$V := \sum_{q=1}^{\infty} V_q < \infty$;*
3. *for each $q \geq 2$ and $n = 1, \dots, q-1$,*

$$\|f_{d,q} \otimes_n f_{d,q}\|_{L^2_s([0,\infty)^{2q-2n})} \underset{d\to\infty}{\to} 0;$$

4. *$\lim_{Q\to\infty} \limsup_{d\to\infty} \sum_{q=Q+1}^{\infty} \mathrm{Var}(I_q(f_{d,q})) = 0$.*

Then, F_d converges in distribution towards the $N(0, V)$ distribution. □

Condition 1,2 and 4 are variance conditions. Condition 3 is a moment condition or equivalently a condition on the decay of tail of the the density function. It is ultimately written in terms of the covariances of the process $\mathbf{Y}_d$ as in Theorem 7.2.4 of [22], see Lemma 5.1.

Hermite expansion of $N_{\mathbf{Y}_d}$: The Hermite expansion of $N_{\mathbf{Y}_d}$ was obtained in [2], see also [17]. In Lemma 6.1 we show that it can be put in the framework of Wiener Chaos.

We introduce the Hermite polynomials $H_n(x)$ by $H_0(x) = 1$, $H_1(x) = x$ and $H_{n+1}(x) = xH_n(x) - nH_{n-1}(x)$. The multi-dimensional (tensorial) versions are, for multi-indexes $\boldsymbol{\alpha} = (\alpha_\ell) \in \mathbb{N}^m$ and $\boldsymbol{\beta} = (\beta_{\ell,k}) \in \mathbb{N}^{m^2}$, and vectors $\mathbf{y} = (y_\ell) \in \mathbb{R}^m$ and $\mathbf{y}' = (y'_{\ell,k}) \in \mathbb{R}^{m^2}$

$$\mathbf{H}_{\boldsymbol{\alpha}}(\mathbf{y}) = \prod_{\ell=1}^{m} H_{\alpha_\ell}(y_\ell), \quad \overline{\mathbf{H}}_{\boldsymbol{\beta}}(\mathbf{y}') = \prod_{\ell,k=1}^{m} H_{\beta_{\ell,k}}(y'_{\ell,k}).$$

It is well known that the standardized Hermite polynomials $\{\frac{1}{\sqrt{n!}} H_n\}$, $\{\frac{1}{\sqrt{\boldsymbol{\alpha}!}} \mathbf{H}_{\boldsymbol{\alpha}}\}$ and $\{\frac{1}{\sqrt{\boldsymbol{\beta}!}} \overline{\mathbf{H}}_{\boldsymbol{\beta}}\}$ form orthonormal bases of the spaces $L^2(\mathbb{R}, \phi_1)$, $L^2(\mathbb{R}^m, \phi_m)$ and $L^2(\mathbb{R}^{m^2}, \phi_{m^2})$ respectively. Here, ϕ_j stands for the standard Gaussian measure on $\mathbb{R}^j$ $(j = 1, m, m^2)$ and $\boldsymbol{\alpha}! = \prod_{\ell=1}^m \alpha_\ell!$, $\boldsymbol{\beta}! = \prod_{\ell,k=1}^m \beta_{\ell,k}!$. Sometimes we write $\boldsymbol{\beta} = (\boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_m)$ with $\boldsymbol{\beta}_\ell = (\beta_{\ell 1}, \dots, \beta_{\ell m}) \in \mathbb{N}^m$ and $\boldsymbol{\beta}! = \prod_{\ell=1}^m \boldsymbol{\beta}_\ell!$. See [22, 24] for a general picture of Hermite polynomials.

Let $f_{\boldsymbol{\beta}}$ $(\boldsymbol{\beta} \in \mathbb{R}^{m^2})$ be the coefficients in the Hermite's basis of the function $f : \mathbb{R}^{m^2} \to \mathbb{R}$ such that

$$f(\mathbf{y}') = |\det(\mathbf{y}')|. \tag{4.8}$$

That is $f(\mathbf{y}') = \sum_{\boldsymbol{\beta} \in \mathbb{R}^{m^2}} f_{\boldsymbol{\beta}} \overline{\mathbf{H}}_{\boldsymbol{\beta}}(\mathbf{y}')$ with

$$f_{\boldsymbol{\beta}} = \frac{1}{\boldsymbol{\beta}!} \int_{\mathbb{R}^{m^2}} |\det(\mathbf{y}')| \overline{\mathbf{H}}_{\boldsymbol{\beta}}(\mathbf{y}') \phi_{m^2}(\mathbf{y}') d\mathbf{y}'.$$

Parseval's Theorem entails $||f||_2^2 = \sum_{q=0}^\infty \sum_{|\boldsymbol{\beta}|=q} f_{\boldsymbol{\beta}}^2 \boldsymbol{\beta}! < \infty$. Moreover, since the function f is even w.r.t. each column, the above coefficients are zero whenever $|\boldsymbol{\beta}_\ell|$ is odd for at least one $\ell = 1, \dots, m$.

Now, consider the coefficients in the Hermite's basis in $L^2(\mathbb{R}, \phi_1)$ for the Dirac delta $\delta_0(x)$. They are $b_{2j} = \frac{1}{\sqrt{2\pi}}(-\frac{1}{2})^j \frac{1}{j!}$, and zero for odd indices, see [15]. Introducing now the distribution $\prod_{j=1}^m \delta_0(y_j)$ and denoting by $b_{\boldsymbol{\alpha}}$ its coefficients it holds

$$b_{\boldsymbol{\alpha}} = \frac{1}{[\frac{\boldsymbol{\alpha}}{2}]!} \prod_{j=1}^m \frac{1}{\sqrt{2\pi}} \left[-\frac{1}{2} \right]^{[\frac{\alpha_j}{2}]}$$

or $b_{\boldsymbol{\alpha}} = 0$ if at least one index α_j is odd.

Proposition 4.1 ([2]). *Let the above notations prevail. We have, in the L^2 sense, that*

$$\bar{N}_d := \frac{N_{\mathbf{Y}_d} - 2d^{m/2}}{2d^{m/4}} = \sum_{q=1}^\infty I_{q,d},$$

where

$$I_{q,d} = \frac{d^{m/4}}{2} \int_{S^m} \sum_{|\boldsymbol{\gamma}|=q} c_{\boldsymbol{\gamma}} \tilde{\mathbf{H}}_{\boldsymbol{\gamma}}(\mathbf{Z}_d(t)) dt,$$

with $\boldsymbol{\gamma} = (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathbb{N}^m \times \mathbb{N}^{m^2}$, $|\boldsymbol{\gamma}| = |\alpha| + |\beta|$, $c_{\boldsymbol{\gamma}} = b_{\boldsymbol{\alpha}} f_{\boldsymbol{\beta}}$ and $\tilde{\mathbf{H}}_{\boldsymbol{\gamma}}(\mathbf{Z}_d(t)) = \mathbf{H}_{\boldsymbol{\alpha}}(\mathbf{Y_d(t)}) \overline{\mathbf{H}}_{\boldsymbol{\beta}}(\overline{\mathbf{Y}}'_{\mathbf{d}}(\mathbf{t}))$. □

In Lemma 6.1 $I_{q,d}$ is written as a stochastic integral with respect to the Brownian motion.

Remark 4.1. *A similar expansion holds for the number of roots $N_{\mathbf{Y}_d}(\mathcal{G})$ of $\mathbf{Y}_d$ on a Borel subset $\mathcal{G}$ of S^m. In fact, in order to obtain the expansion of $N_{\mathbf{Y}_d}(\mathcal{G})$ each factor in the integrand in Kac formula [6]*

$$N_{\mathbf{Y}_d}(\mathcal{G}) = \lim_{\delta \to 0} \int_{\mathcal{G}} |\det \mathbf{Y}'_d(t)| \cdot \frac{1}{(2\delta)^m} \mathbf{1}_{[-\delta,\delta]^m}(\mathbf{Y}_d(t)) dt,$$

is expanded in the Hermite basis. Then, one needs to take the sums out of the integral sign. We have

$$\bar{N}_d(\mathcal{G}) := \frac{N_{\mathbf{Y}_d}(\mathcal{G}) - \mathbb{E}\left(N_{\mathbf{Y}_d}(\mathcal{G})\right)}{d^{m/4}} = \sum_{q=1}^{\infty} I_{q,d}(\mathcal{G}),$$

with

$$I_{q,d}(\mathcal{G}) = \frac{d^{m/4}}{2} \int_{\mathcal{G}} \sum_{|\boldsymbol{\gamma}|=q} c_{\boldsymbol{\gamma}} \tilde{\mathbf{H}}_{\boldsymbol{\gamma}}(\mathbf{Z}_d(t))dt.$$

5. Proof of Theorem 1

Now, let us verify the conditions in Theorem 2. Define

$$G_q(\mathbf{z}) = \sum_{|\boldsymbol{\gamma}|=q} c_{\boldsymbol{\gamma}} \tilde{\mathbf{H}}_{\boldsymbol{\gamma}}(\mathbf{z}) \tag{5.1}$$

so that $I_{q,d} = \frac{d^{m/4}}{2} \int_{S^m} G_q(\mathbf{Z}_d(t))dt$. Mehler's formula, see Lemma 10.7 in [6], shows that $\mathbb{E}\left[\tilde{\mathbf{H}}_{\boldsymbol{\gamma}}(\mathbf{Z}_d(s))\tilde{\mathbf{H}}_{\boldsymbol{\gamma}'}(\mathbf{Z}_d(t))\right]$ can be written as a linear combination of powers of the covariances of the process $\mathbf{Z}_d$ which depend on s,t only through $\langle s,t\rangle$. Hence, we define

$$\mathcal{H}_{q,d}\left(\langle s,t\rangle\right) = \mathbb{E}\left(G_q(\mathbf{Z}_d(s))G_q(\mathbf{Z}_d(t))\right). \tag{5.2}$$

Lemma 6.2 in the Section 6.3 show that $\mathcal{H}_{q,d}$ is an even function.

5.1. Partial sums

In this section we prove points 1,2 and 3 in Theorem 2.
Point 1. We compute the variance of the term corresponding to a fixed q. We have

$$\mathrm{Var}(I_{q,d}) = \frac{d^{m/2}}{4}\mathrm{Var}\left(\int_{S^m} G_q(\mathbf{Z}_d(t))dt\right) = \frac{d^{m/2}}{4}\int_{S^m\times S^m} \mathcal{H}_{q,d}\left(\langle s,t\rangle\right)dsdt.$$

As above, the invariance under isometries of the distribution of $\mathbf{Z}_d$ and (4.2) we get

$$\begin{aligned}\mathrm{Var}(I_{q,d}) &= \kappa_m\kappa_{m-1}\frac{d^{m/2}}{4}\int_0^{\pi} \sin^{m-1}(\theta)\mathcal{H}_{q,d}\left(\cos(\theta)\right)d\theta \\ &= \frac{\kappa_m\kappa_{m-1}}{2}\int_0^{\sqrt{d}\pi/2} d^{(m-1)/2}\sin^{m-1}\left(\frac{z}{\sqrt{d}}\right)\mathcal{H}_{q,d}\left(\cos\left(\frac{z}{\sqrt{d}}\right)\right)dz.\end{aligned}$$

In the last equality we used the change of variables $\theta \mapsto \pi-\theta$ on the interval $[\pi/2,\pi]$, the scaling $\theta = z/\sqrt{d}$ and the fact that $\mathcal{H}_{q,d}$ is even, see Lemma 6.2.

The convergence follows by dominated convergence using for the covariances $\rho_{k,\ell} = \mathbb{E}\left(Z_k(s)Z_\ell(t)\right)$, see (4.4), the bounds in Lemma 4.1 and the domination for $\mathcal{H}_d = \sum_{q=0}^{\infty} \mathcal{H}_{q,d}$ given in (4.6).

POINT 2. Recall from (2.1) that

$$V_\infty = \lim_{d\to\infty} \frac{\mathrm{Var}(N_{\mathbf{Y}_d})}{d^{m/2}} = \lim_{d\to\infty} \sum_{q=0}^{\infty} \mathrm{Var}(I_{q,d}).$$

The second equality follows from Parseval's identity. Thus, by Fatou's Lemma

$$V = \sum_{q=0}^{\infty} V_q = \sum_{q=0}^{\infty} \lim_{d\to\infty} \mathrm{Var}(I_{q,d}) \le V_\infty < \infty.$$

Actually in Corollary 5.1 below the equality is established.

POINT 3. Next Lemma, which proof is postponed to Section 6.2, gives a sufficient condition on the covariances of the process $\mathbf{Z}_d$ in order to verify the convergence of the norm of the contractions. Recall that the law of the process is invariant under isometries, $r_d(s,t) = r_d(\langle s,t\rangle)$, thus, r_d can be seen as a function of one real variable.

Let $g_{q,d} \in L^2_s([0,\infty)^q)$ be such that $I_{q,d} = I^{\mathbf{B}}_q(g_{q,d})$, see Lemma 6.1.

Lemma 5.1. *For $k = 0, 1, 2$, let $r_d^{(k)}$ indicate the k-th derivative of $r_d : [-1,1] \to \mathbb{R}$. If*

$$d^{m/3} \int_0^{\pi/2} \sin^{m-1}(\theta) |r_d^{(k)}(\cos(\theta))| d\theta \underset{d\to+\infty}{\to} 0, \tag{5.3}$$

then, for $n = 1, \dots, q-1$:

$$\|g_{q,d} \otimes_n g_{q,d}\|_2 \to_{d\to\infty} 0.$$

Therefore, it suffices to verify (5.3). For $k = 0, 1, 2$ we have

$$\begin{aligned} d^{m/3} &\int_0^{\pi/2} \sin^{m-1}(\theta) |r_d^{(k)}(\cos(\theta))| d\theta \\ &= d^{m/3} \int_0^{\sqrt{d}\pi/2} \sin^{m-1}\left(\frac{z}{\sqrt{d}}\right) \left| r_d^{(k)}\left(\cos\left(\frac{z}{\sqrt{d}}\right)\right)\right| \frac{dz}{\sqrt{d}} \\ &= \frac{1}{d^{m/6}} \int_0^{\sqrt{d}\pi/2} d^{\frac{m-1}{2}} \sin^{m-1}\left(\frac{z}{\sqrt{d}}\right) \left| r_d^{(k)}\left(\cos\left(\frac{z}{\sqrt{d}}\right)\right)\right| dz. \end{aligned}$$

Now $d^{\frac{m-1}{2}} \sin^{m-1}\left(z/\sqrt{d}\right) \le z^{m-1}$ and taking the worst case in Lemma 4.1 we have $|r_d^{(k)}(z/\sqrt{d})| \le (1+z^2)\exp(-\alpha z^2)$. Hence, the last integral is convergent and (5.3) follows.

5.2. The tail

In this section we deal with Point 4 in Theorem 2. Let π_q be the projection on the q-th chaos $\mathcal{C}_q$ and $\pi^Q = \sum_{q\geq Q} \pi_q$ be the projection on $\oplus_{q\geq Q}\mathcal{C}_q$. We need to bound the following quantity uniformly in d

$$\frac{d^{m/2}}{4}\mathrm{Var}(\pi^Q(N_{\mathbf{Y}_d})) = \frac{1}{4}\sum_{q\geq Q} d^{m/2}\int_{S^m\times S^m}\mathcal{H}_{q,d}(\langle s,t\rangle)dsdt, \tag{5.4}$$

where $\mathcal{H}_{q,d}$ is defined in (5.2).

In order to bound this quantity we split the integral into two parts depending on the distance between s and t. The (geodesical) distance between $s,t\in S^m$ is defined as

$$\mathrm{dist}(s,t) = \arccos(\langle s,t\rangle). \tag{5.5}$$

We divide the integral into the integrals over the regions $\{(s,t) : \mathrm{dist}(s,t) < a/\sqrt{d}\}$ and its complement, a will be chosen later. We do this in the following two subsections.

5.2.1. Off-diagonal term

In this subsection we consider the integral in the rhs of (5.4) restricted to the off-diagonal region $\{(s,t) : \mathrm{dist}(s,t) \geq a/\sqrt{d}\}$. That is,

$$\frac{d^{m/2}}{4}\sum_{q\geq Q}\int_{\{(s,t):\mathrm{dist}(s,t)\geq a/\sqrt{d}\}}\mathcal{H}_{q,d}(\langle s,t\rangle)dsdt.$$

This is the easier case since the covariances of $\mathbf{Z}_d$ are bounded away from 1.

Before continuing our proof we need the following lemma from Arcones ([1], page 2245). Let X be a standard Gaussian vector on $\mathbb{R}^N$ and $h:\mathbb{R}^N\to\mathbb{R}$ a measurable function such that $\mathbb{E}[h^2(X)]<\infty$ and let us consider its L^2 convergent Hermite's expansion

$$h(x) = \sum_{q=0}^{\infty}\sum_{|\mathbf{k}|=q} h_{\mathbf{k}}H_{\mathbf{k}}(x).$$

The Hermite rank of h is defined as

$$\mathrm{rank}(h) = \inf\{\tau :\, \exists\,\mathbf{k}\,, |\mathbf{k}| = \tau\,;\mathbb{E}\,[(h(X)-\mathbb{E}\,h(X))H_{\mathbf{k}}(X)]\neq 0\}.$$

Then, we have

Lemma 5.2 ([1]). *Let $W = (W_1,\dots,W_N)$ and $Q = (Q_1,\dots,Q_N)$ be two mean-zero Gaussian random vectors on $\mathbb{R}^N$. Assume that*

$$\mathbb{E}[W_jW_k] = \mathbb{E}[Q_jQ_k] = \delta_{j,k},$$

for each $1\leq j,k\leq N$. We define

$$r^{(j,k)} = \mathbb{E}[W_jQ_k].$$

Let h be a function on $\mathbb{R}^N$ with finite second moment and Hermite rank τ, $1 \leq \tau < \infty$, define

$$\psi := \max \left\{ \max_{1\leq j\leq N} \sum_{k=1}^{N} |r^{(j,k)}|, \max_{1\leq k\leq N} \sum_{j=1}^{N} |r^{(j,k)}| \right\}.$$

Then

$$|\text{Cov}(h(W), h(Q))| \leq \psi^{\tau} \mathbb{E}[h^2(W)].$$

□

We apply this lemma for $N = m + m^2$, $W = \mathbf{Z}(s)$, $Q = \mathbf{Z}(t)$ and to the function $h(x) = G_q(x)$, defined in (5.1). Recalling that $\rho_{k,\ell}(s,t) = \rho_{k,\ell}(\langle s,t\rangle) = \mathbb{E}\left[Z_k(s)Z_\ell(t)\right]$, the Arcone's coefficient is now

$$\psi(s,t) = \max \left\{ \sum_{1\leq k\leq m+m^2} |\rho_{k,\ell}(s,t)|, \sum_{1\leq \ell\leq m+m^2} |\rho_{k,\ell}(s,t)| \right\}.$$

Thus

$$|\mathcal{H}_{q,d}(\langle s,t\rangle)| \leq \psi(\langle s,t\rangle)^q ||G_q||^2,$$

being $\|G_q\|^2 = \mathbb{E}\left(G_q^2(\zeta)\right)$ for standard normal ζ.

Lemma 5.3. *For f and G_q defined in (4.8) and (5.1) respectively, it holds*

$$||G_q||^2 \leq ||f||_2^2.$$

The proof is postponed to Section 6.3.

We move to the bound of Arcones' coefficient $\psi(\langle s,t\rangle)$. By the invariance of the distribution of $\mathbf{Y}_d$ (and $\mathbf{Z}_d$) under isometries we can fix $s = e_0$ and express t in hyperspheric coordinates (4.1), as above we have $\langle e_0, t\rangle = \cos(\theta)$. Direct computation of the covariances $\rho_{k\ell}$, see Section 6.3, yield that the maximum in the definition of ψ is $|\mathcal{C}| + |A|$, see (4.5). Lemma 4.1 entails that $|\mathcal{C}| + |\mathcal{A}| \leq e^{-\alpha z^2}(1+z)$. For $z = 2$ the bound takes the value $2e^{-4\alpha}$ which is less or equal to one if $\alpha \geq \frac{1}{4}\log 2$, this is always possible because the only restriction that we have is $\alpha < \frac{1}{2}$. Moreover, for δ small enough $e^{-\alpha z^2}(1+z) \geq 1$ if $z < \delta$. This leads to affirm that there exists a $a < 2$ such that for all $z \geq a$ it holds $\mathcal{C} + \mathcal{A} < r_0 < 1$.

These results allow to use the Arcones' result to obtain

$$\begin{aligned}
&\sup_d \sum_{q\geq Q} \frac{d^{m/2}}{4} \int_{\left\{(s,t):\text{dist}(s,t)\geq \frac{a}{\sqrt{d}}\right\}} \mathcal{H}_{q,d}(\langle s,t\rangle) ds dt \\
&\quad = \sup_d \frac{C_m}{4} \left| \sum_{q\geq Q} d^{\frac{m-1}{2}} \int_a^{\sqrt{d}\pi} \sin^{m-1}\left(\frac{z}{\sqrt{d}}\right) \mathcal{H}_d^q \left(\cos\left(\frac{z}{\sqrt{d}}\right)\right) dz \right| \\
&\qquad\qquad \leq C_m ||f||_2^2 \sum_{q\geq Q} r_0^{q-1} \int_a^{\infty} z^{m-1}(1+z)e^{-\alpha z^2} dz \underset{Q\to\infty}{\to} 0.
\end{aligned}$$

5.2.2. Diagonal term

In this subsection we prove that the integral in the rhs of (5.4) restricted to the diagonal region $\{(s,t) : \operatorname{dist}(s,t) < a/\sqrt{d}\}$ tends to 0 as $Q \to \infty$ uniformly in d, $a < 2$ is fixed. That is, we consider

$$\frac{d^{m/2}}{4} \sum_{q \geq Q} \int_{\{(s,t):\operatorname{dist}(s,t)<a/\sqrt{d}\}} \mathcal{H}_{q,d}(\langle s,t\rangle) ds dt.$$

This is the difficult part, we use an indirect argument.

Next proposition gives a convenient partition of the sphere based on the hyperspheric coordinates (4.1). Define the hyperspherical rectangle (HSR for short) with center $x^{(m)}(\tilde{\theta})$ with $\tilde{\theta} = (\tilde{\theta}_1, \ldots, \tilde{\theta}_m)$ and vector radius $\tilde{\eta} = (\tilde{\eta}_1, \ldots, \tilde{\eta}_m)$ as

$$HSR(\tilde{\theta}, \tilde{\eta}) = \{x^{(m)}(\theta) : |\theta_i - \tilde{\theta}_i| < \tilde{\eta}_i, i = 1, \ldots, m\}.$$

Let $T_t S^m$ be the the tangent space to S^m at t. This space can be identified with $t^\perp \subset \mathbb{R}^{m+1}$. Let $\phi_t : S^m \to t^\perp$ be the orthogonal projection over $t^\perp$. The details and the proof are presented in Section 6.1.

Proposition 5.1. *For d large enough, there exists a partition of the unit sphere S^m into HSRs $R_j : j = 1, \ldots, k(m,d) = O(d^{m/2})$ and an extra set E such that*

1. $\operatorname{Var}(N_{\mathbf{Y}_d}(E)) = o(d^{m/2})$.
2. *The HSRs R_j have diameter $O(\frac{1}{\sqrt{d}})$ and if R_j and R_ℓ do not share any border point (they are not neighbours), then* $\operatorname{dist}(R_j, R_\ell) \geq \frac{1}{\sqrt{d}}$.
3. *The projection of each of the sets R_j on the tangent space at its center c_j, after normalizing by the multiplicative factor $\sqrt{d}$, converges to the rectangle $[-1/2, 1/2]^m$ in the sense of Hausdorff metric. That is, the Hausdorff metric of*
$$\left[-\frac{1}{2}, \frac{1}{2}\right]^m \setminus \sqrt{d}\, \phi_{c_j}(R_j)$$

tends to 0 *as* $d \to \infty$.

Set $r = d^{-1/2}$. We can write $S^m = \bigcup_{j=1}^{k(m,r)} R_j \cup E$, and

$$\pi^Q(N_{\mathbf{Y}_d}) = \sum_{q \geq Q} \int_{S^m} G_q(\mathbf{Z}_d(t)) dt$$
$$= \sum_{q \geq Q} \left[\sum_j \int_{R_j} G_q(\mathbf{Z}_d(t)) dt + \int_E G_q(\mathbf{Z}_d(t)) dt \right].$$

By the first item in Proposition 5.1 and (5.2) we have

$$\mathbb{E}\left[(\pi^Q(N_{\mathbf{Y}_d}))^2\right] = \sum_j \sum_\ell \sum_{q \geq Q} \int_{R_j} \int_{R_\ell} \mathcal{H}_{q,d}(\langle s,t\rangle) ds dt + o(d^{m/2}).$$

Actually, in this section we are interested in covering a strip around the diagonal $\{(s,t) \in S^m \times S^m : \mathrm{dist}(s,t) < ar\}$, $a < 2$. Hence, we restrict the sum in the last equation to the set $\{(j,\ell) : |j-\ell| \le 2\}$. Clearly the number of sets verifyng this condition is $O(r^{-1}) = O(\sqrt{d})$ and below we prove that the tail of the variance of $N_{\mathbf{Y}_d}(R_j)/d^{m/2}$ is uniformly in j $O(d^{-m/2})$. Therefore, it remains to bound

$$\sum_{(j,\ell):|j-\ell|<2} \int_{R_j} \int_{R_\ell} \sum_{q\ge Q} \mathcal{H}_{q,d}(\langle s,t\rangle) ds dt$$

$$= \sum_{(j,\ell):|j-\ell|<2} \mathbb{E}\left[\int_{R_j} \sum_{q\ge Q} G_q(\mathbf{Z}_d(s)) ds \cdot \int_{R_\ell} \sum_{q\ge Q} G_q(\mathbf{Z}_d(t)) dt \right]$$

$$\le \sum_{(j,\ell):|j-\ell|<2} \left[\sum_{q\ge Q} \int_{R_j\times R_j} \mathcal{H}_{q,d}(\langle s,t\rangle) ds dt \right]^{1/2} \left[\sum_{q\ge Q} \int_{R_\ell\times R_\ell} \mathcal{H}_{q,d}(\langle s,t\rangle) ds dt \right]^{1/2}.$$

Here we used Cauchy-Schwarz inequality. Fix j, in order to bound

$$\sum_{q\ge Q} \int_{R_j\times R_j} \mathcal{H}_{q,d}(\langle s,t\rangle) ds dt,$$

we note that it coincides with $\mathrm{Var}\left(\pi^Q\left(N_{\mathbf{Y}_d}(R_j)\right)\right)$.

On the other hand, we prove hereunder that there exist some local limit for $\mathbf{Y}_d$ as $d \to \infty$.

At this point it is convenient to work with caps

$$C(s_0, \gamma r) = \{s : d(s, s_0) < \gamma r\}.$$

Note that by the second item in Proposition 5.1 each HSR R_j is included in a cap of radius γr for some γ depending on m.

By the invariance under isometries of the distribution of $\mathbf{Y}_d$, the distribution of the number of roots on a cap $C(s_0, \gamma r)$ does not depend on its center s_0. Thus, without loss of generality we work with the cap of angle γr centered at the east-pole $e_0 = (1, 0, \ldots, 0)$, see Nazarov-Sodin [23]

$$C(e_0, \gamma r) = \{t \in S^m : \mathrm{dist}(t, e_0) < \gamma r\}.$$

We use the local chart $\phi : C(e_0, \gamma r) \to B(0, \sin(\gamma r)) \subset \mathbb{R}^m$ defined by

$$\phi^{-1}(u) = (\sqrt{1 - \|u\|^2}, u), \quad u \in B(0, \sin(\gamma r)),$$

to project this set over the tangent space . Define the random field $\mathcal{Y}_d : B(0,\gamma) \subset \mathbb{R}^m \to \mathbb{R}^m$, as

$$\mathcal{Y}_d(u) = \mathbf{Y}_d(\phi^{-1}(u/r)).$$

Observe that the ℓ coordinates, $\mathcal{Y}_d^{(\ell)}$ say, of $\mathcal{Y}_d$ are independent. Clearly the number of roots of $\mathbf{Y}_d$ on $R \subset C(e_0, \gamma r)$ and the number of roots of $\mathcal{Y}_d$ on $\phi(R/r) \subset B(0, \gamma)$ coincide. That is

$$N_{\mathbf{Y}_d}(R) = N_{\mathcal{Y}_d}(\phi(R/r)).$$

Proposition 5.2. *The sequence of processes $\mathcal{Y}_d^{(\ell)}(u)$ and its first and second order derivatives converge in the finite dimensional distribution sense towards the mean zero Gaussian processes $\mathcal{Y}_\infty$ with covariance function $\Gamma(u,v) = e^{-\frac{||u-v||^2}{2}}$ and its corresponding derivatives.*

Proof. We give a short proof for completeness, see also [23]. The covariance of $Y_\ell(s)$ and $Y_\ell(t)$ is $\langle s,t\rangle^d$, whenever $s,t \in S^m$. In this form we get for $\phi^{-1}(u), \phi^{-1}(v) \in S^m$

$$\left\langle \phi^{-1}(u), \phi^{-1}(v)\right\rangle = \sum_{i=1}^m u_i v_i + \sqrt{1-||u||^2}\sqrt{1-||v||^2}.$$

Using the rescaling we have

$$\left\langle \phi^{-1}\left(\frac{u}{\sqrt{d}}\right), \phi^{-1}\left(\frac{v}{\sqrt{d}}\right)\right\rangle = \frac{1}{d}\sum_{i=1}^m u_i v_i + \sqrt{h}j\sqrt{1-\left\|\frac{v}{\sqrt{d}}\right\|^2}.$$

The Taylor development for $\sqrt{1-x^2}$ gives

$$\begin{aligned}\mathbb{E}\left(\mathcal{Y}_d^{(\ell)}(u)\,\mathcal{Y}_d^{(\ell)}(v)\right) &= \left\langle \phi^{-1}\left(\frac{u}{\sqrt{d}}\right), \phi^{-1}\left(\frac{v}{\sqrt{d}}\right)\right\rangle^d \\ &= \left(1 - \frac{||u-v||^2}{2d} + O\left(\frac{||u||^4}{d^2} + \frac{||v||^4}{d^2}\right)\right)^d \underset{d\to\infty}{\to} e^{-\frac{||u-v||^2}{2}}.\end{aligned}$$

The convergence is uniform over compacts, thus the partial derivatives of this function also converge, towards the derivatives of the limit covariance. Then the claimed result holds in force. □

Remark 5.1. *Using classical criteria on the fourth moment of increments, the weak convergence in the space on continuous functions can be proved. But we do not need it in the sequel.*

Remark 5.2. *The local limit process $\mathcal{Y}_\infty$ has as coordinates $(\mathcal{Y}_\infty^{(1)}, \dots, \mathcal{Y}_\infty^{(m)})$ each of one is an independent copy of the random field with covariance $\Gamma(u) = e^{-\frac{||u||^2}{2}}$, $u \in \mathbb{R}^m$. Then its covariance matrix writes*

$$\tilde{\Gamma}(u) = \operatorname{diag}(\Gamma(u), \dots, \Gamma(u)).$$

The second derivative matrix $\tilde{\Gamma}''(u)$ can be written in a similar way, but here the blocks are equal to the matrix $\Gamma''(u) = (a_{ij})$ where $a_{ij} = e^{-\frac{||u||^2}{2}}H_1(u_i)H_1(u_j)$ if $i \neq j$, and $a_{ii} = e^{-\frac{||u||^2}{2}}H_2(u_i)$. We can adapt the Estrade and Fournier [11] result that says that the second moment of the roots in a compact set of such a process exists if for some $\delta > 0$ we have

$$\int_{B(0,\delta)} \frac{||\tilde{\Gamma}''(u) - \tilde{\Gamma}''(0)||}{||u||^m} du = m\int_{B(0,\delta)} \frac{||\Gamma''(u) - I||}{||u||^m} du < \infty.$$

Since Γ is C^∞ we have

$$||\Gamma''(0) - \Gamma''(u)|| = o(||u||) \text{ as } u \to 0,$$

The convergence of the above integral follows easily using hyperspherical coordinates.

A key fact is that the local limit process, though it can not be defined globally, has the same distribution regardless j. Thus, we bound $\operatorname{Var}\left(\pi^Q\left(N_{\mathbf{Y}_d}(R_j)\right)\right)$ uniformly in j by approximating it with the tail of the variance of the number of zeros of the limit process $\mathcal{Y}_\infty$ on the limit set $[-1/2, 1/2]^m$.

Proposition 5.3. *For all $j \le k(m,d)$ and $\varepsilon > 0$ there exist d_0 and Q_0 such that for $Q \ge Q_0$*

$$\sup_{d > d_0} \mathbb{E}\,[(\pi^Q(N_{\mathbf{Y}_d}(R_j)))^2] < \varepsilon.$$

Proof. Let $R = R_j \subset C(e_0, \gamma r)$, By Remark 4.1, the Hermite expansion holds true also for the number of roots of $\mathbf{Y}_d$ on any subset of S^m. Hence,

$$N_{\mathbf{Y}_d}(R) = \sum_{q=0}^{\infty} d^{\frac{m}{2}} \int_R G_q(\mathbf{Z}_d(t))dt.$$

Let us define $\tilde{R} = \phi(R) \subset B(0, \sin\frac{a}{\sqrt{d}}) \subset \mathbb{R}^m$. It follows that

$$N_{\mathcal{Y}_d}(\tilde{R}) = N_{\mathbf{Y}_d}(R) = \sum_{q=0}^{\infty} d^{\frac{m}{2}} \int_{\tilde{R}} G_q(\mathcal{Y}_d(u), \mathcal{Y}_d'(u))J_\phi(u)du,$$

being $J_\phi(u) = (1 - \|u\|^2)^{-1/2}$ the jacobian. Rescaling $u = v/\sqrt{d}$

$$N_{\mathcal{Y}_d}(\tilde{R}) = \sum_{q=0}^{\infty} \int_{\sqrt{d}\tilde{R}} G_q\left(\mathcal{Y}_d\left(\frac{v}{\sqrt{d}}\right), \mathcal{Y}_d'\left(\frac{v}{\sqrt{d}}\right)\right) J_\phi\left(\frac{v}{\sqrt{d}}\right) dv.$$

Besides, Rice formula, the domination for $\mathcal{H}_{q,d}$ given in (4.6), the convergence of $\mathcal{Y}_d$ to $\mathcal{Y}_\infty$ in Proposition 5.2 and the convergence, after normalization, of $\bar{R}$ to $[-1/2, 1/2]^m$ in Proposition 5.1 yield

$$\operatorname{Var}(N_{\mathcal{Y}_d}(\tilde{R})) \underset{d\to\infty}{\to} \operatorname{Var}\left(N_{\mathcal{Y}_\infty}\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^m\right)\right). \tag{5.6}$$

In fact, writting $\operatorname{Var}(N) = \mathbb{E}\,(N(N-1)) - (\mathbb{E}\,(N))^2 + \mathbb{E}\,(N)$, for the first term we have

$$\begin{aligned}
&\mathbb{E}\left[N_{\mathbf{Y}_d}(R)(N_{\mathbf{Y}_d}(R)-1)\right]\\
&= d^m \int_{\tilde{R}\times\tilde{R}} \mathbb{E}\left[|\det \mathcal{Y}_d'(u)||\det \mathcal{Y}_d'(v)|\,|\mathcal{Y}_d(u)=\mathcal{Y}_d(v)=0\right] p_{u,v}(0,0) J_\phi(u) J_\phi(v)\,du dv\\
&= \int_{\sqrt{d}\tilde{R}\times\sqrt{d}\tilde{R}} \mathbb{E}\left[\left|\det \mathcal{Y}_d'\left(\frac{u}{\sqrt{d}}\right)\right|\left|\det \mathcal{Y}_d'\left(\frac{v}{\sqrt{d}}\right)\right| \mid \mathcal{Y}_d\left(\frac{u}{\sqrt{d}}\right)=\mathcal{Y}_d\left(\frac{v}{\sqrt{d}}\right)=0\right]\\
&\qquad\qquad \cdot p_{\frac{u}{\sqrt{d}},\frac{v}{\sqrt{d}}}(0,0) J_\phi\left(\frac{u}{\sqrt{d}}\right) J_\phi\left(\frac{v}{\sqrt{d}}\right) du dv\\
&\underset{d\to\infty}{\to} \int_{\left[\frac{1}{2},\frac{1}{2}\right]^m\times\left[\frac{1}{2},\frac{1}{2}\right]^m} \mathbb{E}\left[|\det \mathcal{Y}_\infty'(u)||\det \mathcal{Y}_\infty'(v)| \mid \mathcal{Y}_\infty(u)=\mathcal{Y}_\infty(v)=0\right]\\
&\qquad\qquad \cdot p_{\mathcal{Y}_\infty(u),\mathcal{Y}_\infty(v)}(0,0)\,du dv\\
&\qquad = \mathbb{E}\left[N_{\mathcal{Y}_\infty}\left(\left[\frac{1}{2},\frac{1}{2}\right]^m\right)\left(N_{\mathcal{Y}_\infty}\left(\left[\frac{1}{2},\frac{1}{2}\right]^m\right)-1\right)\right] < \infty.
\end{aligned}$$

The remaining terms are easier.

The same arguments show that for all q we have

$$V_{q,d} := \mathrm{Var}(\pi_q(N_{\mathbf{Y}_d}(R))) \underset{d\to\infty}{\to} \mathrm{Var}\left(\pi_q\left(N_{\mathcal{Y}_\infty}\left(\left[-\frac{1}{2},\frac{1}{2}\right]^m\right)\right)\right) =: V_q.$$

Thus, for all Q it follows that $\sum_{q=0}^{Q} V_{q,d} \to_{d\to\infty} \sum_{q=0}^{Q} V_q$. By Parseval's identity, (5.6) can be written as

$$\sum_{q=0}^{\infty} V_{q,d} \underset{d\to\infty}{\to} \sum_{q=0}^{\infty} V_q.$$

Thus, by taking the difference we get

$$\sum_{q>Q} V_{q,d} \underset{d\to\infty}{\to} \sum_{q>Q} V_q. \tag{5.7}$$

Given that the series $\sum_{q=0}^{\infty} V_q$ is convergent, we can choose Q_0 such that for $Q \geq Q_0$ it holds $\sum_{q>Q}^{\infty} V_q \leq \varepsilon/2$. Hence, for this Q_0 and by using (5.7) we can choose d_0 such that for all $d > d_0$ and $Q \geq Q_0$

$$\sum_{q>Q} V_{d,q} \leq \varepsilon.$$

Namely, there exits d_0 such that for $Q \geq Q_0$

$$\sup_{d>d_0} \mathbb{E}\left[(\pi^Q(N_{\mathbf{Y}_d}(R))^2\right] < \varepsilon.$$

The same can be written in the following form: there exists $D(Q)$ a sequence that tends to zero when $Q \to \infty$, such that

$$\sup_{d>d_0} \mathbb{E}\left[(\pi^Q(N_{\mathbf{Y}_d}(R))^2\right] < D(Q).$$

□

Corollary 5.1. *It follows that*

$$V=\sum_{q=0}^{\infty}V_q=V_\infty.$$

See Point 2 in Theorem 2.

6. Technical Proofs

6.1. The partition of the sphere

In this section we describe a convenient essential partition of the unit sphere S^m of $\mathbb{R}^{m+1}$. We use hyperspheric coordinates (4.1) and two speeds

$$r=\frac{1}{\sqrt{d}}\text{ and }\bar{r}=r^{\alpha},\ 0<\alpha<\frac{1}{m}. \tag{6.1}$$

We suppose that r is sufficiently small so that

$$\sin\left(\frac{\bar{r}}{2}\right)\geq\frac{\bar{r}}{4},\quad r\leq\frac{\bar{r}}{2}.$$

Step 1 We begin with θ_1. Set $r_1=r$. Let a be the minimal integer such that $\pi/2-ar_1\leq\bar{r}$. Thus, the segment $[\pi/2-ar_1,\pi/2+ar_1]$ is cut into $2a$ sub-intervals I_{1,i_1}, with centers θ_{1,i_1}, $i_1=1,\dots 2a$. Hence,

$$\{\theta_{1,i_1}:i=1,\dots,2a\}\subset\left[\frac{\bar{r}}{2},\pi-\frac{\bar{r}}{2}\right]. \tag{6.2}$$

Step 2 Depending on the interval I_{1,i_1} in which is located θ_1 we set

$$r_{2,i_1}=\frac{r}{\sin(\theta_{1,i_1})}.$$

Note that because of (6.2) uniformly over all possible values of θ_{1,i_1}

$$\sin(\theta_{1,i_1})\geq\frac{\bar{r}}{4},$$

implying that

$$r_{2,i_1}\leq 4r^{1-\alpha}.$$

We impose again r to be sufficiently small such that $r_{2,i_1}\leq\bar{r}/2$, this is possible by the last inequality and (6.1).

We then decompose the interval of variation of θ_2, $[0,\pi)$, in the same manner using r_{2,i_1} instead of r_1. The intervals are now denoted by I_{2,i_1,i_2}, their centers by θ_{2,i_1,i_2} and their number by a_{2,i_1}.

Step j For $\theta_1\in I_{1,i_1}$, $\theta_2\in I_{2,i_1,i_2}$, $\dots$, $\theta_{j-1}\in I_{j-1,i_1,i_2,\dots,i_{j-1}}$ we set

$$r_{j,i_1,\dots,i_{j-1}}=\frac{r_{j-1,i_1,\dots,i_{j-2}}}{\sin(\theta_{j-1,i_1,i_2,\dots,i_{j-1}})}=\frac{r}{\sin(\theta_{1,i_1})\dots\sin(\theta_{j-1,i_1,i_2,\dots,i_{j-1}})}.$$

By construction, all the sinus in the denominator are greater than $\bar{r}/4$ implying that
$$r_{j,i_1,\ldots,i_{j-1}} \leq 4^{j-1} r^{1-(j-1)\alpha}.$$
We impose again r to be sufficiently small such that $r_{j,i_1,\ldots,i_{j-1}} \leq \bar{r}/2$.

Then we cut again the interval $[0,\pi)$ into sub-intervals in the same manner. The intervals are now denoted by $I_{j,i_1,i_2,\ldots,i_j}$, their centers by $\theta_{j,i_1,i_2,\ldots,i_j}$ and their number by $a_{j,i_1,\ldots i_{j-1}}$.

Step m The last step differs on two points: first we divide the interval $[0,2\pi]$, and second we divide it entirely except rounding problems.

The exceptional set E of the sphere not covered by the sets above is included in the set
$$\bigcup_{i=1}^{m} x^{(m)}\left(\{\theta_i \in [0,\bar{r}] \cup [\pi-\bar{r},\pi]\}\right).$$
Excluding E, we have made an essential partition of the sphere in hyperspherical rectangles (HSR) of the type
$$R(i_1,\ldots,i_m) = \{\theta_1 \in I_{1,i_1}, \theta_2 \in I_{2,i_1,i_2}, \ldots \theta_m \in I_{m,i_1,i_2,\ldots,i_m}\}$$

We are in position to prove Proposition 5.1.

Proof of Proposition 5.1. 1. Since $E \subset \bigcup_{i=1}^{m} x^{(m)}\left(\{\theta_i \in [0,\bar{r}] \cup [\pi-\bar{r},\pi]\}\right)$, its m-dimensional volume $\mathrm{Vol}(E) = O(\bar{r})$ tends to zero.

Using Lemma 4.2
$$\mathrm{Var}(N_{\mathbf{Y}_d}(E)) = o(d^{m/2}).$$

2. Recall that we are using the geodesical distance (5.5). Let $|(i_1,\ldots,i_m)| = i_1 + \ldots + i_m$. We want to prove that If $|(i_1,\ldots,i_m) - (i'_1,\ldots,i'_m)| \geq 2$ then $\mathrm{dist}(R(i_1,\ldots,i_m), R(i'_1,\ldots,i'_m)) \geq r$.

Let us compute the inner product for $\theta' = \theta + \gamma\bar{r}_k e_k$.
$$\begin{aligned}
\left\langle x^{(m)}(\theta), x^{(m)}(\theta')\right\rangle &= 1 + \left\langle x^{(m)}(\theta), x^{(m)}(\theta') - x^{(m)}(\theta)\right\rangle \\
&= 1 + \frac{\gamma^2}{2}\bar{r}_k^2 \left\langle x^{(m)}(\theta), \partial_k^2 x^{(m)}(\theta)\right\rangle + O(\bar{r}_k^2) \\
&= 1 - \frac{\gamma^2}{2}\bar{r}_k^2 \prod_{j=1}^{k-1} \|x^{(m-k)}(\theta_{k+1},\ldots,\theta_m)\|^2 + O(\bar{r}_k^2) \\
&= 1 - \frac{\gamma^2}{2} r^2 + O(\bar{r}_k^2).
\end{aligned}$$

The result follows.

3. Consider a fixed HSR $R(i_1,\dots,i_m)$ with center $\left(\theta_{1,i_1},\dots,\theta_{m,i_1,i_2,\dots,i_m}\right)$ and side lenghts $r_1,\dots,r_{m,i_1,\dots,i_m}$. For short we write $\bar\theta=(\bar\theta_1,\dots,\bar\theta_m)$ for the center and $\bar r_1,\dots,\bar r_m$ for the side lengths.

The generic coordinates of a point on the HSR are

$$\theta=\left(\bar\theta_1+u_1\bar r_1,\dots,\bar\theta_m+u_m\bar r_m\right);\quad \mathbf{u}=(u_1,\dots,u_m)\in\left[-\frac{1}{2},\frac{1}{2}\right]^m.$$

The corresponding cartesian coordinates $x^{(m)}(\theta)$, $x^{(m)}(\bar\theta)$ are given by (4.1). The tangent space is computed by differentiating in (4.1) with respect to $u_1,\dots,u_m$. An orthonormal basis is given by

$$T_k=\begin{pmatrix} 0_{k-1}\\ -\sin(\bar\theta_k)\\ \cos(\bar\theta_k)x^{(m-k)}(\bar\theta_{k+1},\dots,\bar\theta_m)\end{pmatrix};\ k=1,\dots,m.$$

Here, 0_{k-1} stands for a vector of $k-1$ zeros (this is ommited when $k=1$). The projection ϕ can be computed easily on this basis taking inner products. Let us perform one of these inner products, the rest are similar.

$$\left\langle x^{(m)}(\theta)-x^{(m)}(\bar\theta),T_k\right\rangle=-\sin(\bar\theta_k)\Delta_k+\cos(\bar\theta_k)\left\langle x^{(m-k)}(\bar\theta_{k+1},\dots,\bar\theta_m),\bar\Delta_k\right\rangle,$$

with

$$\Delta_k=\prod_{j=1}^{k-1}\sin(\theta_j)\cdot\cos(\theta_k)-\prod_{j=1}^{k-1}\sin(\bar\theta_j)\cdot\cos(\bar\theta_k)$$

and

$$\bar\Delta_k=\prod_{j=1}^{k}\sin(\theta_j)\cdot x^{(m-k)}(\theta_{k+1},\dots,\theta_m)-\prod_{j=1}^{k}\sin(\bar\theta_j)\cdot x^{(m-k)}(\bar\theta_{k+1},\dots,\bar\theta_m).$$

Since the sine and the cosine functions have bounded second derivative for $i=1,\dots,m$

$$\begin{aligned}\sin(\theta_i)&=\sin(\bar\theta_i)+\bar r_iu_i\cos(\bar\theta_i)+O(\bar r_i^2),\\ \cos(\theta_i)&=\cos(\bar\theta_i)-\bar r_iu_i\sin(\bar\theta_i)+O(\bar r_i^2).\end{aligned}$$

The implicit constants in the O notation do not depend on the indexes i_j. Hence, the dominant terms in the differences Δ_k and $\bar\Delta_k$ are those where only one of the θ_j differs from $\bar\theta_j$, the rest are of higher order as $r\to 0$.

Recall that the construction has been performed in such a way that the quantities $\bar r_1,\dots,\bar r_m$ tend to zero uniformly as r tends to 0. Let us study first the case of $\theta_j = \bar{\theta}_j$ for $j \neq k$, in this case

$$\begin{aligned}\left\langle x^{(m)}(\theta) - x^{(m)}(\bar{\theta}), T_k \right\rangle &= -\sin(\bar{\theta}_k) \prod_{j=1}^{k-1} \sin(\bar{\theta}_j) \cdot (-\bar{r}_k u_k \sin(\bar{\theta}_k)) \\ &\quad + \cos(\bar{\theta}_k) \prod_{j=1}^{k-1} \sin(\bar{\theta}_j) \cdot (\bar{r}_k u_k \cos(\bar{\theta}_k)) \\ &\quad \cdot \left\langle x^{(m-k)}(\bar{\theta}_{k+1}, \dots, \bar{\theta}_m), x^{(m-k)}(\bar{\theta}_{k+1}, \dots, \bar{\theta}_m) \right\rangle + O(r^2) \\ &= \prod_{j=1}^{k-1} \sin(\bar{\theta}_j) \cdot \bar{r}_k \cdot u_k + O(r^2) = r u_k + O(r^2).\end{aligned}$$

Here we used that by construction $\prod_{j=1}^{k-1} \sin(\bar{\theta}_j) \cdot \bar{r}_k = r$. By the same arguments one can show that in the case $\theta_k = \bar{\theta}_k$ the terms of the difference are uniformly $o(r)$. Hence, uniformly

$$\frac{1}{r} \left\langle x^{(m)}(\theta) - x^{(m)}(\bar{\theta}), T_k \right\rangle \underset{r \to 0}{\to} u_k.$$

Consequently, since we have a finite number of coordinates, the result on the convergence in the Hausdorff metric follows. □

6.2. Chaotic expansions and contractions

In this section we write the chaotic components $I_{q,d}$ in Proposition 4.1 as multiple stochastic integrals wrt a standard Brownian motion $\mathbf{B}$ and use this fact in order to give a simple sufficient condition in terms of the covariance of $\mathbf{Y}_d$ to guarantee the convergence to 0 of the contractions. For the sake of readability we omit d in the kernels.

Let $\mathbf{B} = \{B(\lambda) : \lambda \in [0, \infty)\}$ be a standard Brownian motion on $[0, \infty)$. By the isometric property of stochastic integrals there exist kernels $h_{t,\ell}$ such that:

$$Y_\ell(t) = \int_0^\infty h_{t,\ell}(\lambda) dB(\lambda). \tag{6.3}$$

The kernels $h_{t,\ell}$ can be computed explicitly from the definition of Y_ℓ writting the random coefficients as integrals wrt the Brownian motion. Taking derivatives (on the sphere), we have that the standardized derivatives can be written as

$$\overline{Y}'_{\ell,k} = \int_0^\infty \bar{h}_{t,\ell,k}(\lambda) dB(\lambda),$$

with

$$\bar{h}_{t,\ell}(\lambda) = \frac{1}{\sqrt{d}} \nabla h_{t,\ell}(\lambda). \tag{6.4}$$

We quickly recall some properties of contractions (4.7) and multiple stochastic integrals, see [24] for details. Note that for $f, g \in L^2([0,\infty)^q)$, $f \otimes_0 g = f \otimes g$ is the tensorial product and $f \otimes_q g = \langle f, g\rangle$ is the inner product in $L^2_s([0,\infty)]^q)$. Besides, if $f = \bar{f}^{\otimes q}$ and $g = \bar{g}^{\otimes q}$, then, $f \otimes_n g = \langle \bar{f}, \bar{g}\rangle^n \bar{f}^{\otimes q-n} \otimes \bar{g}^{\otimes q-n}$ where this time the inner product is in $L^2([0,\infty))$.

Let $h \in L^2([0,\infty))$ and $I_1^{\mathbf{B}}(h) = \int_0^\infty h(\lambda) dB(\lambda)$, then $H_q(I_1^{\mathbf{B}}(h)) = I_q^{\mathbf{B}}(h^{\otimes q})$ where $I_q^{\mathbf{B}}$ is the q-folded multiple stochastic integral wrt $\mathbf{B}$ and H_q the q-th Hermite polynomial. A key property of stochastic integrals is multiplication formula, for $f \in L^2_s([0,\infty)^p)$, $g \in L^2_s([0,\infty)^q)$,

$$I_p^{\mathbf{B}}(f) I_q^{\mathbf{B}}(g) = \sum_{n=0}^{\min\{p,q\}} n! \binom{p}{n} \binom{q}{n} I_{p+q-2n}^{\mathbf{B}}(f \otimes_n g).$$

Note that if $f = \bar{f}^{\otimes p}$, $g = \bar{g}^{\otimes q}$ and $\bar{f}, \bar{g}$ are orthogonal in $L^2([0,\infty))$, then, $I_p^{\mathbf{B}}(f) I_q^{\mathbf{B}}(g) = I_{p+q}^{\mathbf{B}}(f \otimes g)$. Finally, let us mention that if $f \in L^2([0,\infty)^q)$ then $I_q^{\mathbf{B}}(f) = I_q^{\mathbf{B}}(\tilde{f})$ being $\tilde{f}$ the symmetrization of f, that is,

$$\tilde{f}(\lambda_1, \ldots, \lambda_q) = \frac{1}{q!} \sum_{\pi \in \mathcal{P}_q} f(\lambda_{\pi(1)}, \ldots, \lambda_{\pi(q)}),$$

being $\mathcal{P}_q$ the symmetric group of order q.

Next Lemma expresses $I_{q,d}$ as a multiple stochastic integral wrt $\mathbf{B}$.

Lemma 6.1. *Let the notations and assumptions of Proposition 4.1 prevail. Then, $I_{q,d}$ can be written as a multiple stochastic integral*

$$I_{q,d} = I_q^{\mathbf{B}}(g_{q,d}) = \int_{[0,\infty)^q} g_{q,d}(\boldsymbol{\lambda}) dB(\boldsymbol{\lambda});$$

with

$$g_{q,d}(\boldsymbol{\lambda}) = d^{m/4} \sum_{|\boldsymbol{\gamma}|=q} c_{\boldsymbol{\gamma}} \int_{S^m} (\otimes_{\ell=1}^m h_{t,\ell}^{\otimes \alpha_\ell} \otimes \otimes_{k=1}^m \bar{h}_{t,\ell,k}^{\otimes \beta_{\ell,k}})(\boldsymbol{\lambda}) dt,$$

where $h_{t,\ell}$ is defined in (6.3), $\bar{h}_{t,\ell,k}$ is the k-th component of $\bar{h}_{t,\ell}$ given in (6.4).

Proof. Plug in the expressions for Y_ℓ and its derivatives in the formula for $I_{q,d}$ given in Proposition 4.1; use the relation between Hermite polynomials and stochastic integrals; the multiplication formula, the orthogonality of $Y_\ell, \overline{Y}'_{\ell,k}$

and the Fubini stochastic Theorem. Namely,

$$\begin{aligned}
I_{q,d} &= d^{m/4} \sum_{|\boldsymbol{\gamma}|=q} c_{\boldsymbol{\gamma}} \int_{S^m} \prod_{\ell=1}^{m} \left[H_{\alpha_\ell}(I_1^{\mathbf{B}}(h_{t,\ell})) \prod_{k=1}^{m} H_{\beta_{\ell,k}}(I_1^{\mathbf{B}}(\bar{h}_{t,\ell,k})) \right] dt \\
&= d^{m/4} \sum_{|\boldsymbol{\gamma}|=q} c_{\boldsymbol{\gamma}} \int_{S^m} \prod_{\ell=1}^{m} \left[I_{\alpha_\ell}^{\mathbf{B}}(h_{t,\ell}^{\otimes \alpha_\ell}) \prod_{k=1}^{m} (I_{\beta_{\ell,k}}^{\mathbf{B}}(\bar{h}_{t,\ell,k}^{\otimes \beta_{\ell,k}})) \right] dt \\
&= d^{m/4} \sum_{|\boldsymbol{\gamma}|=q} c_{\boldsymbol{\gamma}} \int_{S^m} \prod_{\ell=1}^{m} \left[I_{\alpha_\ell}^{\mathbf{B}}(h_{t,\ell}^{\otimes \alpha_\ell}) I_{\sum_{k=1}^m \beta_{\ell,k}}^{\mathbf{B}}(\otimes_{k=1}^m \bar{h}_{t,\ell,k}^{\otimes \beta_{\ell,k}}) \right] dt \\
&= d^{m/4} \sum_{|\boldsymbol{\gamma}|=q} c_{\boldsymbol{\gamma}} \int_{S^m} I_{\sum_{\ell=1}^m (\alpha_\ell + \sum_{k=1}^m \beta_{\ell,k})}^{\mathbf{B}} (\otimes_{\ell=1}^m h_{t,\ell}^{\otimes \alpha_\ell} \otimes \otimes_{k=1}^m \bar{h}_{t,\ell,k}^{\otimes \beta_{\ell,k}}) dt \\
&= I_{|\boldsymbol{\gamma}|}^{\mathbf{B}} \left(d^{m/4} \sum_{|\boldsymbol{\gamma}|=q} c_{\boldsymbol{\gamma}} \int_{S^m} \otimes_{\ell=1}^m h_{t,\ell}^{\otimes \alpha_\ell} \otimes \otimes_{k=1}^m \bar{h}_{t,\ell,k}^{\otimes \beta_{\ell,k}} dt \right) \\
&= I_q^{\mathbf{B}}(g_{q,d}).
\end{aligned}$$

□

Proof of Lemma 5.1. For simplicity let us write $\tilde{g}_{q,d}(\boldsymbol{\lambda}) = d^{m/4} \int_{S^m} G_{t,q}(\boldsymbol{\lambda}) dt$ with

$$G_{t,q}(\boldsymbol{\lambda}) = \frac{1}{q!} \sum_{\pi \in \mathcal{P}_q} \sum_{|\boldsymbol{\gamma}|=q} c_{\boldsymbol{\gamma}} (\otimes_{\ell=1}^m h_{t,\ell}^{\otimes \alpha_\ell} \otimes \otimes_{k=1}^m \bar{h}_{t,\ell,k}^{\otimes \beta_{\ell,k}})(\boldsymbol{\lambda}_\pi),$$

being $\boldsymbol{\lambda}_\pi = (\lambda_{\pi(1)}, \dots, \lambda_{\pi(q)})$. Note that

$$\tilde{g}_{q,d} \otimes_n \tilde{g}_{q,d}(\boldsymbol{\lambda}_{(q-n)}) = d^{m/2} \int_{S^m \times S^m} [G_{s,q} \otimes_n G_{t,q}](\boldsymbol{\lambda}_{(q-n)}) ds dt.$$

The subscript of the vector $\boldsymbol{\lambda}$ stands for its dimension. Besides. since $G_{t,q}$ is the tensorial product of kernels $h_{t,\ell}, \bar{h}_{\ell,k,d}$ in $L_s^2([0,\infty))$, the last contraction can be expressed in terms of the contractions of these basic kernels $h_{t,\ell}, \bar{h}_{t,\ell,k}$. Besides, according to the isometric property of stochastic integrals we have

$$\begin{aligned}
[h_{s,\ell} \otimes_1 h_{t,\ell}](\lambda) &= \int_0^\infty h_{s,\ell}(\lambda) h_{t,\ell}(\lambda) d\lambda \\
&= \mathbb{E}\left[\int_0^\infty h_{s,\ell}(\lambda) dB(\lambda) \cdot \int_0^\infty h_{t,\ell}(\lambda) dB(\lambda) \right] = \mathbb{E}\left(Y_\ell(s) Y_\ell(t) \right) \\
&= r_d(s,t).
\end{aligned}$$

Similar relations hold for the remaining factors. For instance,

$$\mathbb{E}\left(Y_\ell(s) \overline{Y}_{\ell,k}(t) \right) = \partial_{t_k} r_d(\langle s,t \rangle) = r_d'(\langle s,t \rangle) \partial_{t_k}(\langle s,t \rangle),$$

where ∂_{t_k} means that we fix s and take derivative to the k-th direction.

Hence, we can write

$$[G_{s,q} \otimes_n G_{t,q}](\boldsymbol{\lambda}_{(q-n)}) = \frac{1}{(q!)^2} \sum_{\pi,\pi' \in \mathcal{P}_q} \sum_{|\boldsymbol{\gamma}|=|\boldsymbol{\gamma}'|=q} c_{\boldsymbol{\gamma}} c_{\boldsymbol{\gamma}'} \rho_{q,d}^{(n)}(s,t) \bar{G}_{2q-2n}(\boldsymbol{\lambda}_{\pi,\pi'});$$

where $\rho_{q,d}^{(n)}(s,t)$ is a product of covariances of $(Y_\ell(s), Y_\ell(t))$; $(Y_\ell(s), \overline{Y}'_\ell(t))$ and $(\overline{Y}'_\ell(s), \overline{Y}'_\ell(t))$ with total degree n while $\bar{G}_{2q-2n}(\lambda_{\pi,\pi'})$ is a tensor product of kernels $h_{t,\ell}$ and $\bar{h}_{t,\ell,k}$ with degree $2q-2n$ and the coordinates are permuted according to π and π'. Note that which covariances are involved in the ρ's depend on the indexes $\boldsymbol{\gamma}, \boldsymbol{\gamma}'$.

Therefore, writting $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3, \pi_4)$,

$$\|g_{q,d} \otimes_n g_{q,d}\|_2^2 \leq d^m \frac{1}{(q!)^2} \sum_{\boldsymbol{\pi} \in (\mathcal{P}_q)^4} \sum_{|\boldsymbol{\gamma}|=|\boldsymbol{\gamma}|=q} c_{\boldsymbol{\gamma}} c_{\boldsymbol{\gamma}'} \cdot \int_{(S^m)^4} \rho_{q,d}^{(n)}(s,t) \rho_{q,d}^{(n)}(s',t') \cdot \rho_{q,d}^{(q-n)}(t,t') \rho_{q,d}^{(q-n)}(s,s') ds dt ds' dt'.$$

The variance of Y_ℓ restricted to the sphere is constantly 1, Cauchy-Schwarz inequality implies that we can bound the absolute value of the power of any covariance by the very covariance. Hence, we can bound each term in the last sum by a term of the form (up to a constant)

$$d^m \int_{(S^m)^4} |r_d^{(k_1)}(s,t) r_d^{(k_2)}(s',t') r_d^{(k_3)}(t,t') r_d^{(k_4)}(s,s')| ds dt ds' dt',$$

where $k_j = 0, 1, 2$ indicates a derivative of order $0, 1$ or 2 of r. Since each covariance is a function of the inner product of its arguments, they are invariant under isometries. Thus, consider isometries U_s such that $U_s(s) = e_0$ and $U_{t'}$ such that $U_{t'}(t') = e_0$. Then, (5.3) can be written as

$$d^m \int_{(S^m)^4} |r_d^{(k_1)}(\langle e_0, U_s(t)\rangle) r_d^{(k_2)}(\langle U_{t'}(s'), e_0\rangle)| \cdot |r_d^{(k_3)}(\langle U_{t'}(t), e_0\rangle) r_d^{(k_4)}(\langle e_0, U_s(s')\rangle\, | ds dt ds' dt'.$$

Introducing the, isometric, change of variables $\tau_1 = U_s(t)$, $\tau_2 = U_{t'}(s')$, $\tau_3 = U_{t'}(t)$ and $\tau_4 = s'$ and bounding $|r_d^{(k_4)}(\langle e_0, U_s(s')\rangle)| \leq 1$ we get that the last expression is less or equal than

$$\begin{aligned} d^m & \int_{(S^m)^4} |r_d^{(k_1)}(\langle e_0, \tau_1\rangle) r_d^{(k_2)}(\langle e_0, \tau_2\rangle) r_d^{(k_3)}(\langle e_0, \tau_3\rangle)| d\tau_1 d\tau_2 d\tau_3 d\tau_4 \\ &= C_m d^m \prod_{j=1}^3 \int_{S^m} |r_d^{(k_j)}(\langle e_0, \tau_j\rangle)| d\tau_j = C_m d^m \prod_{j=1}^3 \int_0^\pi \sin^{m-1}(\theta) |r_d^{(k_j)}(\cos(\theta))| d\theta \\ &= C_m d^m \prod_{j=1}^3 \int_0^{\pi/2} \sin^{m-1}(\theta) |r_d^{(k_j)}(\cos(\theta))| d\theta, \end{aligned}$$

where we used and the symmetry wrt $\theta = \pi/2$ of the integrand.

The result follows. □

6.3. Anciliary computations

We start the computations with the covariances of the vector $(\mathbf{Z}_d(s), \mathbf{Z}_d(t))$ defined in (4.3), see [2]. Actually, by the definition of KSS distribution, it suffices to consider

$$\left(Y_\ell(s), Y_\ell(t), \overline{\mathbf{Y}}'_\ell(s), \overline{\mathbf{Y}}'_\ell(t)\right).$$

for a fixed $\ell = 1, \dots, m$. Its variance-covariance matrix can be writen in the following form

$$\left[\begin{array}{c|c|c} A_{11} & A_{12} & A_{13} \\ \hline A_{12}^\top & I_m & A_{23} \\ \hline A_{13}^\top & A_{23}^\top & I_m \end{array}\right],$$

where I_m is the $m \times m$ identity matrix,

$$A_{11} = \begin{bmatrix} 1 & \mathcal{C} \\ \mathcal{C} & 1 \end{bmatrix},\ A_{12} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ -\mathcal{A} & 0 & \cdots & 0 \end{bmatrix},\ A_{13} = \begin{bmatrix} \mathcal{A} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

and $A_{23} = \mathrm{diag}([\mathcal{B}, \mathcal{D}, \dots, \mathcal{D}])_{m\times m}$. The quantities $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ were defined in (4.5).

Gaussian regression formulas, see [6], imply that the conditional distribution of the vector $(\overline{Y}'_\ell(s), \overline{Y}'_\ell(t))$, conditioned on $\mathbf{Y}_d(s) = \mathbf{Y}_d(t) = 0$, is centered normal with variance-covariance matrix given by

$$\left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{12}^\top & B_{22} \end{array}\right],$$

with $B_{11} = B_{22} = \mathrm{diag}([\sigma^2, 1, \dots, 1])$ and $B_{12} = \mathrm{diag}([\sigma^2\rho, \mathcal{D}, \dots, \mathcal{D}])$.

Now we move to the proof of some lemmas.

Lemma 6.2. *The function $\mathcal{H}_{q,d}$ defined in (5.2) is even.*

Proof. We need to explicit the multi-indexes.

$$\begin{aligned} &\mathbb{E}\left[\mathbf{H}_{\boldsymbol{\alpha}}(\mathbf{Y}(s))\mathbf{H}_{\boldsymbol{\beta}}(\overline{\mathbf{Y}}'(s))\mathbf{H}_{\boldsymbol{\alpha}'}(\mathbf{Y}(t))\mathbf{H}_{\boldsymbol{\beta}'}(\overline{\mathbf{Y}}'(t))\right] \\ &\quad = \prod_{\ell=1}^m \mathbb{E}\left[H_{\alpha_\ell}(Y_\ell(s))H_{\alpha'_\ell}(Y_\ell(t)) \cdot \prod_{k=1}^m H_{\beta_{\ell k}}(\overline{Y}'_{\ell k}(s)) \cdot \prod_{k'=1}^m H_{\beta'_{\ell k'}}(\overline{Y}'_{\ell k'}(t))\right] \\ &\qquad = \prod_{\ell=1}^m \mathbb{E}\left[H_{\alpha_\ell}(Y_\ell(s))H_{\alpha'_\ell}(Y_\ell(t))H_{\beta_{\ell 1}}(\overline{Y}'_{\ell 1}(s))H_{\beta'_{\ell 1}}(\overline{Y}'_{\ell 1}(t))\right] \\ &\qquad\qquad \cdot \prod_{j=2}^m \mathbb{E}\left[H_{\beta_{\ell j}}(\overline{Y}'_{\ell j}(s))H_{\beta'_{\ell j}}(\overline{Y}'_{\ell j}(t))\right]. \quad (6.5) \end{aligned}$$

In the second equality we use that the random vectors

$$(Y_\ell(s), Y_\ell(t), \overline{Y}'_{\ell 1}(s), \overline{Y}'_{\ell 1}(t)); \quad (\overline{Y}'_{\ell j}(s), \overline{Y}'_{\ell j}(t)); \quad j \geq 2$$

are independent.

Using Mehler formula, see Lemma 10.7 in [6], we get

$$\mathbb{E}\,[H_{\beta_{\ell j}}(\overline{Y}'_{\ell j}(s))H_{\beta'_{\ell j}}(\overline{Y}'_{\ell j}(t))] = \delta_{\beta_{\ell j}\beta'_{\ell j}}\beta_{\ell j}!\,(\rho''_{\ell j})^{\beta_{\ell j}},$$

where $\rho''_{\ell k} = \rho''_{\ell k}(\langle s,t\rangle) = \mathbb{E}\,(\overline{Y}'_{\ell j}(s)\overline{Y}'_{\ell j}(t))$. Since $\beta_{\ell j}$ is even, this factor is even.

For the first factor, using again Mehler formula we get

$$\mathbb{E}\,[H_{\alpha_\ell}(Y_\ell(s))H_{\alpha'_\ell}(Y_\ell(t))H_{\beta_{\ell 1}}(\overline{Y}'_{\ell 1}(s))H_{\beta'_{\ell 1}}(\overline{Y}'_{\ell 1}(t))] = 0,$$

if $\alpha_\ell + \beta_{\ell 1} \neq \alpha'_\ell + \beta'_{\ell 1}$. Otherwise, consider $\Lambda \subset \mathbb{N}^4$ defined by

$$\Lambda = \{(d_1,d_2,d_3,d_4) : d_1+d_2 = \alpha_\ell,\ d_3+d_4 = \beta_{\ell 1},\ d_1+d_3 = \alpha'_\ell,\ d_2+d_4 = \beta'_{\ell 1}\};$$

then

$$\mathbb{E}\,[H_{\alpha_\ell}(Y_\ell(s))H_{\alpha'_\ell}(Y_\ell(t))H_{\beta_{\ell 1}}(\overline{Y}'_{\ell 1}(s))H_{\beta'_{\ell 1}}(\overline{Y}'_{\ell 1}(t))] = \sum_{(d_i)\in\Lambda} \frac{\alpha_\ell!\alpha'_\ell!\beta_{\ell 1}!\beta'_{\ell 1}!}{d_1!d_2!d_3!d_4!}\rho^{d_1}(\rho')^{d_2}(\rho')^{d_3}(\rho'')^{d_4},$$

where $\rho = \rho(\langle s,t\rangle) = \mathbb{E}\,(Y_\ell(s)Y_\ell(t))$, $\rho' = \mathbb{E}\,(Y_\ell(s)\overline{Y}'_{\ell 1}(t))$, $\rho' = \mathbb{E}\,(\overline{Y}'_{\ell 1}(s)Y_\ell(t))$ and $\rho'' = \mathbb{E}\,(\overline{Y}'_{\ell 1}(s)\overline{Y}'_{\ell 1}(t))$.

Note that the conditions defining the index set Λ and the fact that the indexes α's and β's are even imply that all the d_i's have the same parity. In this form, the first factor in Equation (6.5) is

$$\prod_{\ell=1}^{m} \sum_{(d_i)\in\Lambda} \frac{\alpha_\ell!\alpha'_\ell!\beta_{\ell 1}!\beta'_{\ell 1}!}{d_1!d_2!d_3!d_4!}\rho^{d_1}(\rho')^{d_2}(\rho')^{d_3}(\rho'')^{d_4},$$

Hence, if we change $\langle s,t\rangle$ by $-\langle s,t\rangle$, for each ℓ we have the factor

$$(-1)^{dd_1}\cdot(-1)^{(d-1)(d_2+d_3)}\cdot(-1)^{dd_4} = (-1)^{d(d_1+d_4)+(d-1)(d_2+d_3)} = 1,$$

since the d_i's have the same parity. □

Proof of Lemma 5.3. Recall that if ζ is a standard Gaussian random variable in $\mathbb{R}^{m^2}$, then $||G_q||^2 = \mathbb{E}\,[G_q^2(\zeta)]$. By Hermite polynomials properties, we have

$$\mathbb{E}\,[G_q^2(\zeta)] = \sum_{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|=q} b_{\boldsymbol{\alpha}}^2\boldsymbol{\alpha}!f_{\boldsymbol{\beta}}^2\boldsymbol{\beta}!.$$

The following digression will be useful. Let us consider two sequences b_k and a_k such that $\sum_{k=0}^{\infty} a_k^2 < \infty$ and $b_k^2 \to 0$. We are interested in the sum

$$\sum_{k=0}^{q} a_k^2 b_{q-k}^2 = b_0^2 a_q^2 + b_1^2 a_{q-1}^2 + \ldots + b_q^2 a_0^2.$$

By hypothesis $\sup_k b_k^2 = ||b^2||_\infty < \infty$, thus we get

$$\sum_{k=0}^{q} b_k^2 a_{q-k}^2 \leq ||b^2||_\infty ||a||_2^2. \tag{6.6}$$

Using this elementary fact we can explore the behavior of our sum

$$d_q = \sum_{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|=q} b_{\boldsymbol{\alpha}}^2 f_{\boldsymbol{\beta}}^2 \boldsymbol{\alpha}!\boldsymbol{\beta}! = \sum_{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|=2l} b_{2\alpha_1}^2 \ldots b_{2\alpha_m}^2 f_{(\boldsymbol{\beta}_1,\ldots,\boldsymbol{\beta}_m)}^2 \boldsymbol{\alpha}!\boldsymbol{\beta}!.$$

We affirm that $b_{2\alpha}^2(2\alpha)!$ is decreasing, in fact

$$\frac{b_{2(j+1)}^2(2(j+1))!}{b_{2j}^2(2j)!} = \left[\frac{1}{2}\right]^2 \frac{(2j+1)(2j+2)}{(j+1)^2} = \frac{(j+\frac{1}{2})}{j+1} < 1.$$

Moreover $b_0^2 = \frac{1}{2\pi} < 1$, then $||b||_\infty < 1$. Consider now

$$\mathbb{E}\left[G_q^2(\zeta)\right] = \sum_{k_{m+1}=0}^{q} \sum_{|\boldsymbol{\beta}_2|+\ldots+|\boldsymbol{\beta}_m|=k_{m+1}} \sum_{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}_1|=q-k_{m+1}} b_{\boldsymbol{\alpha}}^2 f_{\boldsymbol{\beta}}^2 \boldsymbol{\alpha}!\boldsymbol{\beta}!.$$

Set $\alpha_i + \beta_{1i} = l_i$, such that $q - k_{m+1} = l_1 + \ldots + l_m$. In this form we obtain

$$\sum_{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}_1|=q-k_{m+1}} b_{\boldsymbol{\alpha}}^2 f_{\boldsymbol{\beta}}^2 \boldsymbol{\alpha}!\boldsymbol{\beta}! = \prod_{i=1}^{m} \sum_{\alpha_i+\beta_{1i}=l_i} b_{\alpha_i}^2 f_{\boldsymbol{\beta}}^2 \alpha_i! \boldsymbol{\beta}_1! \prod_{j=2}^{m} \boldsymbol{\beta}_j!.$$

Using (6.6) it yields

$$\sum_{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}_1|=q-k_{m+1}} b_{\boldsymbol{\alpha}}^2 f_{\boldsymbol{\beta}}^2 \boldsymbol{\alpha}!\boldsymbol{\beta}! \leq \sum_{l_1+\ldots+l_m=q-k_{m+1}} \prod_{i=1}^{m} \sum_{\beta_{1i}=0}^{l_i} f_{\boldsymbol{\beta}}^2 \boldsymbol{\beta}!.$$

This allows us getting the following bound

$$\mathbb{E}\left[G_q^2(\zeta)\right] \leq ||f||_2^2.$$

The result follows. □

References

[1] M. Arcones. Limit theorems for nonlinear functionals of a stationary Gaussian sequence of vectors. Ann. Probab. 22 (1994), no. 4, 2242-2274.

[2] D. Armentano, J-M. Azaïs, F. Dalmao and J. León. On the asymptotic variance of the number of real roots of random polynomial systems. arXiv:1703.08163.
[3] J-M. Azaïs, F. Dalmao and J. León, CLT for the zeros of classical random trigonometric polynomials. Ann. Inst. Henri Poincaré Probab. Stat. 52 (2016), no. 2, 804-820.
[4] J-M. Azaïs, J. León, CLT for crossings of random trigonometric polynomials, Electron. J. Probab. 18 (2013), no. 68, 17 pp.
[5] J-M Azaïs and M. Wschebor. On the roots of a random system of equations. The theorem of Shub and Smale and some extensions. Foundations of Computational Mathematics, 5(2), 125-144.
[6] J-M Azaïs and M. Wschebor. Level sets and extrema of random processes and fields. John Wiley & Sons Inc., Hoboken, NJ, (2009), ISBN: 978-0-470-40933-6.
[7] A. T. Bharucha-Reid and M. Sambandham. Random polynomials. Probability and Mathematical Statistics. Academic Press Inc., Orlando, FL, 1986.
[8] A. Bloch and G. Pólya. On the number of real roots of a random algebraic equation. Proc. Cambridge Philos. Soc., 33:102-114, 1932.
[9] F. Dalmao. Asymptotic variance and CLT for the number of zeros of Kostlan Shub Smale random Polynomials. C.R. Acad. Sci. Paris Ser. I 353 (2015), 1141-1145.
[10] Y. Do and V. Vu. Central limit theorems for the real zeros of Weyl polynomials. arXiv:1707.09276.
[11] A. Estrade and J. Fournier. Number of critical points of a Gaussian random field: Condition for a finite variance. Statistics & Probability Letters Volume 118, November 2016, Pages 94-99.
[12] A. Granville and I. Wigman. The distribution of the zeros of random trigonometric polynomials. Amer. J. Math., 133(2), (2011), 295-357.
[13] M. Kac. On the average number of real roots of a random algebraic equation. Bull. Amer. Math. Soc., 49:314-320, 1943.
[14] E. Kostlan. On the distribution of roots of random polynomials. In The work of Smale in differential topology, From Topology to Computation: Proceedings of the Smalefest,
[15] M. F. Kratz and J. R. León. Hermite polynomial expansion for non-smooth functionals of stationary Gaussian processes: crossings and extremes. Stochastic Process. Appl., 66(2), (1997), 237-252.
[16] T. Letendre. Variance of the volume of random real algebraic submanifolds. arXiv:1608.05658v4
[17] T. Letendre and M. Puchol. Variance of the volume of random real algebraic submanifolds II. arXiv:1707.09771
[18] J.E. Littlewood and A.C. Offord. On the number of real roots of a random algebraic equation. J. London Math. Soc., 13:288-295, 1938.
[19] J.E. Littlewood and A.C. Offord. On the roots of certain algebraic equations. Proc. London Math. Soc., 35:133-148, 1939.
[20] N. B. Maslova. The distribution of the number of real roots of random polynomials. Teor. Verojatnost. i Primenen., 19, (1974), 488-500.

[21] N. B. Maslova, The variance of the number of real roots of random polynomials. (Russian. English summary) Teor. Verojatnost. i Primenen. 19 (1974), 36-51.
[22] I. Nourdin and G. Peccati. Normal approximations with Malliavin calculus. From Stein's method to universality. Cambridge Tracts in Mathematics, 192. Cambridge University Press, Cambridge, (2012), xiv+239 pp. ISBN: 978-1-107-01777-1.
[23] F. Nazarov, M. Sodin. Asymptotic laws for the spatial distribution and the number of connected components of zero sets of Gaussian random functions. Journal of Mathematical Physics, Analysis, Geometry. V12, No. 3 pp 205-278 (2016).
[24] G. Peccati and M. Taqqu. Wiener chaos: moments, cumulants and diagrams. A survey with computer implementation. Supplementary material available online. Bocconi & Springer Series, 1. Springer, Milan; Bocconi University Press, Milan, (2011), xiv+274 pp. ISBN: 978-88-470-1678-1.
[25] M. Shub and S. Smale. Complexity of Bézout's theorem. II. Volumes and Computational algebraic geometry (Nice, 1992), 267-285, Progr. Math., 109, Birkhäuser Boston, Boston, MA, 1993.
[26] M. Sodin, B.Tsirelson. Random complex zeroes. I. Asymptotic normality. Israel J. Math. 144 (2004), 125-149.
[27] M. Wschebor, On the Kostlan-Shub-Smale model for random polynomial systems. Variance of the number of roots. J. Complexity 21 (2005), no. 6, 773-789.